CO452/652: Integer Programming — Winter 2009
Instructor: Chaitanya Swamy
Assignment 4
Due: April 3, 2009 before class

You may use anything proved in class directly. I will maintain a FAQ about the assignment on the course webpage. Acknowledge all collaborators and sources of external help.

The undergraduate (i.e., CO452) students may omit one of the following: Q1(c) or Q7(a), or attempt these as bonus questions.

Q1:
(a) Consider the knapsack polytope $P_{\text{Knap}} = P_I$, where $P = \{ x \in [0,1]^n : a^T x \leq B \}$ with $a \geq 0$ and $B \geq 0$. Recall that we showed that if $C \subseteq \{1,\ldots,n\}$ is a dependent set, i.e., $\sum_{i \in C} a_i > B$, then defining $E(C) := C \cup \{ j : a_j \geq a_i \text{ for all } i \in C \}$, the inequality $\sum_{i \in E(C)} x_i \leq |C| - 1$ is valid for $P_{\text{Knap}}$. Show that this inequality has Chvátal rank at most 1 (relative to $P$). (5 marks)

(b) Let $V = \{1,\ldots,n\}$. Consider the TSP polytope for the $n$-node complete graph:

$$P_{\text{TSP}} = P_I \text{ where } P = \{ x \in \mathbb{R}^n : x(\delta(v)) = 2 \ \forall v \in V, \ x(E(S)) \leq |S| - 1 \ \forall S \neq \emptyset, V, \ 0 \leq x \leq e \}.$$  

A comb is a subgraph induced by a node-set $(H,W_1,W_2,\ldots,W_k)$ satisfying the following conditions: (i) $H \cap W_i \neq \emptyset$, $W_i \setminus H \neq \emptyset$ for all $i$; (ii) $W_i \cap W_j = \emptyset$ for all $i \neq j$; and (iii) $k \geq 3$ and is odd. The set $H$ is called the handle of the comb and the $W_i$s are called the teeth of the comb. Given a comb $(H,W_1,\ldots,W_k)$, prove that the following comb inequality is valid for $P_{\text{TSP}}$ by showing that it has Chvátal rank at most 1.

$$x(E(H)) + \sum_{i=1}^k x(E(W_i)) \leq |H| + \sum_{i=1}^k (|W_i| - 1) - \left\lceil \frac{k}{2} \right\rceil$$  

(5 marks)

(c) Given a graph $G = (V,E)$, consider its stable-set polytope $\text{STAB}(G) := \text{conv} \{ \chi^S : S \subseteq V \text{ is a stable set} \} = P_I$, where $P := \{ x \in \mathbb{R}^n : x_u + x_v \leq 1 \ \forall (u,v) \in E, \ 0 \leq x \leq e \}$. Let $C$ be a clique of $G$. Give a tight bound on the Chvátal rank of the clique-inequality $x(C) \leq 1$ (which is valid for $\text{STAB}(G)$). That is, you should establish lower and upper bounds on the Chvátal rank that are within a constant factor of each other. (One can in fact compute the Chvátal rank exactly.)

You may use the fact that the Chvátal closure of $P$, which is defined as $P' := P \cap \{ x \in \mathbb{R}^n : x \text{ satisfies all rank-1 Chvátal-Gomory inequalities for } P \}$, is a polyhedron, if necessary. Thus, defining $P^{(0)} := P$ and $P^{(i+1)} := (P^{(i)})'$, we obtain that each $P^{(k)}$ is a polyhedron. (5 marks)

Q2: The purpose of this question is to introduce a notion of duality for integer programs based on (nondecreasing) superadditive functions. (Throughout, when we say that $F$ is superadditive, we also require that $F(0) = 0$.) Let (IP): $\max c^T x$ s.t. $x \in P_I$ be an integer program, where $P = \{ x : Ax \leq b, \ x \geq 0 \} \subseteq \mathbb{R}^n$. Let $A$ have $m$ rows, and $A_j \in \mathbb{R}^m$ denote the $j$-th column of $A$. 


(a) Prove the following weak-duality statement. If \( F : \mathbb{R}^m \rightarrow \mathbb{R} \) is a nondecreasing superadditive function such that \( F(A_j) \geq c_j \) for all \( j = 1, \ldots, n \), and \( x \) is a feasible solution to (IP), then \( c^T x \leq F(b) \). \( \quad (2 \text{ marks}) \)

(b) Show that for every inequality \( \alpha^T x \leq \beta \) that is valid for \( P_I \), there exists a nondecreasing superadditive function \( F : \mathbb{R}^m \rightarrow \mathbb{R} \) such that \( F(A_j) \geq \alpha_j \) for all \( j = 1, \ldots, n \) and \( F(b) \leq \beta \). You may use the fact that every valid inequality for \( P_I \) has finite Chvátal rank, and that \( P^{(k)} \) (as defined in Q1(c)) is a polyhedron for all \( k \geq 0 \). \( \quad (7 \text{ marks}) \)

