Bounding Homogeneous Models

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Abstract

A Turing degree $\mathbf{d}$ is *homogeneous bounding* if every complete decidable (CD) theory has a $\mathbf{d}$-decidable homogeneous model $\mathcal{A}$, i.e., the elementary diagram $D^e(\mathcal{A})$ has degree $\mathbf{d}$. It follows from results of Macintyre and Marker that every PA degree (i.e., every degree of a complete extension of Peano Arithmetic) is homogeneous bounding. We prove that in fact a degree is homogeneous bounding *if and only if* it is a PA degree. We do this by showing that there is a single CD theory T such that every homogeneous model of T has a PA degree.

1 Introduction

One of the aims of computable mathematics is to use the tools provided by computability theory to calibrate the strength of theorems and constructions of ordinary mathematics. It is often the case that a given theorem has multiple proofs, all of which seem to require a particular combinatorial principle that can be characterized in degree theoretic terms. It is then natural to attempt to use the methods of computable mathematics to show that the use of this principle (or an effectively equivalent one) is unavoidable. The particular theorem we analyze in this manner is the classical model theoretic result that every countable complete theory

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has a countable homogeneous model. (See Sections 2, 3, and 4 for all definitions. As usual in computable mathematics, we consider only countable languages and structures.) Before describing our results, we set them in context by considering some related results in computable model theory.

It is easy to check that the usual Henkin proof of the completeness theorem given for instance in Marker [25] can be effectivized, to show every complete decidable (CD) theory has a decidable model. However, this construction does not always produce a model that is prime, saturated, or homogeneous. To construct such models we need to control the types realized in the model as in Vaught's epochal 1961 paper [39] on models of complete theories.

Goncharov and Nurtazin [8] and Millar [26] showed that there is a complete atomic decidable (CAD) theory with no decidable (or even computable) prime model, and that such a theory can be chosen with types all computable (TAC), but not uniformly so, because Morley [29] and Millar [26] proved that a CD theory whose types are uniformly computable has a decidable saturated model and hence a decidable prime model. Similarly, Millar [26] constructed a CD theory T with TAC such that T does not have a computable saturated model.

These results raise the question of determining the exact computational complexity of the constructions of prime and saturated models. Goncharov and Nurtazin [8] and also Millar [26] noted that the obvious effectivization of Vaught's construction [39] of a prime model $\mathcal{A}$ of a complete atomic theory T demonstrates that if the theory is also decidable, then $\mathcal{A}$ can be constructed to be $\mathbf{0}'$-decidable. Csima [3] strengthened this result by showing that any CAD theory has a prime model whose elementary diagram has low degree $\mathbf{d}$ (i.e., $\mathbf{d}' = \mathbf{0}'$). (By *degree*, we always mean Turing degree.)

A complementary result can be obtained by considering the *prime bounding* degrees, which are those degrees $\mathbf{d}$ such that every CAD theory has a $\mathbf{d}$-decidable prime model. Csima, Hirschfeldt, Knight, and Soare [4] showed that a Δ^0_2 degree $\mathbf{d}$ is prime bounding if and only if it is nonlow$_2$ (i.e., $\mathbf{d}'' > \mathbf{0}''$). (They also noted that there are low$_2$ prime bounding degrees that are not Δ^0_2, but the exact picture outside the Δ^0_2 degrees is not yet clear.)

In the TAC case the situation is simpler. Hirschfeldt [13] showed that a CAD theory with TAC must have a $\mathbf{d}$-decidable prime model for any $\mathbf{d} > \mathbf{0}$. (This result had been proved earlier for Δ^0_2 degrees $\mathbf{d}$ by Csima [3].)

A degree $\mathbf{d}$ is *saturated bounding* if every CD theory with TAC has a $\mathbf{d}$-decidable saturated model. (The restriction on the complexity of types is essential here, because a saturated model must realize all the types of its theory. Hence, for instance, the existence of nonarithmetical types would force the saturated model to be nonarithmetical.) Macintyre and Marker [23] proved that every PA degree is saturated bounding (see Section 4). Harris [12] proved that this result also follows from work of Jockusch [14], as does the fact that every high degree is saturated bounding. In the other direction, Harris [12] has shown that no c.e. degree $\mathbf{d}$ that is low or even low$_n$ (i.e., $\mathbf{d}^{(\mathbf{n})} = \mathbf{0}^{(\mathbf{n})}$) can be saturated bounding.

Turning finally to homogeneous models, we again have a non-effectivity result. Goncharov [7] showed that there is a CD ω-stable theory with no computable homogeneous model. (A theory T is *ω-stable* if for every $\mathcal{M} \vDash T$ and every countable $X \subseteq M$, there are only countably many types of T over X.) The ω-stability of this theory is particularly interesting since uncountably categorical theories are notable examples of ω-stable theories, but as shown by Harrington [11] and Khisamiev [17], if T is a decidable uncountably categorical theory, then every countable model of T has a decidable copy.

One difference between the homogeneous case and the prime and saturated cases is that there can be only one prime model of a given theory up to isomorphism, and similarly for countable saturated models, but this is not the case for homogeneous models. Indeed, every type of a countable complete theory is realized in some countable homogeneous model, so there is in general no way to bound the complexity of all homogeneous models of a CD theory (since there are such theories with continuum many types). There are several results focusing on the possible degrees of copies of a *given* homogeneous structure $\mathcal{A}$, presented via its *type spectrum*, $\mathbb{T}(\mathcal{A})$, the set of types realized in $\mathcal{A}$, as discussed in Section 3.

By analogy with the prime and saturated cases, we call a degree $\mathbf{d}$ *homogeneous bounding* if every CD theory has a $\mathbf{d}$-decidable homogeneous model. Thus, Goncharov's result in [7] means that $\mathbf{0}$ is not homogeneous bounding. On the other hand, as shown by Macintyre and Marker [23], every PA degree *is* homogeneous bounding, as we now discuss.

