Degree Spectra of Prime Models

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Contents

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Abstract

We consider the Turing degrees of prime models of complete decidable theories. In particular we show that every complete decidable atomic theory has a prime model whose elementary diagram is low. We combine the construction used in the proof with other constructions to show that complete decidable atomic theories have low prime models with added properties.

If we have a complete decidable atomic theory with all types of the theory computable, we show that for every degree $\mathbf{d}$ with $\mathbf{0} < \mathbf{d} \leq \mathbf{0}'$, there is a prime model with elementary diagram of degree $\mathbf{d}$. Indeed, this is a corollary of the fact that if T is a complete decidable theory and L is a computable set of c.e. partial types of T, then for any Δ^0_2 degree $\mathbf{d} > 0$, T has a $\mathbf{d}$-decidable model omitting the nonprincipal types listed by L.

1 Introduction

In computable model theory, we examine the complexity of theorems in model theory using tools from computability theory. For instance, let T be a complete theory. A formula $\varphi(\overline{x})$ is *complete* with respect to T if for every formula $\psi(\overline{x})$ (with the same free variables), exactly one of $T \models (\forall \overline{x})\,[\varphi(\overline{x}) \rightarrow \psi(\overline{x})]$ or $T \models (\forall \overline{x})\,[\varphi(\overline{x}) \rightarrow \neg\psi(\overline{x})]$ holds. A theory T is *atomic* if for every formula $\theta(\overline{x})$ consistent with T, there is a formula $\varphi(\overline{x})$, complete with respect to T, such that $T \models (\forall \overline{x})\,[\varphi(\overline{x}) \rightarrow \theta(\overline{x})]$. A *prime model* of a theory is one that can be embedded into any other model of the theory. For example, the algebraic numbers are a prime model for the theory of algebraically closed fields of characteristic 0. It is a well known result in model theory that every complete atomic theory has a prime model. If we assume that we have a complete decidable atomic theory, we can ask how complicated its prime model must be. Since the theory is complete and decidable, it has a decidable (and, hence, computable) model. Millar [10] ruled out the possibility of there always being a computable *prime* model by constructing a complete decidable atomic theory with no computable prime model. Denisov [4], Drobotun [5], and Millar [10] showed that every complete decidable atomic theory has a prime model computable in $\emptyset'$. In this paper we improve upon this result by showing:

Theorem 1.1 (Prime Model Low Basis Theorem). *Every complete decidable atomic theory has a low prime model, that is, one that is not only $\emptyset'$-computable but whose jump is also $\emptyset'$-computable.*

The construction used in the proof of the Prime Model Low Basis Theorem can be combined with other computability theoretic constructions to show that complete decidable atomic theories have low prime models with added properties. For example, we say that sets A and B form a *minimal pair* if whenever C is such that $C \leq_T A$ and $C \leq_T B$, then C is computable. We show that a complete decidable atomic theory must have a minimal pair of low prime models. We also show that given any low set X, a complete decidable atomic theory must have a low prime model Turing incomparable with X.

A *type* of a theory is a maximal consistent set of formulas on a finite number of variables that extends the theory. In Millar's example of a complete decidable atomic theory with no computable prime model, the theory constructed has the property that all the types of the theory are computable [10]. When examining the prime models of complete decidable atomic theories with all types computable, we know by Millar's example that we cannot be guaranteed a computable prime model. However, there must be prime models that are "almost" computable, as we see by the next theorem.

Theorem 1.2 (Full Basis Theorem for Prime Models). *If T is a complete decidable theory with all types computable, then T has a prime model Turing equivalent to A for every non-computable set $A \leq_T \emptyset'$.*

This theorem is actually a consequence of a theorem we prove on omitting partial types. The classical omitting types theorem tells us that for T a countable consistent theory, if $\{ \Gamma_j(\overline{x}_j) \}_{j \in \omega}$ is a countable family of partial types, all nonprincipal with respect to T, then T has a model omitting all Γ_j. In [11], Millar showed ways in which the classical theorem can and cannot be effectivized. Indeed, while he showed that given a complete decidable theory T, and a computable listing L of a subset Ψ of computable types of T, there is a *decidable* model $\mathfrak{A}$ which omits all nonprincipal types from Ψ, he also showed that there is a complete decidable theory T and a computable set Ψ of computable *partial* types of T such that *no decidable* model $\mathfrak{A}$ of T omits all nonprincipal types in Ψ. In this paper, we show that while we are not guaranteed a decidable model omitting the partial types, we do have one in every other Δ^0_2 degree. That is,

Theorem 1.3. *Let T be a complete decidable theory and let L be a computable set Γ of c.e. partial types of T. Then for any Δ^0_2 degree $\mathbf{d} > 0$, T has a $\mathbf{d}$-decidable model omitting the nonprincipal types listed by L.*

To construct the models for the various theorems, we will build computable trees corresponding to complete decidable theories, so that the infinite paths on the tree will correspond to models of the theory via the usual Henkin construction. To prove the prime model low basis theorem, we use a $0'$ oracle and make finite extensions of a path on the tree. For the theorem on omitting partial types, we build a Δ^0_2 approximation to a path on the tree, using the method of Δ^0_2 permitting.

2 Notation and conventions

2.1 Notation from model theory

We mostly use the notation of Soare [14] for computability theory, Chang and Keisler [1] for basic model theory, and the conventions of Harizanov [6] for computable model theory.

We consider presentations $\mathfrak{A}$ of models with universe ω. For a model $\mathfrak{A}$ we write $D(\mathfrak{A})$ for the atomic diagram, and $D^e(\mathfrak{A})$ for the elementary diagram. We say $\mathfrak{A}$ is *computable* if $D(\mathfrak{A}) \equiv_T 0$, and $\mathfrak{A}$ is *decidable* if $D^e(\mathfrak{A}) \equiv_T 0$. Note that we always have $D(\mathfrak{A}) \leq_T D^e(\mathfrak{A})$. We say $\mathfrak{A}$ is *low* if $D^e(\mathfrak{A})$ (and so $D(\mathfrak{A})$) is low.

