Topological Proof of the Riemann-Hurwitz Theorem

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Definition 1. If C and X are topological spaces, a *covering* of X by C is a continuous surjection $p\colon C \to X$ such that, for every $x \in X$, there is a neighbourhood V of x such that $p^{-1}V$ is a disjoint union of neighbourhoods $U_i, i \in I$, each one homeomorphic to V, with the homeomorphisms given by the restriction of p to each neighbourhood U_i. If $|I| = n$ for every $x \in X$, then C is an *n-sheeted* covering. If, for every $x \in X$, all of X is a possible choice for V, then p is a *trivial* covering, and C is the disjoint union of copies of X.

If $D = \{z : |z| < 1\}$, and $D^o = D - \{0\}$, and $e \in \mathbb{N}$, then $p\colon D^o \to D^o$ by $pw = w^e$ is an e-sheeted covering of D^o. This is true because, for every point $x \in D^o$, say $\alpha^e = z$. Then, if ω is a primitive e^{th} root of unity, there are disjoint neighbourhoods U_i of $\omega^i\alpha$ that are mapped homeomorphically to a neighbourhood V of x. However, if $p\colon D \to D$ by $pw = w^e$, then p is not a covering of D unless $e = 1$. For, if V is any neighbourhood of 0, all components of $p^{-1}V$ that map to V contain 0, and thus if p were a covering, it would be a 1-sheeted covering, which means $e = 1$, for p is bijective.

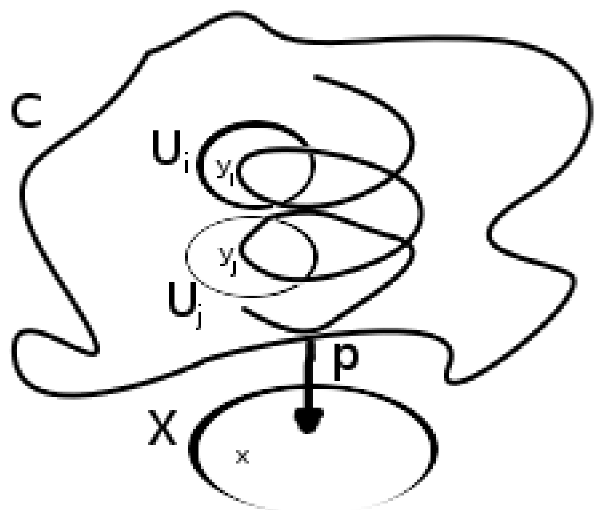

FIGURE 1

Theorem 2. *If X is simply connected, and $p\colon C \to X$ is a covering, then p is a trivial covering.*

Proof. Fix a point $x \in X$ and a path γ originating at x. The preimage under p of γ is unique given a starting point $y \in C$ such that $py = x$. For, by the compactness of γ, γ can be covered with finitely many open sets such that the preimage of each open set is a disjoint union of homeomorphic open sets in C. Say $U \subset C$ is one such open set with $y \in U$. Some of those open sets intersect a neighbourhood of x, so $p^{-1}\gamma \cap U$ can be extended to them. Repeating this process one neighbourhood at a time, a path δ starting at y is found in C such that $p\delta = \gamma$.

Say V is a neighbourhood of x such that $p^{-1}V$ is the disjoint union of U_i such that $pU_i = V$, and the restriction $p_{|U_i}$ is a homeomorphism. Then, in the same manner that paths are extended, U_i can be extended to a neighbourhood W_i such that $pW_i = X$, and the restriction p_{W_i} is a homeomorphism. Assume p is not trivial, then there is a point $x \in X$ such that after performing the extension of U_i, the W_i are not disjoint, otherwise U_i could be chosen to be W_i, which would contradict the non triviality of p. Because W_i are path connected, there is i, j with $i \neq j$ and a path δ from $y_i \in U_i$ to $y_j \in U_j$ such that $py_i = py_j = x$. Then, $p\delta = \gamma$ is a loop in X originating at x. Thus, γ is contractable to a point, x. In a similar manner as to the previous paragraph, the preimage of the homotopy describing the contraction can be taken to be a homotopy of δ with fixed starting and ending points at every stage of the homotopy. Thus, δ is homotopic with a point, but the endpoints are fixed, so the endpoints must be equal, which is a contradiction, as $i \neq j$ implies U_i and U_j are disjoint. $\square$

