Basic Principles of Enumeration.

In this section and the next we'll see several "basic principles" of enumeration and apply them to a variety of problems, of both mathematical and general interest. The basic principles themselves are phrased in the language of sets. For a (finite) set A we will use the notation $|A|$ or $\#A$ to denote the number of elements of A, also called the *cardinality* of A. The empty set is denoted by $\varnothing$ – it is the unique set with no elements.

1. Choices – "AND" versus "OR".

In the next few pages we will often be constructing an object of some kind by repeatedly making a sequence of choices. In order to count the total number of objects we could construct we must know how many choices are available at each step, *but we must know more*: we also need to know how to combine these numbers correctly. A generally good guideline is to look for the words "AND" and "OR" in the description of the sequence of choices available. Here are a few simple examples.

On a table before you are 7 books, 8 magazines, and 5 DVDs.

• *Choose a book and a DVD.* There are 7 choices for book AND 5 choices for DVD: $7 \times 5 = 35$ choices in all.

• *Choose a book or a magazine.* There are 7 choices for book OR 8 choices for magazine: $7 + 8 = 15$ choices in all.

• *Choose a book and either a magazine or a DVD.* There are $7 \times (8+5) = 91$ possible choices.

• *Choose either a book and a magazine, or a DVD.* There are $(7 \times 8) + 5 = 61$ possible choices.

Generally, "AND" corresponds to multiplication and "OR" corresponds to addition. The last two of the above examples show that it is important to determine exactly how the words "AND" and "OR" combine in the description of the problem.

From a mathematical point of view, "AND" corresponds to the Cartesian product of sets. If you choose one element of the set A AND you choose one element of the set B, then this is equivalent to choosing one element of the *Cartesian product of A and B*:

$$A \times B = \{(a, b) : \ a \in A \text{ and } b \in B\},$$

which is the set of all ordered pairs of elements (a, b) with $a \in A$ and $b \in B$. In general, the cardinalities of these sets are related by the formula

$$|A \times B| = |A| \cdot |B|.$$

Similarly, from a mathematical point of view, "OR" corresponds to the union of sets. If you choose one element of the set A OR you choose one element of the set B, then this is equivalent to choosing one element of the *union of A and B*:

$$A \cup B = \{c : \ c \in A \text{ or } c \in B\},$$

which is the set of all elements c which are either in A or in B. It is NOT always true that $|A \cup B| = |A| + |B|$, because any elements in both A and B would be counted twice by $|A| + |B|$. The *intersection of A and B* is the set

$$A \cap B = \{c : \ c \in A \text{ and } c \in B\},$$

which is the set of all elements c which are both in A and in B. What is generally true is that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(This is the first instance of the Principle of Inclusion/Exclusion, which will be discussed in Section 8.) In particular, if $A \cap B = \varnothing$ then $|A \cup B| = |A| + |B|$. Thus, in order to interpret "OR" as addition, it is important to check that the sets of choices A and B have no elements in common. Such a union of sets A and B for which $A \cap B = \varnothing$ is called a *disjoint union* of sets.

When you solve enumeration problems on your own it is usually very useful to describe a choice sequence for constructing the set of objects of interest, paying close attention to the words "AND" and "OR".

2. Permutations.

A *permutation* of a set S is a list of the elements of S exactly once each, in some order. For example, the permutations of the set $\{1, a, X, g\}$ are:

$1aXg$	$a1Xg$	$X1ag$	$g1aX$
$1agX$	$a1gX$	$X1ga$	$g1Xa$
$1Xag$	$aX1g$	$Xa1g$	$ga1X$
$1Xga$	$aXg1$	$Xag1$	$gaX1$
$1gaX$	$ag1X$	$Xg1a$	$gX1a$
$1gXa$	$agX1$	$Xga1$	$gXa1$

To construct a permutation of S we can choose any element v of S to be the first element in the permutation and follow this with any permutation of the set $S \smallsetminus \{v\}$. That is how the table above is arranged – each of the four columns corresponds to one choice of an element of $\{1, a, X, g\}$ to be the first element of the permutation. Within each column the permutations of the remaining elements are listed after the first element.