2.2 $F_n(T)$ and $B_n(T)$

Let $\mathcal{L}$ be a countable relational language, and let T be a complete decidable $\mathcal{L}$-theory. For each $n \in \omega$, let $F_n(T)$ be the set of formulas, $\theta(x_0, x_1, \ldots, x_{n-1})$, with free variables among the variables $\{x_0, \ldots x_{n-1}\}$. Let $B_n(T)$ be the Lindenbaum algebra generated from $F_n(T)$ under the equivalence classes $[\theta(\overline{x})]$ where

$$[\theta(\overline{x}) \equiv \xi(\overline{x})] \iff \vdash_T \ (\forall \overline{x})[\theta(\overline{x}) \leftrightarrow \xi(\overline{x})].$$

Since T is complete and decidable, this equivalence is computable.

2.3 Henkin constructions, $\mathcal{L}_c$ and $\{\theta_i : i \in \omega\}$

We normally build a model $\mathfrak{A}$ of T by a Henkin construction. Thus, our language will usually be $\mathcal{L}_c = \mathcal{L}(T) \bigcup \{c_k : k \in \omega\}$, where $\{c_k : k \in \omega\}$ is a computable set of fresh constants. Let $\{\theta_i : i \in \omega\}$ be an effective enumeration of the sentences in $\mathcal{L}_c$ with the property that if $\theta_i = (\exists x)\varphi(x)$ then $\theta_{i+1} = \varphi(c_k)$ for the least c_k not appearing in any θ_j, $j \leq i$. Let θ^j denote θ or $\neg\theta$ according as $j = 1$ or 0. For $\alpha \in 2^\omega$ define

$$\theta^\alpha = \wedge\{\theta_j^{\alpha(j)} : j < lh(\alpha)\}.$$

We say $\alpha \in 2^\omega$ is a *Henkin path* if whenever $\theta_i = (\exists x)\varphi(x)$ and $\alpha(i) = 1$ we have $\alpha(i+1) = 1$. That is, each path has Henkin witnesses for the existential sentences. Let

$$\mathcal{T} = \{\alpha : \alpha \in 2^{<\omega} \wedge T \cup \theta^\alpha \text{ is consistent} \wedge \alpha \text{ is a Henkin path}\}.$$

For $\alpha \in 2^{<\omega}$ and $j \in \{0,1\}$ let $\alpha\widehat{\ }j$ denote the concatenation of α followed by j, viewed as a string in $2^{<\omega}$. We will identify $i \in \omega$ with θ_i, and $\alpha \in 2^{<\omega}$ with θ^α. Since the theory T is decidable, the tree $\mathcal{T}$ is computable. Let $[\mathcal{T}]$ denote the set of infinite paths of $\mathcal{T}$. Note that every node $\alpha \in \mathcal{T}$ is extendible, because θ^α is consistent. Hence, for any $\alpha \in \mathcal{T}$ there is a computable path $f \in [\mathcal{T}]$ extending α.

The paths in $[\mathcal{T}]$ correspond to elementary diagrams of models of the original theory T. Indeed, since the tree was built to have only Henkin paths, each path in $[\mathcal{T}]$ will have Henkin witnesses. So the model can be obtained using the usual Henkin method, as described in Marker [9].

We are often interested in the atomic diagrams of models we are building. Given the effective enumeration $\{\theta_i \mid i \in \omega\}$ of sentences in $\mathcal{L}_c$ we get an effective sub-enumeration $\{\theta_{i_j} \mid j \in \omega\}$ of the atomic sentences in $\mathcal{L}_c$. Given $\alpha \in \mathcal{T}$ we define the projection of α onto the atomic sentences, $\pi(\alpha)$ to be

$$\pi(\alpha)(j) = \alpha(i_j).$$

Then we define the projection of $\mathcal{T}$ onto atomic sentences by

$$\pi(\mathcal{T}) = \{\pi(\alpha) \mid \alpha \in \mathcal{T}\}.$$

The paths in $[\pi(\mathcal{T})]$ correspond to atomic diagrams of models of the original theory T. Moreover, any node on $\pi(\mathcal{T})$ can be lifted to a node on $\mathcal{T}$ which projects down to it.

2.4 n-types and $S_n(T)$

An *n-type* of T is a maximal consistent set of formulas in $F_n(T)$. Let $S_n(T)$ denote the set of n-types of T (the Stone space of $F_n(T)$), and $S(T) = \bigcup_n S_n(T)$. A *partial type* is a subset of a type. A (partial) type $\Gamma(\overline{x})$ is *nonprincipal* if there is no formula $\theta(\overline{x})$ such that for all $\gamma(\overline{x}) \in \Gamma(\overline{x})$, $T \vdash (\exists \overline{x})(\theta(\overline{x})) \wedge (\forall \overline{x})(\theta(\overline{x}) \rightarrow \gamma(\overline{x}))$.

When building models using the tree $\mathcal{T}$ described in section 2.3, we often want to know how formulas relate to various n-types. For this the following notation is helpful.

Given an $\mathcal{L}_c$ formula θ, and tuples $\overline{c} = \{c_{i_1}, \dots c_{i_n}\}$ and $\overline{x} = \{x_{i_1}, \dots x_{i_n}\}$, let $\theta(\overline{c}/\overline{x})$ be the formula obtained from θ by replacing every occurrence of c_i with x_i, avoiding clashes. Similarly, given $\theta \in F_n(T)$, let $\theta(\overline{x}/\overline{c})$ be the formula obtained from θ by replacing every free occurrence of x_i with c_i.

Given an $\mathcal{L}_c$ formula θ and $\overline{k}$ a finite subset of $\{c_i\}_{i\in\omega}$, let $I = \{i \mid c_i$ appears in θ and $c_i \notin \overline{k}\}$. Then let $\theta^*_\forall(\overline{k}/\overline{x})$ denote the formula $(\forall x_i)_{i\in I}\theta(\overline{k}/\overline{x})$, and let $\theta^*_\exists(\overline{k}/\overline{x})$ denote the formula $(\exists x_i)_{i\in I}\theta(\overline{k}/\overline{x})$.

