CO452/652: Integer Programming — Winter 2009

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Assignment 3

Due: March 9, 2009 before class

You may use anything proved in class directly. I will maintain a FAQ about the assignment on the course webpage. Acknowledge all collaborators and sources of external help.

Q5, marked (*), is a regular question for *graduate (i.e., CO652) students*, and a bonus question for *undergraduate (i.e., CO452) students*.

In all the formulation questions describe *briefly* what your variables and constraints represent.

Q1:

(a) Let $P_i = \{z \in \mathbb{R}^d : T^{(i)}z \leq u^{(i)}\}$ be a non-empty polyhedron for each $i = 1, \ldots, m$. Consider the following mathematical program:

$$\max \quad c^T x \quad \text{s.t.} \quad Ax \leq b \quad \text{and} \quad \max\{z^T x : z \in P_i\} \leq \beta^{(i)} \quad \forall i = 1, \ldots, m. \tag{M-P}$$

Encode (M-P) by an equivalent linear program. Assume that (M-P) is feasible and the optimum value exists. The size of your formulation, that is, the number of variables and constraints, should be polynomial in m, d, and the total number of rows in the $T^{(i)}$ matrices. **(6 marks)**

(b) In class, we considered the following integer program for the traveling salesman problem (TSP) on n nodes. Here e indexes the edges of the n-node complete graph.

$$\begin{aligned}
\min \quad & \sum_e w_e x_e && && \text{(TSP-IP)}\\
\text{s.t.} \quad & x(\delta(v)) = 2 && \text{for all } v = 1, \ldots, n && (1)\\
& x(\delta(S)) \geq 2 && \text{for all } S \subseteq V, 1 \leq |S| \leq n-1 && (2)\\
& x_e \in \{0, 1\} && \text{for all } e.
\end{aligned}$$

The formulation (TSP-IP) has an exponential (in n) number of constraints of the form (2). Show that one can obtain an equivalent formulation of size polynomial in n. **(7 marks)**

You may use the following result. Given an undirected graph $G = (V, E)$ with nonnegative costs $\{c_e\}_{e \in E}$ on the edges, and two distinguished nodes $s, t \in V$, the *min s-t-cut* problem seeks to find a set $S \subseteq V$ such that $s \in S,\ t \notin S$ that minimizes $c(\delta(S))$. (If $s \in S,\ t \notin S$, the partition $(S, V \setminus S)$ is called an s-t cut.) The min s-t cut problem can be modeled by introducing a variable y_e for every edge e that is 1 if $e \in \delta(S)$ and 0 otherwise, and z_v for each node v that is 0 if $v \in S$ (i.e., if v is on the s-side of the s-t cut) and 1 otherwise. We seek to

$$\begin{aligned}
\min \quad & \sum_e c_e y_e \\
\text{s.t.} \quad & z_s = 0, \qquad z_t = 1 \\
& z_u - z_v \leq y_{uv}, \quad z_v - z_u \leq y_{uv} \qquad \text{for all } (u, v) \in E \\
& z, y \geq 0.
\end{aligned}$$

Although, this is a linear program, it is known that the extreme points of the feasible region are {0,1}-vectors, and hence this linear program encodes the min s-t cut problem.

(c) Consider the linear program: $\max\ c^T x$ s.t. $Ax \le b$, where A is an $m \times d$ matrix. Suppose now that the entries of A are not known precisely. But we are told that the (i,j)-th entry of A lies in the interval $[a_{ij} - \Delta_{ij}, a_{ij} + \Delta_{ij}]$ for every i,j, and that in every row i, at most Γ_i of the entries deviate from a_{ij}, where $\Gamma_i \in \{0,1,\ldots,d\}$. In the presence of such uncertainty, one may seek to find the best solution x that remains feasible under *every possible realization* $\tilde{A}$ of the matrix A (satisfying the above conditions). Thus, defining

$$\mathcal{A}_i(\Gamma_i) := \{\tilde{a}_i \in \mathbb{R}^d : \tilde{a}_{ij} \in [a_{ij}-\Delta_{ij}, a_{ij}+\Delta_{ij}]\ \forall j,\ \text{at most } \Gamma_i \text{ of the } \tilde{a}_{ij} \text{ values deviate from } a_{ij}\},$$

we obtain the following *robust optimization problem.*

$$\max\ c^T x \quad \text{s.t.} \quad \tilde{a}_i^T x \le b_i \quad \text{for all } \tilde{a}_i \in \mathcal{A}_i(\Gamma_i),\ \forall i = 1,\ldots m. \tag{Rob-P}$$