Let p_n denote the number of permutations of an n–element set S. The first sentence of the previous paragraph is translated into the equation

$$p_n = n \cdot p_{n-1},$$

provided that n is positive. (In this equation there are n choices for the first element v of the permutation, AND p_{n-1} choices for the permutation of $S \smallsetminus \{v\}$ which follows it.) It is important to note here that each permutation of S will be produced exactly once by this construction.

Since it is easy to see that $p_1 = 1$ (and $p_2 = 2$), a simple proof by induction on n shows the following:

Theorem 2.1. *For every $n \geq 1$, the number of permutations of an n–element set is*

$$n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

The term *n factorial* is used for the number $n(n-1)\cdots 3 \cdot 2 \cdot 1$, and it is denoted by $n!$ for convenience.

We also define $0!$ to be the number of permutations of the 0–element (empty) set $\varnothing$. Since we want the equation $p_n = n \cdot p_{n-1}$ to hold when $n = 1$, and since $p_1 = 1! = 1$, we conclude that $0! = p_0 = 1$ as well.

3. Subsets.

A *subset* of a set S is a collection of some (perhaps none or all) of the elements of S, at most once each and in no particular order.

To specify a particular subset A of S, one has to decide for each element v of S whether v is in A or v is not in A. Thus we have two choices – $v \in A$ OR $v \notin A$ – for each element v of S. If $S = \{v_1, v_2, ..., v_n\}$ has n elements then the total number of choices is 2^n since we have 2 choices for v_1 AND 2 choices for v_2 AND ... AND 2 choices for v_n. Therefore....

Theorem 3.1. *For every $n \geq 0$, the number of subsets of an n–element set is 2^n.*

4. Partial permutations.

A *partial permutation* of a set S is a permutation of a subset of S. More directly, it is a list of some (perhaps none or all) of the elements of S, at most once each and listed in some particular order. We are going to count partial permutations of length k of an n–element set.

First think about the particular case $n = 6$ and $k = 3$, and the set $S = \{a, b, c, d, e, f\}$. A partial permutation of S of length 3 is a list (x, y, z) of elements of S, which must all be different. There are:
6 choices for x (since x is in S), AND
5 choices for y (since $y \in S$ but $y \neq x$), AND
4 choices for z (since $z \in S$ but $z \neq x$ and $z \neq y$).
Altogether there are $6 \cdot 5 \cdot 4 = 120$ partial permutations of $\{a, b, c, d, e, f\}$ of length 3.

This kind of reasoning works just as well in the general case. If S is an n–element set and we want to construct a partial permuation $(v_1, v_2, ..., v_k)$ of elements of S of length k, then there are:

n choices for v_1, AND
$n-1$ choices for v_2, AND
....
$n-(k-2)$ choices for v_{k-1}, AND
$n-(k-1)$ choices for v_k.

This proves the following result.

Theorem 4.1. *For $n, k \geq 0$, the number of partial permutations of length k of an n–element set is $n(n-1)\cdots(n-k+2)(n-k+1)$.*

Notice that if $k > n$ then the number 0 will appear as one of the factors in the product $n(n-1)\cdots(n-k+2)(n-k+1)$. This makes sense, because if $k > n$ then there are no partial permutations of length k of an n–element set. On the other hand, if $0 \leq k \leq n$ then we could also write this product as

$$n(n-1)\cdots(n-k+2)(n-k+1) = \frac{n!}{(n-k)!}.$$

5. k–ELEMENT SUBSETS.

We refine the result of Section 3 by counting subsets of an n–element set S which have a particular cardinality k. So for $n, k \geq 0$ let $\binom{n}{k}$ denote the number of k–element subsets of an n–element set S. Notice that if $k < 0$ or $k > n$ then $\binom{n}{k} = 0$ because in these cases it is impossible for S to have a k–element subset. Thus we need only consider k in the range $0 \leq k \leq n$.