One way to build a homogeneous model of a given theory is via an elementary chain, or a similar iterated extension argument (see [2] or [25]). It is not hard to check that such arguments can be made effective, except for the repeated use of Lindenbaum's Lemma (which states that every consistent set of sentences can be extended to a complete theory), or, equivalently, the use of the completeness theorem for consistent sets of sentences (rather than complete theories). It is well-known that Lindenbaum's Lemma can be carried out effectively in a degree $\mathbf{d}$ (in the sense that every consistent computable set of sentences can be extended to a complete decidable theory) if and only if $\mathbf{d}$ is a PA degree. This is because Lindenbaum's Lemma is easily seen to be equivalent (degree theoretically but also in the sense of reverse mathematics) to Weak König's Lemma (which states that every infinite binary tree has an infinite path). As we explain in Section 4, $\mathbf{d}$ is a PA degree if and only if every computable infinite binary tree has a $\mathbf{d}$-computable path.

Another way to build homogeneous models is by using Scott sets, introduced by Scott [32] to characterize the sets definable in a model of Peano Arithmetic. We will discuss these in Section 4. Such constructions can be found in Macintyre and Marker [23] (who deal with models that are saturated with respect to types coded in a given Scott set) and Ash and Knight [1]. To highlight the interplay between homogeneous models and models of arithmetic, we also include a version in Section 4.

Thus, we see that every PA degree is homogeneous bounding. It also appears that the use of PA degrees (and hence of combinatorial principles effectively equivalent to Weak König's Lemma), is essential to the building of homogeneous models, but we cannot a priori rule out the possibility that a more clever construction might allow us to sidestep the use of Weak König's Lemma. That no such construction exists is part of the import of our main result, which will be proved in Section 5.

Theorem 1.1. *There is a complete decidable theory T such that every countable homogeneous model of T has a PA degree.*

This theorem implies that every homogeneous bounding degree is a PA degree, but it is in fact stronger, since we build a *single* theory T such that the use of PA degrees is necessary to compute even the atomic diagram of a homogeneous model of T. Together with the converse fact mentioned above, we have the following consequence.

Corollary 1.2. *A degree is homogeneous bounding if and only if it is a PA degree.*

In light of Goncharov's result on ω-stable theories mentioned above, it is worth pointing out that the theory T in Theorem 1.1 cannot be made ω-stable, or even atomic (which would be implied by ω-stability). This fact follows from the result mentioned above that every nonlow$_2$ and Δ^0_2 degree is prime bounding. If T is a CAD theory, then let $\mathbf{d}$ be a nonlow$_2$ degree that is not PA (for example, a nonlow$_2$ incomplete c.e. degree). Any $\mathbf{d}$-decidable prime model of T is an example of a homogeneous model of T that does not have a PA degree.

2 Definitions, Notation, and Basic Results

For the most part, we include in this section only the definitions, notation, and basic results that are essential for this paper and that may not be familiar to all readers. For other definitions and as general references we cite [2] and [25] for model theory, [36] and more recently [37] for the computability theory used in computable model theory, and [1, 5, 9, 10, 28] for computable model theory.

We use Φ_e to denote the eth Turing functional. A degree $\mathbf{d}$ is *low* if its first jump is the same as that of the least degree $\mathbf{0}$, that is, $\mathbf{d}' = \mathbf{0}'$. Similarly, $\mathbf{d}$ is *low*$_2$ if $\mathbf{d}'' = \mathbf{0}''$.

A *computable language* is a countable language with an effective presentation of its set of symbols, along with their arities. We consider only countable structures for computable languages. We denote such structures by calligraphic letters such as $\mathcal{A}$, and their universes by the corresponding roman letters such as A. The universe of an infinite countable structure can be identified with ω. As usual, if L is the language of $\mathcal{A}$ and $X \subseteq A$, then L_X is the expansion of L obtained by adding a new constant symbol for each $a \in X$, and $\mathcal{A}_X = (\mathcal{A}, a)_{a \in X}$ is the corresponding expansion of $\mathcal{A}$ to L_X.

A *theory* T in L is a consistent set of sentences in $L = L(T)$ closed under logical consequence. Let $F_n(T)$ denote the set of formulas $\theta(\overline{x})$ of $L(T)$ with n free variables so that $(\exists \overline{x})\theta(\overline{x})$ is consistent with T, and therefore $(\exists \overline{x})\theta(\overline{x}) \in T$ if T is complete.

A *complete type* (or briefly, a *type*) is a maximal consistent set of formulas in a certain fixed number of free variables. For each n let $S_n(T)$ denote the set of n-types of T and let $S(T) = \bigcup_n S_n(T)$. These are sometimes called *pure* types to distinguish them from types defined with parameters $Y \subseteq A$ for some model $\mathcal{A}$ of T. A type $p \in S(T)$ is *realized* in a model $\mathcal{A}$ of T if there exists an $\overline{a} \in A^{<\omega}$ such that $\mathcal{A} \vDash \theta(\overline{a})$ for every $\theta(\overline{x}) \in p$.

We write $\mathcal{A} \equiv \mathcal{B}$ to mean that the structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent, that is, they have the same theory.

We identify a formula with its Gödel number, and say that a set of formulas belongs to a given computability theoretic complexity class $\mathcal{P}$ if the set of Gödel numbers of its elements belongs to $\mathcal{P}$. A theory is *decidable* if it is computable in this sense. It is clear that a computably axiomatizable complete theory is decidable.