Note that for any complete decidable theory T, we have an effective listing of the formulas in $F_n(T)$. We associate each formula with its index in this list. Thus, if φ_e is a partial computable function coding an n-type, $\varphi_e(\theta^*_\forall(\overline{k}/\overline{x}))$ means $\varphi_e(j)$ where j is the index of $\theta^*_\forall(\overline{k}/\overline{x})$ on the list.

3 Degree spectra

Given a model $\mathfrak{A}$, its *degree spectrum* is the collection of the degrees of its isomorphic copies. This notion makes sense for the degrees of both the atomic and elementary diagrams, so we make the following definitions.

Definition 3.1. For a countable structure $\mathfrak{A}$, let

$$dSp^a(\mathfrak{A}) = \{deg(D(\mathfrak{B})) : \mathfrak{B} \cong \mathfrak{A}\}$$
$$dSp^e(\mathfrak{A}) = \{deg(D^e(\mathfrak{B})) : \mathfrak{B} \cong \mathfrak{A}\}$$

We will often refer to the following result of Knight's, which tells us that the degree spectra are upward closed.

Theorem 3.2 (Knight [8]). *Let $\mathfrak{A}$ be a countable structure in a relational language. Then either $dSp^a(\mathfrak{A})$ is a singleton, or $dSp^a(\mathfrak{A})$ is closed upwards.*

Corollary 3.3. *For $\mathfrak{A}$ such that $dSp^e(\mathfrak{A})$ is not a singleton,*
(i) $dSp^e(\mathfrak{A}) \subseteq dSp^a(\mathfrak{A})$.
(ii) $dSp^e(\mathfrak{A})$ is closed upwards.

Proof. (i) This follows immediately from upward closure and the fact that for any structure $\mathfrak{B}$ we always have $D(\mathfrak{B}) \leq_T D^e(\mathfrak{B})$.

(ii) This follows from examining Knight's proof and noting that the same proof works if $D(\mathfrak{A})$ is replaced with $D^e(\mathfrak{A})$ throughout. □

Note that since any two prime models of a theory are isomorphic, and since if $\mathfrak{B}$ is isomorphic to a prime model then it is a prime model, we have that if T is a complete decidable atomic theory and $\mathfrak{A}$ is a prime model of T,

$$dSp^e(\mathfrak{A}) = \{deg(D^e(\mathfrak{B})) \mid \mathfrak{B} \text{ is a prime model of } T\}.$$

Definition 3.4. We say a complete decidable atomic theory T is *nontrivial* if the degree spectra of a prime model of the theory is not a singleton. See Knight [8] for an exact characterization.

The following results tell us about the complexity of the degree spectra of prime models of complete decidable atomic theories.

Theorem 3.5 (Denisov [4], Drobotun [5], Millar [10]). *Let T be a complete, atomic, decidable theory. Then T has a prime model $\mathfrak{A}$ decidable in $0'$.*

The following theorem of Millar states that even though a complete decidable atomic theory has a prime model, and a decidable model, the prime model need not be decidable, or even computable.

Theorem 3.6 (Millar [10]). *There is a complete atomic decidable theory T with no* computable *prime model. Indeed, there is such a theory whose types are all computable.*

4 Prime model low basis theorem

4.1 The basic theorem

¿From theorems 3.5 and 3.6 we see that the degree spectrum of a prime model of a complete decidable atomic theory must contain a $0'$-decidable model, but need not contain a computable model. The following Prime Model Low Basis Theorem shows that in fact such a theory must have a *low* prime model. Indeed, there are many variations of this theorem. We first give a proof of the basic theorem, then show how it can be modified to obtain more results.

Theorem 4.1 (Prime Model Low Basis Theorem). *Every complete decidable atomic theory has a low prime model.*

Proof. Let T be a complete decidable atomic theory. We build $D^e(\mathfrak{A})$ by using a $0'$-oracle to extend a path along the computable tree $\mathcal{T}$ of possible models as described in section 2.3. Enumerate all finite tuples of constants in $\mathcal{L}_c$, and let $\overline{k}$ denote the k^{th} tuple. We wish to meet the requirements:

P_k: $\overline{k}$ realizes a principal type.

Q_k: Whether or not $\varphi_k^{D^e(\mathfrak{A})}(k)\downarrow$ is decided by stage $2k+2$ of the construction.

We will build $\{\alpha_s\}$ such that $\alpha_s \subseteq \alpha_{s+1} \in \mathcal{T}$.

Stage 0: $\alpha_0 = \emptyset$.

Stage $s = 2k+1$: To meet requirement P_k, we want to assure that $\overline{k}$ realizes a principal type. Let $\theta(\overline{x}) \in F_n(T)$ be what $\overline{k}$ satisfies in α_{s-1}, namely $\theta(\overline{x}) = \alpha^*_{s-1_\exists}(\overline{k}/\overline{x})$. Since T is atomic, there is a complete formula in $F_n(T)$ completing θ. We search for an extension of α_{s-1} such that it projects down to such a formula in $F_n(T)$.

For each extension σ of α_{s-1} in $\mathcal{T}$, $0'$ answers the question

$$(\exists j < lh(\sigma))[\sigma(j) = 1 \wedge (\forall \psi \in F_n(T))[T \vdash \theta_j(\overline{x}/\overline{c}) \rightarrow \psi \vee T \vdash \theta_j(\overline{x}/\overline{c}) \rightarrow \neg\psi]].$$

We know there will be a σ for which the answer is "yes". We let α_s be the first such σ.

Stage $s = 2k+2$: To meet requirement Q_k and make $D^e(\mathfrak{A})$ low, ask whether $(\exists \sigma \supseteq \alpha_{s-1})(\exists t)[\sigma \in \mathcal{T} \wedge \varphi^\sigma_{k,t}(k)\downarrow]$. If yes, search for such an extension σ of α_{s-1}, and set $\alpha_s = \sigma$. Otherwise, set $\alpha_s \supset \alpha_{s-1}$, with $\alpha_s \in \mathcal{T}$ and $lh(\alpha_s) > \alpha_{s-1}$.

Let $D^e(\mathfrak{A}) = \lim_s \alpha_s$. This limit exists, as at each stage we extend what we had at the previous stage. At even stages we ensure a proper extension, so it is an infinite path.