To count k–element subsets of S we consider another way of constructing a partial permuation of length k of S. Specifically, we can choose a k–element subset A of S AND a permutation of A. The result will be a permutation of a subset of S of length k. Since every partial permutation of length k of S is constructed exactly once in this way, this translates into the equation

$$\binom{n}{k} \cdot k! = \frac{n!}{(n-k)!}.$$

In summary, we have proved the following result.

Theorem 5.1. *For $0 \leq k \leq n$, the number of k–element subsets of an n–element set is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The numbers $\binom{n}{k}$ are read as "n choose k" and are called *binomial coefficients.*

6. Think of what the numbers mean.

Usually, when faced with a formula to prove, one's first thought is to prove it by algebraic calculations, or perhaps with an induction argument, or maybe with a combination of the two. But often that is not the easiest way, nor is it the most informative. A much better strategy is one which gives some insight into the "meaning" of all of the parts of the formula. If we can interpret all the numbers as counting things, addition as "OR", and multiplication as "AND", then we can hope to find an explanation of the formula by constructing some objects in the correct way.

For example, consider the equation, for any $n \geq 0$:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

This could be proved by induction on n, but many more details would have to be given and the proof would not address the true "meaning" of the formula. Instead, let's interpret everything combinatorially:

- 2^n is the number of subsets of an n–element set, $\{1, 2, ..., n\}$ say;
- $\binom{n}{k}$ is the number of k–element subsets of $\{1, 2, ..., n\}$, for each $0 \leq k \leq n$;
- addition corresponds to "OR" (that is, disjoint union of sets).

So, this formula is saying that choosing a subset of $\{1, 2, ..., n\}$ (in one of 2^n ways) is equivalent to choosing a k–element subset of $\{1, 2, ..., n\}$ (in one of $\binom{n}{k}$ ways) for exactly one value of k in the range $0 \leq k \leq n$. Said that way the formula becomes self–evident, and there is nothing more to prove.

As another example, consider the equation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

where we are using the fact that $\binom{m}{j} = 0$ if $j < 0$ or $j > m$. This equation can be proven algebraically from the formula of Theorem 5.1, and that is a good exercise which I encourage you to try. But a more informative proof interprets these numbers combinatorially as follows:

- $\binom{n}{k}$ is the number of k–element subsets of $\{1, 2, ..., n\}$;
- $\binom{n-1}{k-1}$ is the number of $(k-1)$–element subsets of $\{1, 2, ..., n-1\}$;
- $\binom{n-1}{k}$ is the number of k–element subsets of $\{1, 2, ..., n-1\}$;
- addition corresponds to disjoint union of sets.

So, this equation is saying that choosing a k–element subset A of $\{1, 2, ..., n\}$ is equivalent to either choosing a $(k-1)$–element subset of $\{1, 2, ..., n-1\}$ or a k–element subset of $\{1, 2, ..., n-1\}$. This is perhaps not as clear as the previous example, but the two cases depend upon whether the chosen k–element subset A

of $\{1, 2, ..., n\}$ is such that $n \in A$ OR $n \notin A$. If $n \in A$ then $A \setminus \{n\}$ is a $(k-1)$–element subset of $\{1, 2, ..., n-1\}$, while if $n \notin A$ then A is a k–element subset of $\{1, 2, ..., n-1\}$. This construction explains the correspondence, proving the formula.

This principle – interpreting equations combinatorially and proving the formulas by describing explicit correspondences between sets of objects – is one of the most important and powerful ideas in enumeration. We'll have a lot of practice using this way of thinking in the next few weeks.

Incidentally, the equation in the second example above is a very useful recurrence relation for computing binomial coefficients quickly. Together with the facts

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$$

and $\binom{n}{0} = \binom{n}{n} = 1$ it can be used to grind out any number of binomial coefficients without difficulty. The resulting table is known as *Pascal's Triangle*:

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

7. Multisets.

Imagine a bag which contains a large number of marbles of three colours – red, green, and blue, say. The marbles are all indistinguishable from one another except for their colours. There are N marbles of each colour, where N is very, very large (more precisely we should be considering the limit as $N \to \infty$). If I reach into the bag and pull out a handful of 11 marbles, I will have r red marbles, g green marbles, and b blue marbles, for some nonnegative integers (r, g, b) such that $r + g + b = 11$. How many possibile outcomes are there?