The *atomic* (or *open*) *diagram* of a structure $\mathcal{A}$, denoted by $D(\mathcal{A})$, is the set of all atomic sentences and negations of atomic sentences of L_A true in $\mathcal{A}_A$. The *elementary diagram* of $\mathcal{A}$, denoted by $D^e(\mathcal{A})$, is the set of all sentences of L_A true in $\mathcal{A}_A$. A structure $\mathcal{A}$ is *computable* if $D(\mathcal{A})$ is computable, and *decidable* if $D^e(\mathcal{A})$ is computable. More generally, for a degree $\mathbf{d}$, a structure $\mathcal{A}$ is $\mathbf{d}$-*computable* if $D(\mathcal{A})$ is $\mathbf{d}$-computable, and $\mathbf{d}$-*decidable* if $D^e(\mathcal{A})$ is $\mathbf{d}$-computable. It is conventional to define the *degree* of a structure $\mathcal{A}$ to be the Turing degree of the *atomic* diagram $D(\mathcal{A})$, not the elementary diagram $D^e(\mathcal{A})$.

A structure $\mathcal{A}$ is *automorphically trivial* if there is a finite $F \subset A$ such that every permutation of A fixing F pointwise is an automorphism of $\mathcal{A}$. Automorphically trivial structures are rather uninteresting from the point of view of computable model theory, since for each such structure $\mathcal{A}$ there is a degree $\mathbf{d}$ such that every copy of $\mathcal{A}$ (with universe ω) has degree $\mathbf{d}$ (and if the language is finite, then $\mathbf{d} = \mathbf{0}$). Knight [18] showed that if $\mathcal{A}$ is not automorphically trivial and has a $\mathbf{d}$-computable copy, then $\mathcal{A}$ has a copy whose degree is exactly $\mathbf{d}$. (This result was proved earlier for models of Peano Arithmetic by Solovay and Marker [24].) The same proof yields the analogous result for $D^e(\mathcal{A})$ in place of $D(\mathcal{A})$.

An important tool for proving completeness and decidability of a theory is quantifier elimination. A theory T has *quantifier elimination* if for every formula $\varphi(\overline{x})$ there is a quantifier-free formula $\psi(\overline{x})$ such that $T \vdash \varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$. If T has no constant symbols, then this definition needs to be slightly amended, since L has no quantifier-free sentences; in this case, for a sentence φ, we only require the existence of a quantifier-free formula $\psi(x)$ of one free variable such that $T \vdash \varphi \leftrightarrow \psi(x)$. We will use the following well-known criterion for a theory having quantifier-elimination.

Theorem 2.1 (See [25, Corollary 3.1.6]). *Suppose that for all $\mathcal{A}, \mathcal{B} \models T$ and all common substructures $\mathcal{C}$ of $\mathcal{A}$ and $\mathcal{B}$, the structures $\mathcal{A}$ and $\mathcal{B}$ satisfy the same existential sentences with parameters from C. Then T has quantifier elimination.*

A theory satisfying the hypothesis of Theorem 2.1 is called *submodel complete*.

3 Homogeneous Structures

Definition 3.1. A countable structure $\mathcal{A}$ is *homogeneous* if it satisfies the following property. Let $\overline{a} = (a_0, \ldots, a_{n-1})$ and $\overline{b} = (b_0, \ldots, b_{n-1})$ be finite sequences of elements of A such that

$$(\mathcal{A}, a_0, \ldots, a_{n-1}) \equiv (\mathcal{A}, b_0, \ldots, b_{n-1}) \qquad (*)$$

(i.e., $\overline{a}$ and $\overline{b}$ realize the same n-type). Then

$$(\forall a_n \in A)(\exists b_n \in A)[\ (\mathcal{A}, a_0, \ldots, a_{n-1}, a_n) \ \equiv \ (\mathcal{A}, b_0, \ldots, b_{n-1}, b_n)\ ].$$

It is not hard to show that $\mathcal{A}$ is homogeneous if and only if for any two sequences $\overline{a}$ and $\overline{b}$ satisfying $(*)$ there is an automorphism of $\mathcal{A}$ taking $\overline{a}$ to $\overline{b}$. As mentioned above, every countable complete theory has a countable homogeneous model. Prime models and countable saturated models are examples of homogeneous models.

3.1 Type Spectra and Uniqueness of Homogeneous Models

Definition 3.2. Let T be a theory and let $\mathcal{A}$ be a model of T. The *type spectrum* $\mathbb{T}(\mathcal{A})$ of $\mathcal{A}$ is the set of all (pure) types of T realized in $\mathcal{A}$. That is,

$$\mathbb{T}(\mathcal{A}) = \{\ p\ :\ p \in S(T)\ \wedge\ \mathcal{A} \text{ realizes } p\ \}.$$

We write $\mathbb{T}_n(\mathcal{A})$ for $\mathbb{T}(\mathcal{A}) \cap S_n(T)$, the set of n-types of T realized in $\mathcal{A}$.

(Some authors in computable model theory use $S(\mathcal{A})$ in place of $\mathbb{T}(\mathcal{A})$, but this conflicts with the standard usage in ordinary model theory. Given a structure $\mathcal{A}$ and a set $Y \subseteq A$, Marker [25, p. 115] defines $S_n^{\mathcal{A}}(Y)$ to be the set of n-types in the theory of $\mathcal{A}_Y$. Our use of $\mathbb{T}(\mathcal{A})$ differs from Marker's for several reasons: (1) we consider only pure types in the original language and do not allow any extra constant symbols to be added; (2) we consider only those types actually *realized* in $\mathcal{A}$, not merely those consistent with the theory of $\mathcal{A}_Y$; and hence (3) $\mathbb{T}_n(\mathcal{A})$ is not necessarily closed, even though $S_n^{\mathcal{A}}(Y)$ is always closed in the usual Cantor set topology on $2^{<\omega}$. Marker has no notation for our $\mathbb{T}(\mathcal{A})$.)

The most pleasing and useful property of homogeneous models and of the set of pure types $\mathbb{T}(\mathcal{A})$ is the following Uniqueness Theorem for Homogeneous Models.