Definition 3. A Riemann surface is a connected surface locally homeomorphic to $\mathbb{C}$ with analytic change of coordinant maps. More formally, a topological space X is a Riemann surface if for all $x, y \in X$, there are open neighbourhoods V_x of x and V_y of y in X, and open sets U_x and U_y in $\mathbb{C}$, and homeomorphisms $\phi\colon U_x \to V_x$ and $\psi\colon U_y \to V_y$ such that $\psi^{-1}\phi$ and $\phi^{-1}\psi$ are analytic where they are defined (they will be defined when, and only when $V_y \cap V_x \neq \varnothing$.) The functions ϕ and ψ are called coordinant maps. The functions $\psi^{-1}\phi$ and $\phi^{-1}\psi$ are called change of coordinant maps.

Definition 4. A function $f\colon X \to Y$ between Riemann surfaces is *analytic* if for every $P \in X$, with $Q = fP$, and open sets U_P and U_Q of $\mathbb{C}$ and coordinant maps $\phi\colon U_P \to V_P$, and $\psi\colon U_Q \to V_Q$, where V_P is a neighbourhood of P, and V_Q is a neighbourhood of Q, the change of coordinant map $\psi^{-1}f\phi$ is analytic.

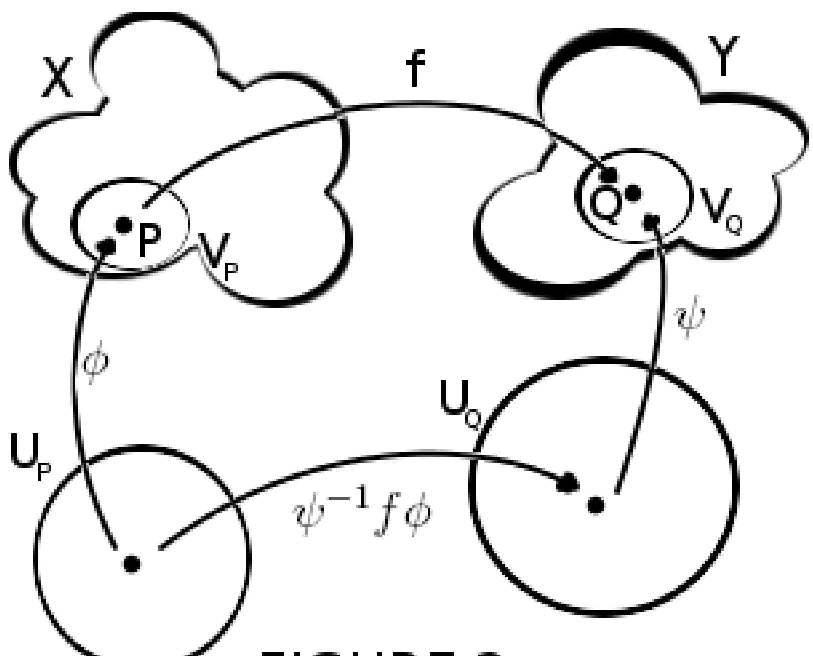

FIGURE 2

Theorem 5. *If the situation around P is as in figure 2, ϕ and ψ can be chosen so that*

$\psi^{-1} f \phi w = w^e$ *for some* $e \in \mathbb{N}$.

Proof. Assume U_P and U_Q contain the origin, and that $\phi 0 = P$ and $\psi 0 = Q$. This is possible because U_P and U_Q are open, so ϕ and ψ can be composed with the appropreate translations, which are analytic homeomorphisms. Write $\psi^{-1} f \phi = h(z) = \sum_{k=0}^{\infty} a_k z^k$. Let $e \in \mathbb{N}$ be the smallest number such that $a_e \neq 0$. So, $h(z) = z^e g(z)$ where $g(z) = \sum_{k=0}^{\infty} a_{e+k} z^k$, and $g(0) \neq 0$. So, $\frac{g(0)}{a_e} = 1$. The e^{th} root function is defined around 1, so $k(z) = (\frac{g(z)}{a_e})^{\frac{1}{e}}$ is analytic in a neighbourhood of 0. Also, $h(z) = z^e g(z) = (zk(z))^e$. The derivative $(zk(z))' = k(z) + zk'(z)$, so $(zk(z))'|_0 = k(0) \neq 0$, so $z \mapsto zk(z)$ is invertible around 0 by the analytic inverse function theorem. Let w be it's inverse, then $\psi^{-1} f \phi w = (zk(z))^e = w^e$. The restriction of $\psi^{-1} f \phi$ to $U_P - \{0\}$ is an e-sheeted covering of $U_Q - \{0\}$. But, ϕ and ψ are homeomorphisms, so they preserve the disjointness of neighbourhoods, and they are bijective, so f is e to 1, and the preimage under f of a neighbourhood in $V_Q - Q$ is the disjoint union of e neighbourhoods. Therefore, f restricted to V_P is an e-sheeted covering of V_Q. So, e does not depend on the choice of ϕ and ψ. The number e is called the *ramification index* of f at P, denoted $e(P)$, and if $e > 1$, then P is called a *ramification point* of f. Let R be the set of all ramification points of f, and let $S = fR$. ◻