Assume that (Rob-P) is feasible and the optimum value exists, and argue that (Rob-P) can be encoded by a linear program of size polynomial in m and d. **(7 marks)**

Q2: Given a complete directed graph on n nodes (that is, there is an arc (i,j) for every pair of nodes i,j) with costs $\{c_{ij}\}_{i,j}$ on the arcs, the *linear ordering problem* is to find a permutation $\pi : \{1,\ldots,n\} \mapsto \{1,\ldots,n\}$ of the nodes that maximizes $\sum_{i,j:\pi(i)<\pi(j)} c_{ij}$. Using binary ordering variables δ_{ij} to indicate if $\pi(i) < \pi(j)$, this can be formulated as

$$\begin{aligned}
\max \quad & \sum_{i,j} c_{ij}\delta_{ij} && && \text{(LO-IP)}\\
\text{s.t.} \quad & \delta_{ij} + \delta_{ji} = 1 && \text{for all } i < j && (3)\\
& \delta_{j_1 j_2} + \cdots + \delta_{j_r j_1} \le |C| - 1 && \text{for all cycles } C = \{j_1,\ldots,j_r\} && (4)\\
& \delta_{ij} \in \{0,1\} && \text{for all } i,j. && (5)
\end{aligned}$$

Let $Q := \{\delta \in \mathbb{R}_+^{n(n-1)} : (3),(4) \text{ hold}\}$, and $P_{\text{LO}} = Q_I$ be the convex hull of feasible solutions to (LO-IP).

(a) Let Q' be the polyhedron obtained from Q by dropping the inequalities (4) for $|C| \ge 4$. Show that $Q' = Q$. **(6 marks)**

(b) Prove that $\dim(P_{\text{LO}}) = \frac{n(n-1)}{2}$, and the inequalities (4) for $|C| = 3$ define facets of P_{LO}. **(9 marks)**

Q3: Let $P = \{x : Ax \le b\}$ be a non-empty pointed polyhedron, where A is a rational matrix. Let $a_1^T,\ldots,a_m^T$ be the rows of A, and suppose that $a_i^T x \ge 0$ for all $x \in P$. Give a mixed integer program (MIP) of size polynomial in m and d whose feasible solutions correspond to extreme points of P. That is, any feasible solution to your MIP should map to an extreme point of P, and conversely, any extreme point of P should map to a feasible solution to your MIP. **(10 marks)**

Q4: In the popular puzzle Sudoku, which many of you might be familiar with, there is a 9×9 grid, with some squares already filled in with a numbers from $1,\ldots,9$. The goal is to complete the grid by filling in each square with a number from 1 to 9, so that each row, each column, and each 3×3 grid (see below) contains all the numbers from $1,\ldots,9$ (so each number from $1,\ldots,9$ appears

1	2	6						
			1				9	
	4			6		2		5
		5	2		4			9
	7						5	
4			7		6	3		
2		9		7			8	
	8				3			
						9	3	4

1	2	6	9	3	5	8	4	7
8	5	7	1	4	2	6	9	3
9	4	3	8	6	7	2	1	5
3	1	5	2	8	4	7	6	9
6	7	2	3	1	9	4	5	8
4	9	8	7	5	6	3	2	1
2	3	9	4	7	1	5	8	6
5	8	4	6	9	3	1	7	2
7	6	1	5	2	8	9	3	4

Figure 1: A 9×9 Sudoku puzzle, and its solution.

exactly once in every row, column and 3×3 grid). A Sudoku puzzle is guaranteed to have a *unique* solution. Figure 1 gives an example puzzle, and its solution.

One can easily generalize the puzzle to an arbitrary $N \times N$ grid, where $N = n^2$ for an integer n. We number the columns $1, \ldots, N$ from left to right, and the rows $1, \ldots, N$ from top to bottom. Call an $n \times n$ grid, a *principal* $n \times n$ *grid* if its bottommost and rightmost square (i, j) is such that both i and j are multiples of n. In the $N \times N$ puzzle, again some of the squares are filled initially with numbers from $1, \ldots, N$, and we need to fill in the remaining squares so that every row, every column, and every principal $n \times n$ grid contains all the numbers from 1 to N. (Thus, the standard Sudoku puzzle is the case where $N = 9$, $n = 3$.) As before, a puzzle is supposed to have a unique solution.

(a) Formulate an integer program (IP) of size polynomial in n that given an $N \times N$ ($N = n^2$) Sudoku puzzle checks if the puzzle has a unique solution. You may choose any convenient way for representing the input puzzle. Your IP solution should (i) yield a valid completion of the puzzle if one exists; and (ii) if there are two or more valid completions (i.e., the puzzle does not have a unique solution) then it should yield two distinct valid completions. **(7 marks)**

(b) Formulate an integer program that generates an $N \times N$ ($N = n^2$) Sudoku puzzle with the fewest number of filled squares. As mentioned earlier, a valid Sudoku puzzle is required to have *exactly one* valid completion. You may choose any convenient way for specifying the output puzzle.