Theorem 3.3 (Morley and Keisler [30]). *Given a countable complete theory T and homogeneous models $\mathcal{A}$ and $\mathcal{B}$ of T of the same cardinality,*

$$\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B}) \implies \mathcal{A} \cong \mathcal{B}.$$

See Marker [25, Theorem 4.3.23] for a proof of the general case, and [25, Theorem 4.2.15] for the countable case, which is all we need in this paper.

Remark 3.4. If $\mathcal{A}$ is a countable homogeneous model of a theory T then

(i) $\mathcal{A}$ is prime iff $\mathbb{T}(\mathcal{A}) = S^P(T)$, the set of principal types of $S(T)$; and

(ii) by the Uniqueness Theorem 3.3, $\mathcal{A}$ is saturated iff it is *weakly saturated* (i.e., $\mathbb{T}(A) = S(T)$).

3.2 The Five Homogeneity Conditions on a Type Spectrum

If T is a complete theory and $\mathcal{C} \subseteq S(T)$ is a given set of types, what conditions on $\mathcal{C}$ must be satisfied so that there is a homogeneous model $\mathcal{A}$ of T with $\mathbb{T}(\mathcal{A}) = \mathcal{C}$? Goncharov [6] and Peretyat'kin [31] (see also [5]) gave a set of five conditions that completely answers this question.

Theorem 3.5 (Goncharov [6], Peretyat'kin [31]). *Let T be a complete theory and $\mathcal{C} \subseteq S(T)$ be a countable set satisfying conditions 1–5 below. Then T has a countable homogeneous model $\mathcal{A}$ with $\mathbb{T}(\mathcal{A}) = \mathcal{C}$.*

1. ***$\mathcal{C}$ is closed under permutations of variables.***
If $p(x_0, \ldots, x_{n-1}) \in \mathcal{C}$ is an n-type and σ is a permutation of $\{0, \ldots, n-1\}$, then $p(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}) \in \mathcal{C}$.

2. ***$\mathcal{C}$ is closed under forming subtypes.***
If $p(x_0, \ldots, x_{n-1}) \in \mathcal{C}$ and $m < n$, then $p(x_0, \ldots, x_{n-1}) \restriction \{x_0, \ldots, x_{m-1}\}$ (the restriction of p to formulas in which only $x_0, \ldots, x_{m-1}$ appear free) is also in $\mathcal{C}$.

3. ***$\mathcal{C}$ is closed under unions of types on disjoint sets of variables.***
If $p = p(x_0, \ldots, x_{n-1})$ and $q = q(x_0, \ldots, x_{n-1})$ are such that $p, q \in \mathcal{C}$, then there is an $r = r(x_0, ..., x_{2n-1}) \supseteq p(x_0, \ldots, x_{n-1}) \cup q(x_n, \ldots, x_{2n-1})$ such that $r \in \mathcal{C}$.

4. ***$\mathcal{C}$ is closed under amalgamation of types.***
If $p(x_0, \ldots, x_{n-1}, x_n)$ and $q(x_0, \ldots, x_{n-1}, x_n)$ are such that $p, q \in \mathcal{C}$, and p and q both extend the same n-type $s(x_0, \ldots, x_{n-1})$, then there is an $r = r(x_0, \ldots, x_{n-1}, x_n, x_{n+1}) \supseteq p(x_0, \ldots, x_{n-1}, x_n) \cup q(x_0, \ldots, x_{n-1}, x_{n+1})$ such that $r \in \mathcal{C}$.

5. ***$\mathcal{C}$ is closed under amalgamation of types with formulas.***
If $p = p(x_0, \ldots, x_{n-1}) \in \mathcal{C}$ and the formula $\theta(x_0, \ldots, x_{n-1}, x)$ is consistent with p, then there is a $q(x_0, \ldots, x_{n-1}, x) \in \mathcal{C}$ such that $p \cup \{\theta\} \subseteq q$.

3.3 Type Spectra of Decidable Homogeneous Structures

Morley [29] asked under what conditions a countable homogeneous model has a decidable homogeneous (isomorphic) copy. Morley [29] observed the following obvious implications.

Remark 3.6. If a countable homogeneous model $\mathcal{B}$ has a decidable (isomorphic) copy $\mathcal{A}$ (and hence $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$), then:

(i) Every element of $\mathbb{T}(\mathcal{A})$ is computable.

(ii) There is a uniformly computable listing of all the types in $\mathbb{T}(\mathcal{A})$.

Definition 3.7. Let $\mathcal{A}$ be a homogeneous model such that every element of $\mathbb{T}(\mathcal{A})$ is computable and $\mathbf{d}$ be a degree. A $\mathbf{d}$-*basis* for $\mathcal{A}$ is a listing $\{p_i\}_{i\in\omega}$ of $\mathbb{T}(\mathcal{A})$, possibly with repetitions, along with a $\mathbf{d}$-computable function g such that $g(i)$ is a Δ^0_0 index for p_i.

We view g as defining a matrix whose rows are the types p_i. It is not sufficient to have a g that merely uniformly $\mathbf{d}$-computably specifies the rows (i.e., $g(i, x) = p_i(x)$), all of which are computable, but rather $g(i)$ must specify an index such that $\Phi_{g(i)} = p_i$, from which a nonoracle Turing machine can compute p_i. Now Remark 3.6 asserts that if $\mathcal{B}$ has a decidable copy then there must exist *some* listing of $\mathbb{T}(\mathcal{B})$ that forms a $\mathbf{0}$-basis, although not every listing of $\mathbb{T}(\mathcal{B})$ need have this property.

