MATH 135 Fall 2007

Proofs, Part III

Some miscellaneous induction examples...

Example

Prove that $\sum_{r=1}^{n} f_r^2 = f_n f_{n+1}$ for all $n \in \mathbb{P}$.

We prove this by induction. Perhaps surprisingly, we do not need strong induction.

<u>Base Case</u>

If $n = 1$, $\sum_{r=1}^{1} f_r^2 = f_1^2 = 1$ and $f_1 f_2 = 1$, so the result is true.

<u>Induction Hypothesis</u>

Assume the result is true for $n = k$, for some $k \in \mathbb{P}$, $k \geq 1$.

<u>Induction Conclusion</u>

Consider $n = k + 1$

$$\begin{aligned} \sum_{r=1}^{k+1} f_r^2 &= \left[\sum_{r=1}^{k} f_r^2\right] + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \qquad \text{(by Induction Hypothesis)} \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2} \end{aligned}$$

so the result is true for $n = k + 1$.
By POMI, the result is true for all $n \in \mathbb{P}$.

Example

If $n + 1$ concrete blocks are placed in n knapsacks, then one knapsack must contain at least two concrete blocks.

We prove this now by induction, and use a different approach later.

<u>Base Case</u>

$n = 1 :$ 1 knapsack, 2 concrete blocks

<u>Induction Hypothesis</u>

Assume the result holds for $n = k$, for some $k \in \mathbb{P}$, $k \geq 1$.

<u>Induction Conclusion</u>

Consider $n = k + 1$.
We have $k + 1$ knapsacks and $k + 2$ concrete blocks.

<u>Case 1</u>

If knapsack 1 contains two or more concrete blocks, we are done.

<u>Case 2</u>

If knapsack 1 contains one concrete block, we have k knapsacks and $k+1$ concrete blocks remaining. By Induction Hypothesis, one of these last k knapsacks contains two or more concrete blocks.

Case 3
If knapsack 1 contains 0 concrete blocks, we are left with k knapsacks and $k+2$ concrete blocks. So certainly $k+1$ concrete blocks are put in k knapsacks (plus one more!).
By Induction Hypothesis, one of these last k knapsacks contains two or more concrete blocks.

Therefore the result holds for $n = k+1$.
Therefore the result holds for all $n \in \mathbb{P}$ by POMI.

Note
This is called the "Pigeonhole Principle". Can you generalize to n boxes, $mn+1$ objects?

Example
An $m \times n$ rectangle of $N = mn$ chocolate squares is to be broken into unit squares. Prove that $mn-1$ breaks are always needed.

We prove this by induction on N, the total number of squares.

Base Case
$N = 1$: 0 cuts are needed.

Induction Hypothesis
Assume the result holds for all bars of any dimensions of up to K squares. (Note that this is a Strong hypothesis.)

Induction Conclusion
Consider $N = K+1$.
Take such a bar and break it into two pieces of sizes X and Y chocolate squares. (We have to make a first break somehow.)
Note $X+Y = K+1$ and $X, Y \geq 1$ so $X, Y \leq K$.
By our (Strong) Induction Hypothesis, these two new bars take $X-1$ and $Y-1$ breaks to separate into unit squares. (In order to separate the whole bar into unit squares, we must separate each of these bits into unit squares.)
Thus, the total number of breaks required is

$$1 + (X-1) + (Y-1) = X + Y - 1 = K + 1 - 1 = N - 1$$

Therefore the result holds for $N = K+1$
Therefore the result holds for all $N \in \mathbb{P}$, by POSI.

Example
Prove that $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n \cdot (n+1)} = 1 - \dfrac{1}{n+1}$.
Try this on your own by induction – it's fairly standard.

How about this: $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ (try putting R.S. back together)

So:

$$
\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} &= \left[\frac{1}{1}-\frac{1}{2}\right]+\left[\frac{1}{2}-\frac{1}{3}\right]+\cdots+\left[\frac{1}{n}-\frac{1}{n+1}\right] \\
&= \frac{1}{1}-\frac{1}{n+1}
\end{aligned}
$$

How do we know for sure? What really happens inside "$\cdots$"?
Are we totally comfortable with this?
We are using induction tacitly here. In mathematics, we often use induction without saying so.