Verification:

Since the construction is $0'$-computable, $D^e(\mathfrak{A}) \leq_T 0'$. At stage $2k+1$, we extend to meet requirement P_k. Thus each tuple $\overline{k}$ realizes a principal type, and so $\mathfrak{A}$ is a prime model. At stage $2k+2$ we meet requirement Q_k. Since all Q_k are met, $D^e(\mathfrak{A})$ is low. Indeed, $0'$ can follow the construction to stage $2k+2$, so $0'$ can compute $\{k \mid \varphi_k^{D^e(\mathfrak{A})}(k)\downarrow\}$. □

Remark 1. *The Prime Model Low Basis Theorem does not follow immediately from the Low Basis Theorem.*

Indeed, suppose that one could always build a Π^0_1 class of prime models of a complete decidable theory. By a result of Jockusch and Soare [7], given the degree of any completion of Peano arithmetic and any Π^0_1 class, there is a path in the class computable in the degree. Since there are completions of Peano arithmetic of low degree, this would show that each theory would have a prime model computable in *the same* low degree. That is to say, there is a low degree such that every complete atomic decidable theory has a prime model decidable in that degree. We call degrees with this property *prime bounding*. In [3], Csima, Hirschfeldt, Knight, and Soare show that a Δ_2^0 degree is prime bounding if and *only if* it is not low$_2$. Thus there can be no low prime bounding degree, so we have a contradiction.

Conjecture 4.2 (Clote [2]). *There is a complete and decidable atomic theory T such that $D^e(\mathfrak{A}) \geq_T 0'$ for every prime model $\mathfrak{A}$ of T.*

Corollary 4.3. *Clote's conjecture is false.*

Proof. By the prime model low basis theorem, every complete decidable atomic theory has a low prime model $\mathfrak{A}$, and so for that $\mathfrak{A}$, $D^e(\mathfrak{A}) \not\geq_T 0'$. □

Can we replace $0'$ in Clote's conjecture by some other $X >_T 0$? By Theorem 4.1 such X must be low. We answer this question in the following section.

4.2 Avoiding low X

The methods of the previous theorem can be combined with other requirements to yield stronger results.

This next theorem shows that we can avoid the cone above and below a low set X. Since $D(\mathfrak{A}) \leq_T D^e(\mathfrak{A})$, we avoid X by $D^e(\mathfrak{A})$ above and $D(\mathfrak{A})$ below. Note that we only use that $\mathfrak{A}$ has no computable prime model to ensure $D(\mathfrak{A}) \not\leq_T X$.

Theorem 4.4. *For all complete decidable atomic theories T with no computable prime model and for all $0 <_T X$ low, there is a low prime model $\mathfrak{A}$ of T such that $X \not\leq_T D^e(\mathfrak{A})$, and $D(\mathfrak{A}) \not\leq_T X$.*

Proof. Let T be a complete decidable atomic theory. As in Theorem 4.1, we build $D^e(\mathfrak{A})$ by using a $0'$-oracle to extend a path along the computable tree $\mathcal{T}$ of possible models. Again we let $\overline{k}$ denote the k^{th} possible tuple of constants. We wish to meet the requirements:

P_k: $\overline{k}$ realizes a principal type.

Q_k: Whether or not $\varphi_k^{D^e(\mathfrak{A})}(k)\downarrow$ is decided by stage $4k+2$ of the construction.

N_k: $X \neq \varphi_k^{D^e(\mathfrak{A})}$

M_k: $D(\mathfrak{A}) \neq \varphi_k^X$

We will build $\{\alpha_s\}$ such that $\alpha_s \subseteq \alpha_{s+1} \in \mathcal{T}$.

Stage 0: $\alpha_0 = \emptyset$.

Stage $s = 4k+1$*:* Do as in stage $2k+1$ of Theorem 4.1.

Stage $s = 4k+2$*:* Do as in stage $2k+2$ of Theorem 4.1.

Stage $s = 4k+3$*:* Ask whether

$$(\exists t)(\exists y)(\exists \sigma \supseteq \alpha_{s-1})[\sigma \in \mathcal{T} \wedge \varphi_{k,t}^{\sigma}(y) \downarrow \neq X(y)].$$

This is a $0'$ (or X') question. If no, set $\alpha_s = \alpha_{s-1}$. If yes, search for such t, y, and σ, and set $\alpha_s = \sigma$.

Stage $s = 4k+4$*:* Given α_{s-1}, the current approximation to the elementary diagram, we can compute $\pi(\alpha_{s-1})$, an approximation to the *atomic* diagram, as described in Section 2.3. Then ask whether

$$(\exists t)(\exists y)(\exists \tau)(\exists \sigma \supseteq \pi(\alpha_{s-1}))[\sigma \in \pi(\mathcal{T}) \wedge \tau \subseteq X \wedge \sigma(y) \neq \varphi_{k,t}^{\tau}(y) \downarrow].$$

This is a $0'$ (or X') question. If yes, search for such an extension σ of $\pi(\alpha_{s-1})$, and let $\alpha_s \in \mathcal{T}$ be such that $\pi(\alpha_s) = \sigma$. Such an α_s exists by the definition of $\pi(\mathcal{T})$. Otherwise, set $\alpha_s = \alpha_{s-1}$.

Let $D^e(\mathfrak{A}) = \lim_s \alpha_s$. This limit exists, and is infinite.

Verification:

As in the previous theorem, requirements P_k and Q_k are met at stages $4k+1$ and $4k+2$. So $\mathfrak{A}$ is a low prime model.

Lemma 1. *Requirement N_k is met for all k.*

Proof. Assume for a contradiction that $X = \varphi_k^{D^e(\mathfrak{A})}$. Then at stage $4k+3$ the answer to $(\exists t)(\exists y)(\exists \sigma \supseteq \alpha_{s-1})[\sigma \in \mathcal{T} \wedge \varphi_{k,t}^{\sigma}(y) \downarrow \neq X(y)]$ must have been "no". Thus $(\forall t)(\forall y)(\forall \sigma \supseteq \alpha_{s-1})[\sigma \in \mathcal{T} \rightarrow [\varphi_{k,t}^{\sigma}(y) \uparrow \vee \varphi_{k,t}^{\sigma}(y) \downarrow = X(y)]]$. Since $X = \varphi_k^{D^e(\mathfrak{A})}$ is total, and since $\mathcal{T}$ is computable, we can compute $X(y)$ by looking at $\varphi_{k,t}^{\sigma}(y)$ for varying $\sigma \in \mathcal{T}$ and t and noting that once this converges, it converges to $X(y)$. □

It follows immediately from the lemma that $X \not\leq_T D^e(\mathfrak{A})$.