The size of your formulation need not be polynomially bounded in n. **(8 marks)**

Q5: (*) Let $G = (V, E)$ be a connected graph. A (straight-line) planar drawing of G is a mapping $f : V \mapsto \mathbb{R}^2$ such that for each edge (u, v), the line segment joining $f(u)$ and $f(v)$ (which represents the edge (u, v)) does not contain $f(w)$ for any node $w \neq u, v$. Note that if $|V| \geq 3$, this implies that $f(u) \neq f(v)$ for distinct nodes u and v. The number of crossings of such a planar drawing is the number of (unordered) pairs of edges whose corresponding line segments intersect. The *rectilinear* *crossing number* of G, denoted $\text{rcr}(G)$, is the smallest number of crossings of a planar drawing of G. (A planar graph has rectilinear crossing number equal to 0.)

Let M be a large enough integer. Formulate an integer program to compute the M-grid crossing number of G, which is the smallest number of crossings of a planar drawing of G, where each node u is mapped to a (distinct) point (x_u, y_u), with $x_u, y_u \in \{1, \ldots, M\}$. (Assume M is large enough that a feasible planar drawing exists.) You may use both continuous and integer variables in your formulation. The size of your formulation should be polynomial in M and $|V|$. **(15 marks)**

Remark. The definition of $\text{rcr}(G)$ does not make it clear that there exists an optimum planar drawing where the nodes are mapped to rational points in $\mathbb{R}^2$, but this is easy to infer via a perturbation argument. Thus, there exists some (large) integer M such that the M-grid crossing number of G is equal to $\text{rcr}(G)$.
Determining $\text{rcr}(G)$ is a major computational open problem even for moderately-sized graphs. In fact, even the value of $\text{rcr}(K_{18})$, where K_{18} is the complete graph on 18 nodes, is not known. I do not know if the approach of writing an integer program yields a computationally feasible approach.

Open Questions: Recall the formulation (TSP-IP) for the traveling salesman problem (TSP) on n nodes that was considered in class. Here e indexes the edges of the n-node complete graph. Let P_{TSP} denote the convex hull of the feasible solutions to (TSP-IP). Any tour has n edges, thus if G is a connected graph on n-nodes that *does not* contain a Hamiltonian cycle then any tour may contain at most $n-1$ edges of G, so

$$x(E(G)) \leq n - 1 \qquad (*)$$

is a valid inequality for P_{TSP}. Let $\mathcal{G}_n$ denote the collection of all n-node connected non-Hamiltonian graphs. There are certain known classes of graphs for which (*) is a facet-defining inequality for P_{TSP}. It seems unlikely that one would be able to obtain a "good" characterization of all (or a "rich enough" subset of) graphs for which (*) yields a facet of P_{TSP}. However, even the answer to the following basic question seems to be not known:

$$\text{Is } P_{\text{TSP}} = \{x : x \text{ satisfies (1)}, \ \ (*) \text{ for all } G \in \mathcal{G}_n, \quad 0 \leq x_e \leq 1 \text{ for all } e\}?$$

Does equality hold above if we add the subtour-elimination constraints (2)?

Both these questions are more likely to have negative answers (and it may not be hard to give counterexamples), which motivates one to consider the following strengthening of (*). Given *any* graph G on at most n nodes (connected or not), define $T(G)$ to be the maximum value of $|T \cap E(G)|$ among all tours T (on n-nodes). Clearly $T(G) \leq n-1$ for all $G \in \mathcal{G}_n$. Now one can strengthen (*) as follows:

$$x(E(G)) \leq T(G), \qquad (**)$$

which (by definition) is a valid inequality for P_{TSP}. Notice that (2) (in the form $x(E(S)) \leq |S| - 1$) is an inequality of this form obtained by taking G to be the complete graph on the node-set S. In fact, it is not hard to see that *every* valid inequality $\alpha^T x \leq \beta$ for P_{TSP}, where α is a {0,1}-vector, is implied by some constraint of the form (**). Again, it is an open question whether

$$P_{\text{TSP}} = \{x : x \text{ satisfies (1)}, \ \ (**) \text{ for all graphs } G, \quad 0 \leq x_e \leq 1 \text{ for all } e\}?$$

The above observations make it tempting to conjecture that the answer is positive (and if so, it may not be hard to prove this).