Theorem 6. *If* $f\colon X \to Y$ *is a surjective analytic function between compact Riemann Surfaces, then the following hold:*

1. *The set of ramification points of* X *is finite. Also, the number of points in* $f^{-1}Q$ *is finite for every* $Q \in Y$.

2. *The function* f *restricted to* $X - R$ *is an* n*-sheeted covering of* $Y - S$.

3. *For every* $Q \in Y$, $\displaystyle\sum_{P \in f^{-1}(Q)} e(P) = n$.

Proof. 1. Since X is compact, it is enough to show that R is discrete. For, if R were not discrete, there would be a convergent sequence P_k in R. Say figure 2 is the situation around P. Then, U_P and U_Q are Riemann surfaces, and $\psi^{-1} f \phi$ is an analytic function from U, a neighbourhood of 0 in U_P to U_Q. But, $P_k \to P$ so there is a P_m in ϕU. However, P_k is a ramification point of f, so f is many to 1 on $\phi_{P_m} U_{P_m} - P_m \cap f \phi U$, so P_m is a ramification point of $\psi^{-1} f \phi$. This is a contradiction, for $f^{-1} P_m \neq 0$, and 0 is the only possible ramification point of $\psi^{-1} f \phi$.

Similarly, $f^{-1}Q$ is discrete. For if $P_k \to P$, and $fP_k = Q$, then $fP = Q$ by continuity of f. Say figure 2 is the situation around P. Then, there is a $P_m \in V_P$ such that $\phi_{-1} P_m = w_0 \neq 0$, and $fP_m = Q$. Let ϕ_{P_m} be a coordinant map that takes 0 to P_m, and maps onto a neighbourhood U_{P_m} of P_m. Then, $U_P \cap U_{P_m} \neq \varnothing$, and $0 = \psi_Q^{-1} f \phi_{P_m} 0$ and $\psi_Q^{-1} f \phi_{P_m} 0 = \psi_Q^{-1} f \phi \phi^{-1} \phi_{P_m} 0$ and $\psi_Q^{-1} f \phi \phi^{-1} \phi_{P_m} 0 = \psi_Q^{-1} f \phi w_0 = w^e \neq 0$.

2. Say $Q \notin S$. It suffices to find a neighbourhood of Q whose preimage is the disjoint union of n homeomorphic copies of itself. By the first part of this theorem, $f^{-1}Q$ is finite, so let $f^{-1}Q = \{P_1, \ldots, P_n\}$. Choose U_i open such that $P_i \in U_i$, and if $V_i = fU_i$ then $V_i \cap S = \varnothing$. There is a neighbourhood of Q, V contained in $V_1 \cap \ldots \cap V_n$ such that if $U_i' = U_i \cap f^{-1}V$, then f restricted to U_i' is a homeomorphism onto V. Clearly, $U_i' \subset f^{-1}V$, so $U_1 \cup \ldots \cup U_n \subset f^{-1}V$. By contradiction, assume no such neighbourhood V existed. This amounts to stating that for a basis of neighbourhoods of Q, the reverse inclution does not hold. That is, there are neighbourhoods N_k of Q such that $\cap_k N_k = \{Q\}$ and there is $P_k' \in f^{-1}N_k$ but $P_k' \notin U_1' \cup \ldots \cup U_n'$. But X is compact, so there is a convergent subsequence $P_{k_i}' \to P' \in X$. As $\cap_k N_k = \{Q\}$, the $f^{-1}N_k$ are contained in the preimage of smaller and smaller neighbourhoods of Q. So, $fP_k \to Q$, and by continuity of f, $fP' = Q$. This implies that $P' = P_j$ for some j. So, $P' \in U_j$. But, $P_{k_i} \notin U_j$ for all i, and U_j a neighbourhood of P', so P_{k_i} cannot converge to P'.