Lemma 2. *Requirement M_k is met for all k.*

Proof. Assume for a contradiction that $D(\mathfrak{A}) = \varphi_k^X$. Then at stage $4k+4$ the answer to $(\exists t)(\exists y)(\exists \tau)(\exists \sigma \supseteq \beta)[\sigma \in \pi(\mathcal{T}) \wedge \tau \subseteq X \wedge \sigma(y) \neq \varphi_{k,t}^{\tau}(y)\downarrow]$ was "no". So

$$(\forall t)(\forall y)(\forall \tau)(\forall \sigma \supseteq \beta)[(\sigma \in \pi(\mathcal{T}) \wedge \tau \subset X \wedge \varphi_{k,t}^{\tau}(y)\downarrow) \rightarrow \varphi_{k,t}^{\tau}(y) = \sigma(y)].$$

Now since $D(\mathfrak{A}) = \varphi_k^X$ is total, this shows that $D(\mathfrak{A})$ was the *unique* extension of $\pi(\alpha_{s-1})$ in the computable tree $\pi(\mathcal{T})$. Thus $D(\mathfrak{A})$ is a computable prime model of T, a contradiction. □

It follows immediately from the lemma that $D(\mathfrak{A}) \not\leq_T X$.

□

Corollary 4.5 (Csima, Knight). *For every complete decidable theory T with a prime model, and for every $X >_T 0$, there is a prime model $\mathfrak{A}$ of T such that $X \not\leq_T D^e(\mathfrak{A})$. The result also follows from work of Knight's in [8].*

Proof. If X is low, this follows immediately from Theorem 4.4. If X is not low, then by the Low Basis Theorem for Prime Models 4.1, X has a low prime model $\mathfrak{A}$. But then, as X is not low, $X \not\leq_T D^e(\mathfrak{A})$. □

4.3 Minimal pair

Theorem 4.6. *If T is a nontrivial complete decidable atomic theory, then T has a minimal pair of low prime models.*

Proof. Let T be a nontrivial complete decidable atomic theory. If T has a computable prime model, then since it is nontrivial it must have a minimal pair of low prime models by upward closure. So we assume that T has no computable prime model.

As in the previous two theorems, we build $D^e(\mathfrak{A})$ and $D^e(\mathfrak{B})$ by using a $0'$-oracle to extend a paths along the computable tree $\mathcal{T}$ of possible models. We again let $\overline{k}$ denote the k^{th} tuple of constants. We wish to meet the requirements:

$P_k^{\mathfrak{A}}$: $\overline{k}$ realizes a principal type in $\mathfrak{A}$.

$P_k^{\mathfrak{B}}$: $\overline{k}$ realizes a principal type in $\mathfrak{B}$.

$Q_k^{\mathfrak{A}}$: Whether or not $\varphi_k^{D^e(\mathfrak{A})}(k)\downarrow$ is decided by stage $3k+2$ of the construction.

$Q_k^{\mathfrak{B}}$: Whether or not $\varphi_k^{D^e(\mathfrak{B})}(k)\downarrow$ is decided by stage $3k+2$ of the construction.

N_k: $\varphi_k^{D^e(\mathfrak{A})} = \varphi_k^{D^e(\mathfrak{B})} = f$ total $\Rightarrow f$ computable.

Remark 2 (Posner). *If $D^e(\mathfrak{A}) \neq D^e(\mathfrak{B})$ are both noncomputable, and if requirement N_k is met for all $k \in \omega$, then $D^e(\mathfrak{A})$ and $D^e(\mathfrak{B})$ form a minimal pair.*

Proof. Let $n_0 \in D^e(\mathfrak{A}) - D^e(\mathfrak{B})$ (or $n_0 \in D^e(\mathfrak{B}) - D^e(\mathfrak{A})$) . For each i and j there is an index k such that

$$\varphi_k^X(x) = \begin{cases} \varphi_i^X(x) & \text{if } n_0 \in X, \\ \varphi_j^X(x) & \text{otherwise.} \end{cases}$$

Then if $\varphi_i^{D^e(\mathfrak{A})} = \varphi_j^{D^e(\mathfrak{B})} = f$ total, we have $\varphi_k^{D^e(\mathfrak{A})} = \varphi_k^{D^e(\mathfrak{B})} = f$ total, and so f is computable since N_k was met. □

We will build $\{\alpha_s\}$ and $\{\beta_s\}$ such that $\alpha_s \subseteq \alpha_{s+1} \in \mathcal{T}$ and $\beta_s \subseteq \beta_{s+1} \in \mathcal{T}$.

Construction:

Stage 0: Let $\alpha_0 \neq \beta_0$ be incompatible nodes in $\mathcal{T}$.

Stage $s = 3k+1$: For each of α_s and β_s, do as in stage $2k+1$ of Theorem 4.1.

Stage $s = 3k+2$: For each of α_s and β_s, do as in stage $2k+2$ of Theorem 4.1.

Stage $s = 3k+3$: Ask whether

$$(\exists \alpha' \supseteq \alpha_{s-1})(\exists \beta' \supseteq \beta_{s-1})(\exists n)(\exists s)[\alpha' \in \mathcal{T} \wedge \beta' \in \mathcal{T} \wedge \varphi_{k,s}^{\alpha'}(n)\downarrow \neq \varphi_{k,s}^{\beta'}(n)\downarrow].$$

If yes, extend to these. If no, let $\alpha_s = \alpha_{s-1}$, $\beta_s = \beta_{s-1}$.

Let $D^e(\mathfrak{A}) = \lim_s \alpha_s$, and $D^e(\mathfrak{B}) = \lim_s \beta_s$. These limits exist, and are infinite.