(c) Deduce from parts (a) and (b) that if (IP) has an optimal solution, then its optimal value is equal to
\[
\min F(b) \quad \text{s.t.} \quad F(A_j) \geq c_j \forall j = 1, \ldots, n, \quad F(0) = 0, \quad F : \mathbb{R}^m \rightarrow \mathbb{R} \text{ is nondecr., superadditive.} \quad (6 \text{ marks})
\]

Q3: This question considers a generalization of the lifting procedure (not lift-and-project) described in class for strengthening valid inequalities, where we simultaneously lift more than one variable at a time. Let \( P \subseteq [0,1]^n \) be a polyhedron with \( P_I \neq \emptyset \). Let \( J \subseteq \{1, \ldots, n\} \), and \( z \) be a vector in \( \{0,1\}^J \). Define \( S := \{ x \in P_I : x_j = z_j \text{ for all } j \in J \} \) and suppose \( S \neq \emptyset \). Suppose \( \sum_{j \notin J} \pi_j x_j \leq \delta \) is a valid inequality for \( S \). Consider the set
\[
Q := \left\{ \alpha \in \mathbb{R}^J : \sum_{j \in J} \alpha_j (x_j - z_j) + \sum_{j \notin J} \pi_j x_j \leq \delta \quad \text{is valid for } P_I \right\}.
\]
Prove that \( Q \) is a non-empty polyhedron, and it is pointed iff \( \text{proj}_J(P_I) \) is full-dimensional (i.e., \( \dim(\text{proj}_J(P_I)) = |J| \)), where \( \text{proj}_J(P_I) \) denotes the projection of \( P_I \) onto the \( (x_j)_{j \in J} \)-space. Argue that if \( \hat{\alpha} \) is an extreme point of \( Q \), then the inequality \( \sum_{j \in J} \hat{\alpha}_j x_j + \sum_{j \notin J} \pi_j x_j \leq \delta + \sum_{j \notin J} \hat{\alpha}_j z_j \) defines a face of \( P_I \) of dimension at least \( \dim(\{ x \in S : \sum_{j \notin J} \pi_j x_j = \delta \}) + |J| \). \( \quad (15 \text{ marks}) \)

Q4: Let \( K \) be the set of solutions to
\begin{align*}
2x_1 - 2x_2 & \leq 1 \quad (1) \\
2x_1 - 2x_2 & \geq -1 \quad (2) \\
0 & \leq x_1 \leq 1 \quad (3) \\
0 & \leq x_2 \leq 1. \quad (4)
\end{align*}

(a) Using the Balas-Ceria-Cornuérjols lift-and-project method compute \( P_1(K) \). Also compute \( N(K) \) using the Lovász-Schrijver lift-and-project procedure.

(b) Verify (geometrically) that \( P_1(K) \) is indeed equal to \( \text{conv}(\{ x \in K : x_1 \in \{0,1\} \}) \), and that \( N(K) \subseteq P_1(K) \cap P_2(K) \). \( \quad (15 \text{ marks}) \)

Q5: Consider the integer program
\[
\min x_{n+1} \quad \text{s.t.} \quad 2x_1 + 2x_2 + \cdots + 2x_n + x_{n+1} = n, \quad x \in \{0,1\}^{n+1}.
\]
Prove that a branch-and-bound algorithm that branches by setting a fractional variable to 0 or 1 will require the enumeration of an exponential (in \( n \)) number of subproblems when \( n \) is odd. \( \quad (10 \text{ marks}) \)
Q6: Consider an undirected graph $G = (V, E)$ with distinct vertices $s, t$ and nonnegative edge costs $\{c_e\}$. Call an $s$-$t$ path is odd if it contains an odd number of edges. Show that one can find an minimum-cost odd $s$-$t$-path in time polynomial in the input length. (7 marks)

(Hint: Let $G_1, G_2$ be two disjoint copies of $G$, and $u_i, i = 1, 2$ denote the copy of $u$ in $G_i$. Let $G'$ be the graph obtained by taking the union of $G_1$ and $G_2 \setminus \{s_2, t_2\}$ (i.e., the graph obtained by removing $s_2, t_2$, and their incident edges, from $G_2$). Add suitable edges connecting the nodes of $G_1$ and $G_2$, and give these edges suitable costs so that a minimum-cost perfect matching in $G'$, if one exists, corresponds to a minimum-cost odd $s$-$t$ path in $G$. You may use the fact that minimum-cost perfect matchings in arbitrary (i.e., not necessarily bipartite) graphs can be computed in polynomial time.)

(b) Given a graph $G$ with nonnegative edge costs $\{c_e\}$, the Maxcut problem is to find a set $\emptyset \neq S \subseteq V$ that maximizes $c(\delta(S))$. An odd circuit is a cycle with an odd number of edges and no repeated nodes. Consider the following polyhedron.

$$P := \{x \in \mathbb{R}^E : \ x(C) \leq |C| - 1 \ \text{ for every odd circuit } C; \ 0 \leq x \leq e\}.$$  

Show that $\max c^T x \ s.t. \ x \in P$ is equal to the optimal value of the Maxcut problem on $G$. Show that one can solve $\max c^T x \ s.t. \ x \in P$ in polynomial time. (8 marks)

(c) (Bonus part) There was an error in this question, which has been corrected below.