The word "multiset" is meant to suggest a set in which the objects can occur more than once. For example, the outcome $(4, 5, 2)$ in the above situation corresponds to the "set" $\{R, R, R, R, G, G, G, G, G, B, B\}$ in which R is a red marble, G is a green marble, and B is a blue marble. This is an 11–element multiset with elements of three types. The number of these multisets is the solution to the above problem.

In general, if there are t types of element then a *multiset of size n with elements of t types* is a sequence of nonnegative integers $(m_1, ..., m_t)$ such that

$$m_1 + m_2 + \cdots + m_t = n.$$

The interpretation is that m_i is the number of elements of the multiset which are of the i–th type, for each $1 \leq i \leq t$.

Theorem 7.1. *For any $n \geq 0$ and $t \geq 1$, the number of n–element multisets with elements of t types is*

$$\binom{n+t-1}{t-1}.$$

Proof. Think of what that number means! By Theorem 5.1 $\binom{n+t-1}{t-1}$ is the number of $(t-1)$–element subsets of an $(n+t-1)$–element set. So, let's write down a row of $(n+t-1)$ circles from left to right:

O O O O O O O O O O O O O

and cross out some $t-1$ of these circles to choose a $(t-1)$–element subset:

O O O O X O O O O O X O O

Now the $t-1$ crosses chop the remaining sequence of n circles into t segments of consecutive circles. (Some of these segments might be empty, which is to say of length zero.) Let m_i be the length of the i–th segment of consecutive O-s in this construction. Then $m_1 + m_2 + \cdots + m_t = n$, so that $(m_1, m_2, ..., m_t)$ is an n–element multiset with t types. Conversely, if $(m_1, m_2, ..., m_t)$ is an n–element multiset with t types then write down a sequence of m_1 O-s, then an X, then m_2 O-s, then an X, and so on, finishing with an X and then m_t O-s. The positions of the X-s will indicate a $(t-1)$–element subset of the positions $\{1, 2, ..., n+t-1\}$.

The construction of the above paragraph shows how to translate between $(t-1)$–element subsets of $\{1, 2, ..., n+t-1\}$ and n–element multisets with t types of element. This one–to–one correspondence completes the proof of the theorem. □

To answer the original question of this section, the number of 11–element multisets with elements of 3 types is $\binom{11+3-1}{3-1} = \binom{13}{2} = 78$.

8. The Principle of Inclusion/Exclusion.

In a vase is a bouquet of flowers. Each flower is (at least one of) fresh, fragrant, or colourful:

(a) 11 flowers are fresh;
(b) 7 flowers are fragrant;
(c) 8 flowers are colourful;
(d) 6 flowers are fresh and fragrant;

(e) 5 flowers are fresh and colourful;
(f) 2 flowers are fragrant and colourful;
(g) 2 flowers are fresh, fragrant, and colourful.
How many flowers are in the bouquet?

The Principle of Inclusion/Exclusion is a systematic method for answering such questions, which involve overlapping conditions which can be satisfied (or not) in various combinations.

For a small problem like the one above we can work backwards as follows:
(h) from (g) and (f) there are 0 flowers which are fragrant and colourful but not fresh;
(i) from (g) and (e) there are 3 flowers which are fresh and colourful but not fragrant;
(j) from (g) and (d) there are 4 flowers which are fresh and fragrant but not colourful;
(k) from (c)(g)(h)(i) there are 3 flowers which are colourful but neither fresh not fragrant;
(ℓ) from (b)(g)(h)(j) there is 1 flower which is fragrant but neither fresh nor colourful;
(m) from (a)(g)(i)(j) there are 2 flowers which are fresh but neither fragrant nor colourful.