Verification:

As in the previous theorems, requirements $P_k^{\mathfrak{A}}$, $P_k^{\mathfrak{B}}$, $Q_k^{\mathfrak{A}}$, and $Q_k^{\mathfrak{B}}$ are met at stages $4k+1$ and $4k+2$. So $\mathfrak{A}$ and $\mathfrak{B}$ are low prime models.

Lemma 3. *Requirement N_k is met for all k.*

Proof. Suppose $f = \varphi_k^{D^e(\mathfrak{A})} = \varphi_k^{D^e(\mathfrak{B})}$ is total. We can compute f from α_{3k+3} as follows: To compute $f(n)$, compute $\varphi_{k,s}^{\sigma}$ for possible extensions $\sigma \supseteq \alpha_{3k+3}$. Once one of these converges to a value (and eventually one must, as $f = \varphi_k^{D^e(\mathfrak{A})}$ is total), that value is $f(n)$. Indeed, since $f = \varphi_k^{D^e(\mathfrak{B})}$, there is some extension τ of β_{3k+3} and some t such that $\varphi_{k,t}^{\tau}(n) \downarrow$. Now if $\varphi_{k,s}^{\sigma}(n) \downarrow$ for some s and some $\sigma \supseteq \alpha_{3k+3}$, then by the construction $\varphi_{k,s}^{\sigma}(n) \downarrow = \varphi_{k,t}^{\tau}(n) \downarrow = f(n)$. □

At stage 0 we guarantee $D^e(\mathfrak{A}) \neq D^e(\mathfrak{B})$. Since T has no computable prime model, $D^e(\mathfrak{A})$ and $D^e(\mathfrak{B})$ are not computable. So by Remark 2 and Lemma 3, $D^e(\mathfrak{A})$ and $D^e(\mathfrak{B})$ form a minimal pair.

□

5 n-jump degrees

Definition 5.1. [Jockusch] For each $n \geq 0$, if $\{deg(D(\mathfrak{B}))^{(n)} : \mathfrak{B} \cong \mathfrak{A}\}$ has a least element $\mathbf{d}$, then we say that $\mathbf{d}$ is the *n-jump degree* of $\mathfrak{A}$.

Theorem 5.2. *If T is a complete decidable theory with a prime model $\mathfrak{A}$, but no computable prime model, then $\mathfrak{A}$ has no 0-jump degree.*

Proof. Follows immediately from Corollary 4.5 on avoiding cones. □

Theorem 5.3. *If T is a complete decidable theory with prime model $\mathfrak{A}$, then $\mathfrak{A}$ has n-jump degree $\mathbf{0}^{(n)}$ for all $n \geq 1$.*

Proof. By Theorem 4.1, T has a low prime model $\mathfrak{A}$. So, for $n \geq 1$, $deg(D(\mathcal{A}))^{(n)} = 0^{(n)}$. □

6 Δ_2^0 permission

In the next two theorems, we will build models computable in particular Δ_2^0 degrees. To do this we use the method of Δ_2^0 permission. For a nice description of this method, see Miller [12].

Given a Δ_2^0 set C, let $\{C_s\}_{s\in\omega}$ be a computable approximation of C. For $s > 0$, let

$$x_s = \max\{x \mid (\exists t < s)[x \leq t \wedge C_s \restriction x = C_t \restriction x]\}$$

$$t_s = \min\{t \mid x_s \leq t < s \wedge C_s \restriction x_s = C_t \restriction x_s\}.$$

So x_s is the greatest length of agreement of C_s with any preceeding stage and t_s is that preceeding stage (or first such stage if there is more than one.)

Lemma 4. *To build $A \leq_T C$, it is enough to ensure $A_s \upharpoonright x_s = A_{t_s} \upharpoonright x_s$ for all s.*

Proof. To compute $A \upharpoonright x$, find a stage s such that $C_s \upharpoonright x = C \upharpoonright x$. Then $A \upharpoonright x = A_s \upharpoonright x$. □

We say a stage s is a *true stage* of the approximation $\{C_s\}$ to C if the length of agreement of C_s with C is greater than the corresponding length of agreement for every preceeding stage. That is,

$$(\exists x \leq s)[C_s \upharpoonright x = C \upharpoonright x \wedge (\forall t)[x \leq t < s \rightarrow C_t \upharpoonright x \neq C \upharpoonright x]].$$

Note that if we meet the hypothesis of Lemma 4 when building A, then the true stages of A will be the same as the true stages for C. Note also that if s is a true stage then t_s is the previous true stage.

When building A to satisfy certain properties, we will want to show that C permits us to do things often enough. We make use of the following lemma. For a proof see [12].

Lemma 5 (Δ^0_2 Permission). *Let $s_0 = 0, s_1, s_2, \ldots$ be the true stages of a computable approximation $\{C_s\}_{s\in\omega}$ of C, with $s_i < s_{i+1}$ for all i. Let $\{n_s\}_{s\in\omega}$ be a nondecreasing unbounded computable sequence. If $\{q \mid n_{s_q} > x_{s_q}\}$ is finite, then C is computable.*

7 When the types are all computable

The different forms that the degree spectrum of various structures can take has been widely studied. It was asked whether a structure could have the property that its degree spectrum was exactly all the non-computable degrees. Slaman [13] and Wehner [15] each constructed such countable first order structures, answering the question in the affirmative. The structures that they constructed were built particularly to answer the question, and were not previously studied. Miller [12] constructed a linear ordering whose degree spectrum restricted to the Δ^0_2 degrees is exactly the non-computable Δ^0_2 degrees. We now show the same for prime models of any complete decidable theories whose types are all computable.

Theorem 7.1. *Let T be a complete decidable theory with all types computable, and with a prime model but no computable prime model. Then T has prime models in all non-computable Δ^0_2 degrees.*

Proof. Deferred to the end of section 8. □

8 Omitting Partial Types

The following is the classical theorem on omitting types.

Theorem 8.1 (Omitting Types). *Let T be a countable consistent theory. If $\{ \Gamma_j(\overline{x}_j) \}_{j\in\omega}$ is a countable family of partial types, all nonprincipal with respect to T, then T has a model omitting all Γ_j.*

Millar gives the following theorem on effectively omitting *complete* types.