   Note that n is constant, for if $Q_1, Q_2 \in Y$, there is a path connecting them that does not intersect S. This path is simply connected, so its covering space is trivial. Its covering space is n-sheeted at Q_1, so it must be n-sheeted at Q_2. Define the *degree* of f to be n, and denote this quantity $\deg f$.

3. If $Q \in Y$, let $f^{-1}Q = \{P_1, \ldots, P_m\}$. Then there are neighbourhoods U_i of P_i and V_i of Q such that $f \colon U_i \to V_i$ and there are coordinant maps such that $\psi^{-1} f \phi_{P_i} w = w^{e(P_i)}$. Then, f is $e(P_i)$ to 1 from $U_i - 0$ to $U_Q - 0$. There is a neighbourhood of Q, V contained in $\bigcap_i = 1^m V_i$ such that $V \cap S$ contains either nothing, or possibly Q. Then, by the second part of this theorem, f is n to 1 on $V - \{Q\}$. So, $\sum_{i=1}^m e(P_i) = n$.

$\square$

Theorem 7 (Riemann-Hurwitz). *If $f \colon X \to Y$ is a surjective analytic function between Riemann surfaces, then Y is triangularisable implies that X is triangularisable, and $2g_X - 2 = \deg f(2g_y - 2) + \sum_{P \in X}(e(P) - 1)$ where g_X is the genus of X, and g_Y is the genus of Y.*

Proof. Refine the triangularisation of Y so that it contains all of S as its vertices, and each edge and face contains only one element from S. This is possible as S is finite by the first part of the previous theorem. The triangularisation of X will be constructed to consist of the preimage under f of the edges, faces, and vertices of Y. Clearly, the preimage under f

of a vertex of the triangularisation of Y are vertices of X. Let T be an edge or face of Y, and φ the homeomorphism of T with an interval or triangle. Let T^o be the corresponding open line or open edge. Since each point of S is a vertex, $T^o \cap S = \varnothing$. So, by the second part of the previous theorem, f is an n-sheeted covering of T^o. Also, T^o is simply connected, so by the first theorem of covering spaces, f is a trivial covering, so the preimage of an edge or face is a disjoint union of n edges or faces, $U_1, \dots U_n$. In the case where $T \cap S = \varnothing$, T is still covered trivially by f, so the closure of each component U_i maps to T and is homeomorphic with an edge or face by the composition φf. If $T \cap S$ is not empty, it contains one point Q. Then, $T - \{Q\}$ is covered trivially by f, and its preimage is the disjoint union $U_1 - P, \dots U_n - P$ where each $U_i - P$ is homeomorphic by φf to an edge or face with a point removed. Also, as T contains only one point of S, φf extends to a bijection from T to an interval or triangle by $\varphi f P = \varphi Q$. To show it is a homeomorphism, assume figure 2 is the situation around P. Then, $|w_n| \to 0 \Leftrightarrow |w_n^{e(P)}| \to 0$, and $P_n \to P \Leftrightarrow \phi^{-1} P_n \to 0 \Leftrightarrow f P_n = Q$. So, U_i is homeomorphic with an interval or a triangle, and thus the preimage of T is n edges or faces.

If Y has a triangularisation with e edges, f faces, and v vertices, the preimage of the triangularisation of Y is a triangularisation of X with ne edges and ne faces. Let Q be a vertex in the triangularisation of Y. By part 3 of the previous theorem, $n - \sum_{P \in f^{-1}Q} e(P) = 0$. So, $n - \sum_{P \in f^{-1}Q}(e(P) - 1) = |f^{-1}Q|$. So, the induced triangularisation of X has

$$\sum_{Q \text{ a vertex}} \left(n - \sum_{P \in f^{-1}Q} (e(P) - 1) \right) \text{ vertices.}$$

But $e(P) - 1 = 0$ unless $P \in R$, and each P is in the preimage of only 1 point, and each element of S is a vertex of the triangularisation of Y, so summing over elements in the preimage of each vertex is the same as summing over all of R, or all of X. So, the number of vertices in the induced triangularisation of X is $nv - \sum_{P \in X}(e(P) - 1)$. So, $2g_X - 2 = nv - ne + nf - \sum_{P \in X}(e(P) - 1) = \deg f(2 - 2g_Y) - \sum_{P \in X}(e(P) - 1)$. $\square$