The total number of flowers is counted by the disjoint union of the cases (g) through (m); that is $2 + 0 + 3 + 4 + 3 + 1 + 2 = 15$.

A *Venn diagram* is extremely useful for organizing this calculation. (Making diagrams in LaTeX is a real pain. Please see the diagram Venn3.JPG linked from the CO220 homepage.) Item (g) in the original data gives the number of flowers (2) counted in the central triangle. The subsequent steps (h) to (m) calculate the rest of the numbers in the diagram, moving outwards from the center.

The above works very well for three properties (fresh, fragrant, colourful) but becomes increasingly difficult to apply as the number of properties increases. Consider this alternative formula:

$$(a) + (b) + (c) - (d) - (e) - (f) + (g) = 11 + 7 + 8 - 6 - 5 - 2 + 2 = 15.$$

This looks much easier to apply, and it gives the right answer, always. Why? That is the Principle of Inclusion/Exclusion, which we now explain in general.

Let A_1, A_2,... A_t be finite sets. We want a formula for the cardinality of the union of these sets $A_1 \cup A_2 \cup \cdots \cup A_t$. First a bit of notation: if S is a nonempty subset of $\{1, 2, ..., t\}$ then let A_S denote the intersection of the sets A_i for all $i \in S$. So, for example, with this notation we have $A_{\{2,3,5\}} = A_2 \cap A_3 \cap A_5$.

Theorem 8.1. *Let A_1, A_2,..., A_t be finite sets. Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_t| = \sum_{\varnothing \neq S \subseteq \{1,2,\ldots,t\}} (-1)^{|S|-1} |A_S|.$$

(In this formula the summation is over all nonempty subsets of $\{1, 2, .., t\}$.)

We'll prove this theorem a little bit later on.

9. Combinatorial Probabilities.

We can reinterpret counting problems in terms of probabilities by making one additional hypothesis. That hypothesis is that **every possible outcome is equally likely**. The exact definition of what is an "outcome" depends on the particular problem. If Ω denotes a (finite) set of all possible outcomes, then any subset E of Ω is what a probabilist calls an *event*. The probability that a randomly chosen outcome from Ω is in the set E is $|E|/|\Omega|$ exactly because every outcome has likelyhood $1/|\Omega|$ of being chosen, and there are $|E|$ elements in E. Here are a few examples to illustrate these ideas.

• *What is the probability that a random subset of* $\{1, 2, ..., 8\}$ *has at most* 3 *elements?*
Here an outcome is a subset of $\{1, 2, ..., 8\}$, and there are $2^8 = 256$ such subsets. The number of subsets of $\{1, 2, ..., 8\}$ with at most 3 elements is

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} = 1 + 8 + 28 + 56 = 93.$$

So the probability in question is

$$\frac{93}{256} = 0.363281...$$

to six decimal places.

• *What is the probability that a random permutation of the set* $\{a, b, c, d, e, f\}$ *contains the letters* fad *as a consecutive subsequence?*
Here an outcome is a permutation of $\{a, b, c, d, e, f\}$, and there are $6! = 720$ such permutations. Those permutations of this set which contain fad as a consecutive subsequence can be constructed uniquely as the permutations of the set $\{b, c, e, fad\}$, so there are $4! = 24$ of these. Thus, the probability in question is

$$\frac{24}{720} = \frac{1}{30} = 0.03333...$$

• *What is the probability that a randomly chosen* 2*–element multiset with* t *types of element has both elements of the same type?*
The outcomes are the 2–element multisets with t types, numbering

$$\binom{2+t-1}{t-1} = \binom{t+1}{t-1} = \binom{t+1}{2} = \frac{(t+1)t}{2}$$

in total. Of these, exactly t of them have both elements of the same type – choose one of the t types and take two elements of that type. Thus, the probability in question is

$$\frac{2t}{(t+1)t} = \frac{2}{t+1}.$$

The values for the first few t are given in the following table to four decimal places:

t	1	2	3	4	5	6	7
	1.0000	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500