Theorem 8.2 (Millar [11], Effective Omitting Types). *Let T be a complete decidable theory, and let L be a computable listing of a subset Ψ of computable types of T. Then there is a* decidable *model $\mathfrak{A}$ which omits all nonprincipal types from Ψ.*

He then showed that you cannot effectively omit *partial* types.

Theorem 8.3 (Millar [11], Non-Omitting Partial Types). *There is a complete decidable theory T and a computable set Ψ of computable partial types of T such that* no decidable *model $\mathfrak{A}$ of T omits all nonprincipal types in Ψ.*

We can, however, omit partial types below any nonzero Δ^0_2 degree, as shown in the following theorem.

Theorem 8.4. *Let T be a complete decidable theory and let L be a computable set Γ of c.e. partial types of T. Then for any Δ^0_2 degree $\mathbf{d} > 0$, T has a $\mathbf{d}$-decidable model omitting the nonprincipal types listed by L.*

Proof. Let C be any Δ^0_2 set. We want to show that T has a model $\mathfrak{A}$ omitting the nonprincipal types listed in L such that $D^e(\mathfrak{A}) \leq_T C$.

We use Δ^0_2-permission. Let $\{C_s\}_{s\in\omega}$ be a computable approximation of C. For $s > 0$, let x_s and t_s be as in Section 6. We let $\mathcal{T}$ be the tree as in 2.3, and let $\mathcal{T}^s$ denote the tree up to length s.

Enumerate all finite subsets of $\{c_0, c_1, c_2, \ldots\}$ and let $\overline{k}$ denote the k^{th} finite subset.

We build $D^e(\mathfrak{A})$ by stages. At each stage, α_s will be a node on $\mathcal{T}$, so that $\{\alpha_s\}_{s\in\omega}$ will be a Δ^0_2 approximation to an infinite path on $\mathcal{T}$. That is, $\{\alpha_s\}_{s\in\omega}$ will approximate $D^e(\mathfrak{A})$ for some $\mathfrak{A}$. We use notation as described in Section 2, associating formulas θ with their indices. We must meet the requirements:

$R_{\langle e,k\rangle}$: If $e \in L$ and φ_e codes a non-principal type, then $\overline{k}$ does not realize φ_e in $\mathfrak{A}$.

To ensure $D^e(\mathfrak{A}) \leq_T C$, we ensure $\alpha_s \restriction x_s = \alpha_{t_s} \restriction x_s$, for t_s and x_s as in Section 6. At each stage, we define *restraint* functions $r(n,s)$, which preserve the initial segment of α_s used to meet requirement n. Note that, since we are doing Δ^0_2 permitting, at stage s we look at $r(n,t_s)$, not $r(n,s-1)$, as we only need to preserve what happened at the previous true stage.

We will say that σ *is not consistent with* $\overline{k}$ *realizing* φ_e *at stage* s if

$$(\exists \theta < lh(\sigma))[\sigma(\theta) = 0 \wedge \varphi_{e,s}(\theta^*_\forall(\overline{k}/\overline{x})) \downarrow= 1].$$

Note that once this holds for some σ and s, it holds for all $\tau \supseteq \sigma$ and all $t \geq s$. Say $\overline{k}$ is an n-tuple. We are checking if φ_e codes an n-type realizing $\overline{k}$. So we convert θ into $\theta^*_\forall(\overline{k}/\overline{x})$ which is in $F_n(T)$. We use $\forall$ because then no matter what assignment is made on the other variables outside $\overline{k}$ in θ, there is a dissagreement. *Construction:*

Stage 0: Let $r(n,0) = \emptyset$ for all n.

Stage s: Compute x_s, t_s, $\alpha_{t_s} \restriction x_s$, and $\mathcal{T}^s$. Find the least $\langle e,k\rangle < s$ such that

(1) $e \in L$ and α_{t_s} is consistent with $\overline{k}$ realizing φ_e. That is,

$$\neg(\exists \theta < lh(\alpha_{t_s}))[\alpha_{t_s}(\theta) = 0 \wedge \varphi_{e,s}(\theta^*_\forall(\overline{k}/\overline{x})) \downarrow= 1].$$

(2) There is a $\sigma \in \mathcal{T}^s$, $\sigma \supseteq \alpha_{t_s} \restriction x_s, \sigma \supseteq r(n,t_s)$ for $n < \langle e,k\rangle$ such that σ does not let $\overline{k}$ realize φ_e. That is,

$$(\exists \theta < lh(\sigma))[\sigma(\theta) = 0 \wedge \varphi_{e,s}(\theta^*_\forall(\overline{k}/\overline{x})) \downarrow= 1].$$

If such $\langle e,k\rangle$ and σ exist, set $r(\langle e,k\rangle, s) = \sigma$. Set $r(n,s) = r(n,t_s)$ for $n < \langle e,k\rangle$, $r(n,s) = \emptyset$ for $n > \langle e,k\rangle$. Let $\alpha_s = \sigma$. Note that α_s is not consistent with $\overline{k}$ realizing φ_e.

If no such $\langle e,k\rangle$ and σ exist, let $\alpha_s = \alpha_{t_s}$ and set $r(n,s) = r(n,t_s)$ for all n.

Lemma 6. $\lim_{s\to\infty} \alpha_s$ *converges to a path* $f \in [\mathcal{T}]$ *and* $f \leq_T C$.

Proof. Follows from $\alpha_s \restriction x_s = \alpha_{t_s} \restriction x_s$, and Lemma 4. □

Let $\mathfrak{A}$ be the model whose elementary diagram is obtained from f. Let $s_0 = 0 < s_1 < s_2 < s_3 < \cdots$ be the true stages of C, and so of $D^e(\mathfrak{A})$.

Lemma 7. $\lim_{q\to\infty} r(n, s_q) < \infty$ *for all* n.

Proof. Suppose the result holds for $m < n = \langle e, k\rangle$. Let q be such that for $m < n$, $\lim_{p\to\infty} r(m, s_p) = r(m, s_q)$. Now, by construction, $r(n, s_q) = r(n, t_{s_q}) = r(n, s_{q-1})$ *unless* requirement $n = \langle e, k\rangle$ received attention at stage s_q. By (2) of the construction, the action would be preserved by those of lower priority. □

Let $r(n) = \lim_{q\to\infty} r(n, s_q)$.

