CO452/652: Integer Programming — Winter 2009

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Assignment 2

Due: February 13, 2009 before class

You may use anything proved in class directly. I will maintain a FAQ about the assignment on the course webpage. Acknowledge all collaborators and sources of external help.

Q5, marked (*), is a regular question for *graduate (i.e., CO652) students*, and a bonus question for *undergraduate (i.e., CO452) students*.

Q0: (DO NOT HAND THIS IN) Let $P \neq \emptyset$ be a polyhedron. Prove that the following are equivalent: (a) P is a polytope; (b) P is bounded; (c) $\text{charcone}(P) = \{0\}$.

Q1: Let $P \subseteq \mathbb{R}^d$ be a non-empty polyhedron. Prove the following statements.

(a) $\text{charcone}(P) = \{y \in \mathbb{R}^d : x + \lambda y \in P \text{ for all } x \in P, \lambda \geq 0\}$.

(b) If $P = \{x : Ax \leq b\}$, then $\text{charcone}(P) = \{x : Ax \leq 0\}$.

(c) If $P = Q + C$, where Q is a polytope and C is a polyhedral cone, then $\text{charcone}(P) = C$.

(d) If $\max\ c^T x$ s.t. $x \in P$ has an optimal solution, then $c^T y \leq 0$ for all $y \in \text{charcone}(P)$.

(15 marks)

Q2: Consider the polyhedron $P \subseteq \mathbb{R}^2$ given by

$$\begin{aligned} x_1 - x_2 &\leq 0 && (1)\\ -x_1 + x_2 &\leq 1 && (2)\\ 2x_2 &\geq 5 && (3)\\ 8x_1 - x_2 &\leq 16 && (4)\\ x_1 + x_2 &\geq 4 && (5)\\ x &\in \mathbb{R}^2. \end{aligned}$$

(a) Find the dimension of P.

(b) Find an interior point of P (if one exists).

(c) Describe all the faces of P.

(d) Consider each of the faces $F_i = \{x \in P : a_i^T x = b_i\}$ for $i = 1, \ldots, 5$. What is the dimension of F_i? Which inequalities define facets of P?

(e) Give an irredundant system of inequalities that describe P.

(15 marks)

Q3: Let $P = \{x : Ax \leq b\} \subseteq \mathbb{R}^d$ be a rational polyhedron with $\mathbb{Z}(P) := P \cap \mathbb{Z}^d \neq \emptyset$. Given $c \in \mathbb{R}^d$, consider the linear program (LP): $\max\ c^T x$ s.t. $x \in P$ and the integer program (IP): $\max\ c^T x$ s.t. $x \in \mathbb{Z}(P)$.

(a) Prove that (LP) is unbounded iff (IP) is unbounded. **(7 marks)**

(b) Show that there exist vectors $\ell, u \in \mathbb{R}^d$ such that, for any $c \in \mathbb{R}^d$ for which (IP) is not unbounded, the optimal value of (IP) is equal to max $c^T x$ s.t. $x \in \mathbb{Z}(P),\ \ell \le x \le u$. **(8 marks)**

Q4: Let $P \subseteq \mathbb{R}^d$ be a non-empty polyhedron.

(a) Show that P can be written (uniquely) as $\text{lin}(P) + P'$ where P' is a pointed polyhedron that lies in a space orthogonal to $\text{lin}(P)$, that is, $v^T x = 0$ for all $v \in \text{lin}(P), x \in P'$. **(6 marks)**

(b) Let $F \subseteq P$. Show that F is a face of P of dimension $\dim(\text{lin}(P)) + k$ iff $F = F' + \text{lin}(P)$ where F' is a face of P' of dimension k. **(9 marks)**

Q5: (*) A polyhedron $P \subseteq \mathbb{R}^d$ is said to be of *antiblocking type* if $P \neq \emptyset,\ P \subseteq \mathbb{R}^d_+$, and if $x \in P$ and $0 \le y \le x$ implies that $y \in P$. The *antiblocker* of an antiblocking polyhedron $P \subseteq \mathbb{R}^d_+$ is defined by $A(P) := \{z \in \mathbb{R}^d_+ : z^T x \le 1 \text{ for all } x \in P\}$.

(a) Let $P \subseteq \mathbb{R}^d_+$ be a polyhedron of antiblocking type. Show that $A(P)$ is also a polyhedron of antiblocking type. Show that P must be of the form $\{x \in \mathbb{R}^d : Ax \le b,\ x \ge 0\}$, where A is a nonnegative matrix, and $b \ge 0$. Deduce that $A(A(P)) = P$. **(5 marks)**

(b) Let $P = \{x \in \mathbb{R}^d_+ : Ax \le e\}$ where A is a nonnegative (non-zero) matrix with rows $a_1^T, a_2^T, \ldots, a_m^T$ and e is the all-1s vector. Show that $A(P) = \{z \in \mathbb{R}^d_+ : \exists \lambda \ge 0 \text{ s.t. } z \le \sum_{i=1}^m \lambda_i a_i,\ \sum_{i=1}^m \lambda_i = 1\}$ (note that P is of antiblocking type). Show that $a_i^T x \le 1$ is a facet-defining inequality of P iff a_i is an extreme point of $A(P)$ that is maximal in $A(P)$ (that is, $z \in A(P)$ and $z \ge a_i$ implies that $z = a_i$). **(8 marks)**

(c) Let A be a $m \times d$ nonnegative matrix with rows $a_1^T, \ldots, a_m^T$, and B be a $k \times d$ nonnegative matrix with rows $b_1^T, \ldots, b_k^T$. Prove that

$$\max \{w^T a_1, w^T a_2, \ldots, w^T a_m\} \quad = \quad \min\ y^T e \quad \text{s.t.} \quad y^T B \ge w^T,\ y \ge 0 \qquad \text{for all } w \in \mathbb{R}^d_+$$

iff

$$\max \{w^T b_1, w^T b_2, \ldots, w^T b_k\} \quad = \quad \min\ y^T e \quad \text{s.t.} \quad y^T A \ge w^T,\ y \ge 0 \qquad \text{for all } w \in \mathbb{R}^d_+.$$

(7 marks)