Morley's question [29] was whether every countable homogeneous structure $\mathcal{B}$ for which $\mathbb{T}(\mathcal{B})$ has a $\mathbf{0}$-basis must have a decidable copy. This question was answered negatively by Millar [27], Peretyat'kin [31], and Goncharov [6], who gave examples of homogeneous structures with uniformly computable type spectra (i.e., $\mathbf{0}$-bases) but no computable copy. Lange [20] recently extended this result by building, for each low$_2$ and Δ^0_2 degree $\mathbf{d}$, a homogeneous structure with uniformly computable type spectrum (i.e., a $\mathbf{0}$-basis) but no $\mathbf{d}$-computable copy. Lange [19] showed that her result is the best possible one of this kind for Δ^0_2 degrees, by proving that for every nonlow$_2$ and Δ^0_2 degree $\mathbf{d}$ and every homogeneous structure $\mathcal{A}$ with uniformly computable type spectrum, $\mathcal{A}$ has a $\mathbf{d}$-decidable copy. Lange [19] also showed that every homogeneous structure with a $\mathbf{0}'$-basis has a copy whose elementary diagram has low degree. This result implies the theorem of Csima [3] mentioned above that every CAD theory has a low prime model. For further discussion of these results, see [21, 22].

3.4 The Effective Extension Property

Morley's question does have a positive answer if we require not only that $\mathbb{T}(\mathcal{B})$ have a $\mathbf{0}$-basis but in addition that the basis have an "effective extension property" (EEP), which is the effective analogue of property 5 of Theorem 3.5. This EEP condition allows us to amalgamate an n-type p and an $(n+1)$-ary formula consistent with p to effectively obtain an $(n+1)$-type extending both.

Definition 3.8. For an n-type $p(\overline{x}) \in S_n(T)$ and an $(n+1)$-ary formula $\theta(\overline{x}, y) \in F_{n+1}(T)$, we say θ is *consistent with* p if $(\exists y)\theta(\overline{x}, y) \in p(\overline{x})$.

Goncharov [6] and Peretyat'kin [31] gave the following key criterion for the existence of a decidable copy of a homogeneous structure. Fix a language L and let $\{\theta_j\}_{j\in\omega}$ be an effective list of all formulas in L, namely of $\bigcup_n F_n(L)$. Let $\overline{x}$ denote $(x_0, \ldots, x_{n-1})$.

Theorem 3.9 (EEP Theorem; Goncharov [6], Peretyat'kin [31])**.** *Let $\mathcal{A}$ be a countable homogeneous structure for L. Then $\mathcal{A}$ has a decidable copy if and only if there exist*

1. *a* $\mathbf{0}$*-basis* $\{p_i\}_{i\in\omega}$ *for* $\mathbb{T}(\mathcal{A})$*, and*
2. *a computable binary function f such that for all n,*
   - *for every n-type $p_i(\overline{x}) \in \mathbb{T}_n(\mathcal{A})$ and*
   - *for every $(n+1)$-ary formula $\theta_j(\overline{x}, x_n) \in F_{n+1}(T)$ consistent with $p_i(\overline{x})$,*

   f gives us an $(n+1)$-type $p_{f(i,j)} \in \mathbb{T}_{n+1}(\mathcal{A})$ extending both, that is,

$$p_i(\overline{x}) \cup \{\theta_j(\overline{x}, x_n)\} \subseteq p_{f(i,j)}(\overline{x}, x_n).$$

If so, we say the $\mathbf{0}$*-basis* $\{p_i\}_{i\in\omega}$ *for* $\mathbb{T}(\mathcal{A})$ *has the* effective extension property (EEP).

The proof of the EEP Theorem 3.9 uses the Uniqueness Theorem 3.3 for Homogeneous Models. The EEP Theorem can be relativized in the usual way to any degree $\mathbf{d}$. For example, in Lange's low basis theorem [19] mentioned above we are given a homogeneous model $\mathcal{A}$ of a CD theory T and a $\mathbf{0}'$-basis for $\mathbb{T}(\mathcal{A})$. Lange builds a new basis Y for $\mathbb{T}(\mathcal{A})$ and simultaneously an extension function f for Y such that for some low degree $\mathbf{d}$: (1) Y is a $\mathbf{d}$-basis for $\mathbb{T}(\mathcal{A})$; (2) f is $\mathbf{d}$-computable; and (3) f is an extension function for Y as defined in the EEP Theorem. Now using f and Y, the relativization of the EEP Theorem to $\mathbf{d}$ gives us a $\mathbf{d}$-decidable copy of $\mathcal{A}$. Note that in this case the types in Y are still computable, so we are relativizing the presentation of the members of Y and the effective extension function, but not the rows themselves. However, it should be noted that in general it is not possible to obtain such a basis given a $\mathbf{d}$-decidable homogeneous structure $\mathcal{B}$; in this case, the EEP theorem guarantees only the existence of a listing $\{p_i\}_{i\in\omega}$ of $\mathbb{T}(\mathcal{B})$ along with a $\mathbf{d}$-computable function g such that $g(i)$ is a $\Delta_0^{\mathbf{d}}$ index for p_i.

4 Models of Peano Arithmetic and Scott Sets

In this section we develop properties of the PA degrees and relate them to Scott sets. We also give a proof, similar to the one in Macintyre and Marker [23], that every PA degree is homogeneous bounding.

4.1 Trees and Π^0_1-Classes

A *tree* is a subset of $2^{<\omega}$ that is closed under initial segments. A *path* P of a tree $\mathcal{T}$ is a maximal set of nodes of $\mathcal{T}$ that is linearly ordered by the containment relation. Let $[\mathcal{T}]$ denote the set of all infinite paths of $\mathcal{T}$. Let $\{\tau_n\}_{n\in\omega}$ be a fixed computable 1-1 numbering of $2^{<\omega}$. We identify each τ_n with n and in this way consider trees and paths as subsets of ω.