Lemma 8. *For all* $\langle e, k\rangle$, *if* φ_e *is a nonprincipal partial type listed in* L, *then* $\overline{k}$ *does not realize* φ_e *in* $\mathfrak{A}$.

Proof. Suppose for a contradiction that we have $\langle e, k\rangle$ such that φ_e is a nonprincipal partial type listed in L realized by $\overline{k}$ in $\mathfrak{A}$. We say θ is an s*-escape* if $\varphi_{e,s}(\theta^*_\forall(\overline{k}/\overline{x})) \downarrow= 1$ and $(\exists\tau)(lh(\tau) = \theta)$ such that τ is consistent with $\overline{k}$ realizing $\varphi_{e,s}$, and $\tau\widehat{\ }0 \in \mathcal{T}^s$. That is, an s-escape marks the first location where there is an extension not consistent with $\overline{k}$ realizing φ_e at stage s. Let $n_s > n_{s-1}$ be the least θ such that θ is an s-escape and for all other paths β of length θ on $\mathcal{T}$ consistent with $\overline{k}$ realizing $\varphi_{e,s}$, there is an s-escape $m_\beta \geq \theta$. If no such θ exists, set $n_s = n_{s-1}$.

Clearly $n_{s+1} \geq n_s$. Since φ_e is nonprincipal, for any β consistent with $\overline{k}$ realizing $\varphi_{e,s}$ there exists $t \geq s$, $m_\beta > lh(\beta)$ such that m_β is a t-escape. Indeed, if there is no escape, then all extensions of β are consistent with $\overline{k}$ realizing φ_e. So for all $\gamma \in \varphi_e$, $T \models (\exists\overline{x})(\theta^{\beta*}_\forall(\overline{x})) \wedge (\forall\overline{x})(\theta^{\beta*}_\forall(\overline{x}) \to \gamma(\overline{x}))$ contradicting φ_e non-principal. Given a fixed escape, there are finitely many ways the partial type can go, so for each of these finitely many possibilities, there will eventually be an escape. Hence the sequence $\{n_s\}$ is unbounded.

Now let $r = \max\{r(n) \mid n < \langle e, k\rangle\}$. Suppose $s_q > r$. Then, since s_q is a a true stage and $\overline{k}$ realizes φ_e in $\mathfrak{A}$, (1) holds for $\langle e, k\rangle$ at stage s_q, and (2) does not.

For n_{s_q}, look at the paths which were considered. As φ_e was realized, $D^e(\mathfrak{A})$ must have gone through one of them, say β. Since $s_q \geq r$, for no σ such that $m_\beta \in \sigma$ does $\sigma \supseteq \alpha_{t_{s_q}} \upharpoonright x_{s_q}$. So if $m_\beta > x_{s_q}$ then as s_q is a true stage this would contradict m_β being an s-escape. Thus $m_\beta \leq x_{s_q}$, and so $n_{s_q} \leq m_\beta \leq x_{s_q}$.

So $\{q \mid n_{s_q} > x_{s_q}\}$ is finite. So by Lemma 5, C is computable, a contradiction. □

□

We now prove Theorem 7.1.

Theorem 8.5 (Theorem 7.1). *Let T be a complete decidable theory with all types computable, and with a prime model but no computable prime model. Then T has prime models in all non-computable Δ^0_2 degrees.*

Proof. Proceed as in the proof of Theorem 8.4, but consider all e, not just $e \in L$. We are no longer guaranteed that φ_e codes a type, just that it is c.e. But at any point when φ_e does not behave as an n-type, we may consider requirement $R_{\langle e,k \rangle}$ met. □

References

[1] C.C. Chang and H.J. Keisler, *Model Theory*, 3rd edn., Stud. Logic Found. Math., vol. 73, North-Holland, Amsterdam, 1990 [1st edn. 1973, 2nd edn. 1977].

[2] P. Clote, A note on decidable model theory, Model Theory and Arithmetic, Springer Lecture Notes in Mathematics 890 (1981), 134-142.

[3] B. F. Csima, D. R. Hirschfeldt, J. F. Knight, and R. I. Soare, Bounding prime models, to appear.

[4] A. S. Denisov, Homogeneous $0'$-elements in structural pre-orders, Algebra and Logic, vol. 28 (1989), pp. 405–418.

[5] B. N. Drobotun, Numerations of simple models, Siberian Math. J. 18 (1977), no. 5, 707–716 (1978)

[6] V. S. Harizanov, Pure computable model theory, in *Handbook of Recursive Mathematics* (Yu. L. Ershov, S. S. Goncharov, A. Nerode, J. B. Remmel, eds.), Stud. Logic Found. Math., vol. 138–139, Elsevier Science, Amsterdam, 1998, pp. 3–114.

[7] C. G. Jockusch, Jr. and R. I. Soare, Π^0_1 Classes and degrees of theories, Trans. A.M.S., vol. 173, pp. 33–56.

[8] J. F. Knight, Degrees coded in jumps of orderings, J. Symbolic Logic, vol. 51 (1986), pp. 1034–1042.

[9] D. Marker, *Model Theory: An Introduction*, Springer-Verlag, New York, 2002.

[10] T. S. Millar, Foundations of recursive model theory, Ann. Math. Logic, vol. 13 (1978), 45–72.

[11] T.S. Millar, Omitting types, type spectrums, and decidability, J. Symbolic Logic, vol. 48 (1983), 171–181.

[12] R. G. Miller, The Δ_2^0-Spectrum of a Linear Order, The Journal of Symbolic Logic, vol. 66 (2001) 470-486.

[13] T. Slaman; Relative to any Nonrecursive Set, Proceedings of the American Mathematical Society, vol. 126 (1998), 2117-2122.

[14] R. I. Soare, *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*, Springer-Verlag, Heidelberg, 1987.

[15] S. Wehner; Enumerations, Countable Structures, and Turing Degrees, Proceedings of the American Mathematical Society, vol. 126 (1998), 2131-2139.

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