A semidefinite program (SDP) is an optimization problem involving a symmetric matrix $X$ that has the following form:

$$\max \sum_{i,j} c_{ij} X_{ij} \ s.t. \ \sum_{i,j} a_{ij}^{(\ell)} X_{ij} \leq b^{(\ell)} \ \forall \ell = 1, \ldots, k, \ X \succeq 0 \quad \text{(SDP)}$$

where $X \succeq 0$ denotes the constraint that $X$ is required to be positive semidefinite (PSD). Consider the following semidefinite-programming relaxation for the Maxcut problem.

$$\max \sum_{e=(u,v) \in E} c_e \left(\frac{1 - z_u^T z_v}{2}\right) \ \text{s.t.} \ z_u^T z_u = 1 \ \text{for all } u \in V. \quad \text{(MC-SDP)}$$

This is a semidefinite program because if we use $X$ to denote $ZZ^T$, where $Z$ is an $n \times d$ matrix (for some $d$) with rows $z_u^T$ for $u = 1, \ldots, n$, then substituting $X_{uv}$ for $z_u^T z_v$, we obtain a problem of the form (SDP). Moreover, if $X$ is a PSD matrix representing a solution to this resulting SDP, then by a well-known result called the Cholesky decomposition, we can write $X = ZZ^T$ for some $n \times d$ matrix $Z$; hence, $X$ encodes a solution to (MC-SDP). (MC-SDP) is a relaxation of the Maxcut problem, because given any cut $(S, V \setminus S)$ we can set $z_u$ for all $u \in S$ to some common unit vector, and $z_v$ for all $v \notin S$ to the opposite unit vector, so that the objective function of (MC-SDP) evaluates precisely to $c(\delta(S))$.

Now define $K \subset \mathbb{R}^{E+V}$ as the set of feasible solutions to the following system.

$$d_e \geq x_u - x_v, \quad d_e \geq x_v - x_u, \quad d_e \leq x_u + x_v, \quad d_e \leq 2 - x_u - x_v \quad \forall e = (u,v) \in E,$$

$$0 \leq d_e, x_u \leq 1 \quad \forall e \in E, u \in V.$$
The integer program \( \max \sum_{e} c_e d_e \) s.t. \((d, x) \in \mathbb{Z}(K)\) is a valid formulation for the MAXCUT problem, where \( x_u \) indicates which side of the cut (the 0-side or 1-side) \( u \) is on, and \( d_e \) thus encodes if edge \( e \) is cut. Let \( M^+(K) \) be the convex set in the higher-dimensional space obtained by applying the semidefinite version of the Lovász-Schrijver procedure to \( K \). Prove that \( M^+(K) \) yields a relaxation for MAXCUT that is at least as strong as (MC-SDP) by showing that any point in \( M^+(K) \) maps to a solution to (MC-SDP) of no smaller value. (Thus, the maximum value of \( c^T d \) over points in \( M^+(K) \) is at most the optimal value of (MC-SDP).) (10 marks)

Q7: In this question, we compare the Chvátal-Gomory (CG) procedure for generating valid inequalities with the Balas-Ceria-Cournuélols (BCC) lift-and-project method.

(a) Consider again the sable-set polytope \( \text{STAB}(G) \) for a graph \( G \), the polyhedron \( P \) defined in Q1(c), which we now denote as \( K \), and a clique inequality \( x(C) \leq 1 \) obtained from a clique \( C \) of \( G \). Show that starting with the polyhedron \( K \), one requires at least \(|C| - 3\) sequential applications of the BCC lift-and-project method (no matter what sequence of variables is chosen) before we obtain a polyhedron for which this clique inequality is valid. (5 marks)

(b) (Bonus part) Consider the polyhedron

\[
K := \{(x, y) \in \mathbb{R}^2 : x \leq B, \ x \leq B^2 y, \ x \geq 0, \ 0 \leq y \leq 1\},
\]

where \( B \) is a positive integer. Notice that \( x \leq B^2 y \) denotes a big-\( M \) constraint that, for integer \( y \), forces \( y = 1 \) if \( x > 0 \), and thus, \( \mathbb{Z}(P) = \{(0, 0)\} \cup \{(x, 1) : 0 \leq x \leq B, \ x \in \mathbb{Z}\} \). Observe that this big-\( M \) constraint can be strengthened to \( x \leq By \), that is, \( x \leq By \) is valid for \( K_I \). It is easy to see that \( P_y(K) = K_I \). (Although we defined the lift-and-project operators in the context pure \{0,1\}-IPs, one can also apply them to (mixed) IPs where only a subset of the variables are \{0,1\}-variables. The only difference is that now only the \{0,1\}-variables \( x_j \) are candidates for multiplying our constraint-system by \( x_j \) and \((1 - x_j)\); the linearization, and projection steps are unchanged.)

Show however that the Chvátal-rank of \( x \leq By \) is at least \( \gamma B - \delta \) for some constants \( \gamma, \delta, \gamma > 0 \). (10 marks)