Definition 4.1. (i) A class $\mathcal{C} \subseteq 2^\omega$ is a Π^0_1-*class* if there is a computable tree $\mathcal{T} \subseteq 2^{<\omega}$ such that $\mathcal{C}$ is the set of infinite paths through $\mathcal{T}$, that is,

$$\mathcal{C} = \{\, f \,:\, (\forall n)\,[\, f \restriction n \;\in\; \mathcal{T}\,]\,\}.$$

(ii) A class $\mathcal{B} \subseteq 2^\omega$ is a *basis* for Π^0_1 if for every nonempty Π^0_1-class $\mathcal{C}$ there exists a $B \in \mathcal{B} \cap \mathcal{C}$.

For example the Low Basis Theorem [15] proves that the sets of low degree form a basis for Π^0_1.

4.2 Characterizations of the PA Degrees

The structure $\mathcal{N} = (\omega, +, \cdot, S, 0)$ is computable but has no decidable copy. *Peano Arithmetic (PA)* is the effectively axiomatizable (but not decidable) first order theory containing basic axioms about the arithmetic operations plus an infinite list of axioms for induction. A *standard* model of arithmetic is a structure isomorphic to $\mathcal{N}$, and a model of PA is *nonstandard* otherwise. Tennenbaum [38] showed that there is no computable nonstandard model of PA.

As mentioned above, a Turing degree $\mathbf{d}$ is a *PA degree* if it is the degree of a complete extension of PA. The following well-known equivalent characterizations of PA degrees were proved by Scott [32], Shoenfield [33], and Jockusch and Soare [15, 16]. For a discussion see Simpson [34].

Theorem 4.2. *The following are equivalent for any Turing degree* $\mathbf{d}$.

(i) *The* $\mathbf{d}$*-computable sets form a basis for* Π^0_1. *(That is, every infinite computable binary tree has an infinite* $\mathbf{d}$*-computable path.)*

(ii) *If* U *and* V *are disjoint c.e. sets, then there is a* $\mathbf{d}$*-computable set* S *such that* $U \subseteq S$ *and* $V \cap S = \emptyset$. *(Such a set* S *is called a* separating set *for* U *and* V*.)*

(iii) *The degree* $\mathbf{d}$ *is the degree of a complete extension of PA.*

(iv) *The degree* $\mathbf{d}$ *is the degree of the elementary diagram* $D^e(\mathcal{A})$ *of a nonstandard model* $\mathcal{A}$ *of PA.*

(v) *The degree* $\mathbf{d}$ *is the degree of the atomic diagram* $D(\mathcal{A})$ *of a nonstandard model* $\mathcal{A}$ *of PA.*

Note that by (iii) and the Low Basis Theorem [15], there are low PA degrees.

We briefly discuss the relationship between these properties. Note that the classes in (ii) and (iii) are clearly Π^0_1, so (i) implies (ii) and (iii). As mentioned above, Solovay and Marker [24] showed that the degrees in (v) are closed upwards; this result was generalized by Knight [18], and holds for the degrees in (iv) as well. It is also easy to show that the degrees in (iii) are closed upwards. From these facts it follows that (iv) implies (iii) and (v). Furthermore, we can Henkinize a theory without changing its degree. Hence, we can show that the degrees of complete extensions of PA correspond to the degrees of elementary diagrams of nonstandard models of PA, so (iii) implies (iv).

Clearly, (ii) implies (iii) because if U is the set of Gödel numbers of sentences provable from PA and V is the set of Gödel numbers of sentences refutable from PA, then U and V are disjoint c.e. sets whose separating sets correspond to completions of PA. These sets U and V will play a role in our proof of Theorem 1.1 where we construct a CD theory T such that any homogeneous model of T can compute a separating set for U and V, i.e., can compute a path of the tree of separating sets.

The power of (v) (which is not directly used in this paper) is revealed in Definition 4.4 below, with the use of a nonstandard number to code an infinite set by divisibility with respect to a certain set of primes.

The main remaining implication of Theorem 4.2 is the direction (iii) implies (i). Let T be a complete extension of PA and $\mathcal{T} \subseteq 2^\omega$ an infinite computable tree. A node $\tau \in \mathcal{T}$ is *extendible* (on $\mathcal{T}$) if there are infinitely many nodes on $\mathcal{T}$ extending τ, or equivalently by compactness if there is some infinite path $f \in [\mathcal{T}]$ that extends τ. It suffices to show that if T has computed an extendible node $\tau \in \mathcal{T}$ then T can choose an extendible immediate extension $\tau' = \tau\frown i$ for $i = 0$ or 1.

If node $\tau_n \in 2^{<\omega}$ is *not* extendible then for some k there are no nodes on $\mathcal{T}$ of length k that extend τ_n. Let $N(\mathbf{n}, \mathbf{k})$ be the quantifier-free formal statement in the language of PA that asserts that node τ_n has no extensions of $\mathcal{T}$ of length k, where $\mathbf{n}$ and $\mathbf{k}$ are the formal numerals corresponding to integers n and k. If this statement is true in the standard model $\mathcal{N}$, then PA (and hence T) proves $N(\mathbf{n}, \mathbf{k})$ and hence $(\exists x)N(\mathbf{n}, x)$. The main difficulty is that T can add extra sentences of the form $(\exists x)N(\mathbf{n}, x)$ that are not in PA and hence lie to us about the nonextendibility of τ_n. To overcome this problem we ask T a Rosser style question,

$$\theta : \quad (\forall x)[N(\mathbf{n_0}, x + 1) \implies N(\mathbf{n_1}, x)],$$

where n_i is the index of $\tau_n\frown i$ for $i = 0$ or 1. This sentence θ asserts roughly that for any string extending $\tau_n\frown 1$ there is a longer one extending $\tau_n\frown 0$.

If $\tau_n{}^\frown 0$ is nonextendible, then for some k the quantifier-free sentence $N(\mathbf{n_0}, \mathbf{k})$ is true in $\mathcal{N}$ and provable in PA, and hence is in T. But τ_n is extendible, so in this case $\tau_n{}^\frown 1$ is extendible, and hence $\theta \notin T$

On the other hand, if $\tau_n{}^\frown 1$ is nonextendible, then for some k the sentence $N(\mathbf{n_1}, \mathbf{k})$ is provable in PA, and hence is in T. But in this case $\tau_n{}^\frown 0$ is extendible, so for each $j \leqslant k+1$, the sentence $\neg N(\mathbf{n_0}, \mathbf{j})$ is provable in PA, and hence is in T. From this it follows immediately that $\theta \in T$.

Thus, if $\theta \in T$ then $\tau_n{}^\frown 0$ is extendible, so we can define $\tau_n' = \tau_n{}^\frown 0$. Similarly, if $\theta \notin T$, then $\tau_n{}^\frown 1$ is extendible, so we can define $\tau_n' = \tau_n{}^\frown 1$.

4.3 Scott Sets

Scott [32] introduced Scott sets to classify the collection of sets definable in a model of Peano Arithmetic. They also play an important role in reverse mathematics, since they are the ω-models of WKL_0, the subsystem of second order arithmetic consisting of the base system RCA_0 plus Weak König's Lemma. (See [35] for the relevant definitions.)

Definition 4.3. A nonempty set $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a *Scott set* if the following conditions are satisfied for all $A, B \subseteq \omega$:

(i) $[\, A \in \mathcal{S} \wedge B \in \mathcal{S} \,] \implies A \oplus B \in \mathcal{S}$,

(ii) $[\, A \in \mathcal{S} \wedge B \leqslant_{\mathrm{T}} A \,] \implies B \in \mathcal{S}$, and

(iii) $[\, T \in \mathcal{S}$ is an infinite tree $] \implies (\exists P \in \mathcal{S})[\, P \in [\mathcal{T}] \,]$.

The crucial condition (iii) asserts that $\mathcal{S}$ is a basis for the collection of Π^0_1-classes computable in any member $A \in \mathcal{S}$. Another way to look at (iii) is that any consistent set of axioms in $\mathcal{S}$ has a completion in $\mathcal{S}$. (This fact follows from the effective equivalence between Lindenbaum's Lemma and Weak König's Lemma discussed above.)

The collection of arithmetical sets is a Scott set, but there are many others. One way to obtain Scott sets is to consider nonstandard models of Peano Arithmetic.

Definition 4.4. Let $\mathcal{A}$ be a nonstandard model of PA. The *Scott set of* $\mathcal{A}$ (also called the *standard system of* $\mathcal{A}$), denoted by $\mathcal{SS}(\mathcal{A})$, is the collection of all sets of the form

$$\{n \in \omega : \mathcal{A}_A \models \text{“}a \text{ is divisible by the } n\text{th prime number”}\}$$

for $a \in A$.

It can be verified (see [1]) that $\mathcal{SS}(\mathcal{A})$ is a Scott set, and that its members are exactly the subsets of ω that are *representable* in $\mathcal{A}$, that is, those sets X such that

$$X = \{n \in \omega : \mathcal{A}_A \models \psi(\bar{c}, \mathbf{n})\}$$

for some formula $\psi(\bar{c}, x)$ with parameters $\bar{c}$.

An *enumeration* of a countable set $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a binary relation ν such that $\mathcal{S} = \{\nu_0, \nu_1, \ldots\}$, where ν_i is $\{n : (i, n) \in \nu\}$. Let $\mathbf{d}$ be a Turing degree. If $\mathcal{A}$ is a nonstandard model of PA of degree $\mathbf{d}$, then clearly $\mathcal{SS}(\mathcal{A})$ has a $\mathbf{d}$-computable enumeration.

Definition 4.5. Let $\mathcal{S}$ be a countable Scott set. An *effective enumeration* of $\mathcal{S}$ is an enumeration ν of $\mathcal{S}$ together with binary functions f and g and a unary function h such that f, g, and h witness the fact that $\mathcal{S}$ is a Scott set, in the sense that for every $i, j \in \omega$:

(i) $\nu_{f(i,j)} = \nu_i \oplus \nu_j$,

(ii) $\nu_{g(i,e)} = \Phi_e^{\nu_i}$, and

(iii) ν_i is an infinite tree $\Rightarrow \nu_{h(i)} \in [\nu_i]$.

This effective enumeration is $\mathbf{d}$-*computable* if ν, f, g, and h are all $\mathbf{d}$-computable.

Macintyre and Marker [23] used the relativized form of Theorem 3.9 to establish the following result.

Theorem 4.6 (Macintyre and Marker [23])**.** *If a countable Scott set has an enumeration that is* $\mathbf{d}$*-computable, then it has an* effective $\mathbf{d}$*-computable enumeration.*

In particular, if $\mathcal{A}$ is a nonstandard model of PA of degree $\mathbf{d}$, then $\mathcal{SS}(\mathcal{A})$ has an *effective* $\mathbf{d}$-computable enumeration.

4.4 PA Degrees Are Homogeneous Bounding

We can now put the information in the previous sections together to establish the following result.

Theorem 4.7 (Macintyre and Marker [23])**.** *Every PA degree is homogeneous bounding.*

Proof. Let $\mathbf{d}$ be a PA degree. Let $\mathcal{A}$ be a nonstandard model of arithmetic of degree $\mathbf{d}$, and consider the Scott set $\mathcal{S} = \mathcal{SS}(\mathcal{A})$. Note that $\mathcal{S}$ has a $\mathbf{d}$-computable enumeration. Let T be a complete decidable theory. Recall that the types of T can be coded by sets of integers in an effective way. Let $\mathcal{C}$ be the set of all complete types of T coded by sets in $\mathcal{S}$. The set $\mathcal{C}$ can be viewed as the intersection of the set of all (codes for) finite complete types of T with $\mathcal{S}$. We can now verify that $\mathcal{C}$ satisfies conditions 1–5 of Theorem 3.5 in the following way: It is clear that conditions 1–5 hold for the set of all types of T. For $\mathcal{C}$, conditions 1 and 2 are satisfied by property (ii) in the definition of a Scott set. For condition 3 use (ii) to get $q(x_n, ..., x_{2n-1}) \in \mathcal{C}$ from $q(x_0, ..., x_{n-1}) \in \mathcal{C}$. Then, by (i) and (ii), $p(x_0, ..., x_{n-1}) \cup q(x_n, ..., x_{2n-1}) \in \mathcal{C}$, so, by (ii) and the fact that T is decidable, the tree of all possible extensions of this partial type is in $\mathcal{S}$, and thus, by (iii), there is a path of this tree in $\mathcal{S}$. This path is an $r \in \mathcal{C}$ as required by condition 3. Conditions 4 and 5 hold by similar arguments. Hence, there is a countable homogeneous model $\mathcal{A}$ of T whose type spectrum is $\mathcal{C}$.

By Theorem 4.6, $\mathcal{S}$ has an effective $\mathbf{d}$-computable enumeration. In particular, there is a uniform $\mathbf{d}$-computable procedure for finding paths through trees in $\mathcal{S}$. Therefore, there is a $\mathbf{d}$-computable procedure that, given an n-type $p = p(x_0, \ldots, x_{n-1})$ in $\mathcal{C}$ and a formula $\theta(x_0, \ldots, x_{n-1}, x)$ consistent with p, finds an $(n+1)$-type extending both p and θ. This is because such an $(n+1)$-type can be found as a path through a tree in $\mathcal{S}$. Thus, $\mathcal{C}$ has the $\mathbf{d}$-effective extension property, which, by Theorem 3.9 relativized to $\mathbf{d}$, implies that $\mathcal{A}$ has a $\mathbf{d}$-decidable isomorphic copy $\mathcal{B}$. The model $\mathcal{B}$ is a $\mathbf{d}$-decidable homogeneous model of T. $\square$

5 Proof of Theorem 1.1

In this section we prove our main result, by building a complete decidable theory T such that every homogeneous model of T has a PA degree. We describe such a theory T in the language L with equality, infinitely many unary predicate symbols P_i, $i \in \omega$, infinitely many binary predicate symbols R_i, $i \in \omega$, a unary predicate symbol D, and a binary predicate symbol E. For a formula φ, we write φ^1 for φ and φ^0 for $\neg\varphi$. For $\sigma \in 2^{<\omega}$, we write $P^\sigma x$ for

$$\bigwedge\nolimits_{i<|\sigma|} P_i^{\sigma(i)} x,$$

and $R^\sigma xy$ for

$$\bigwedge\nolimits_{i<|\sigma|} R_i^{\sigma(i)} xy.$$

Let U be the set of Gödel numbers of sentences provable from PA and let V be the set of Gödel numbers of sentences refutable from PA. By Theorem 4.2, any degree that can compute a separating set for U and V is a PA degree. Fix computable enumerations $\{U_s\}_{s\in\omega}$ and $\{V_s\}_{s\in\omega}$ of U and V, respectively.

The idea of this proof is to define the theory T in such a way that

1. if $\mathcal{A} \vDash T$ then $E^{\mathcal{A}}$ is an equivalence relation, and if $c, d \in A$ are in different $E^{\mathcal{A}}$-equivalence classes and satisfy exactly the same P_i, then the set $\{i : R_i^{\mathcal{A}}(c, d)\}$ is a separating set for U and V; and

2. every homogeneous model of T must contain such elements c and d.

These conditions imply that if $\mathcal{A} \vDash T$ is homogeneous, then the atomic diagram of $\mathcal{A}$ can compute a separating set for U and V, and hence has a PA degree.

We first describe T informally. The basic axioms for T (axiom groups I–V below) state that the following facts hold. The binary predicate E is an equivalence relation dividing the universe into two equivalence classes. The unary predicate D

holds of exactly two elements a and b, which are in different equivalence classes, and no other atomic formulas hold of a, b, or pairs involving these elements. For each $\sigma \in 2^{<\omega}$ there are infinitely many x in each equivalence class such that $P^\sigma x$ holds. The binary predicates R_i are symmetric, and can hold of a pair (x, y) only if x and y are in different equivalence classes.

The heart of T is given by following collection of axioms (axiom group VI below). For every $\sigma \in 2^\omega$ and every i, if $P^\sigma x$, $P^\sigma y$, and $\neg Exy$, then

$$i \in U_{|\sigma|} \Rightarrow R_i xy$$

and

$$i \in V_{|\sigma|} \Rightarrow \neg R_i xy.$$

What this means is that if $P^\sigma x$, $P^\sigma y$, and $\neg Exy$, and at stage $|\sigma|$ we know that i is in one of U or V, then we require this fact to be coded into the atomic 2-type of (x, y). We will see that this is enough to ensure that T has the properties mentioned above.

To ensure completeness and decidability, T also has a collection of axioms (axiom group VII below) ensuring that anything not ruled out by the previously described axioms and describable by a quantifier-free formula must hold of some tuple of elements.

After defining T more formally, we will be able to show that it is consistent, complete, and decidable. Let $\mathcal{A}$ be a homogeneous model of T and let $a, b \in A$ be the elements of $D^{\mathcal{A}}$. It will not be hard to show that a and b have the same 1-type, so that by homogeneity there is an automorphism f of $\mathcal{A}$ such that $f(a) = f(b)$. Let $c \in A$ be such that $E^{\mathcal{A}}ac$ and let $d = f(c)$. Then c and d are in different equivalence classes but $P_i^{\mathcal{A}}x \Leftrightarrow P_i^{\mathcal{A}}y$ for all i. So if $i \in U$ then letting σ be such that $i \in U_{|\sigma|}$ and c and d satisfy P^σ, we see that there is an axiom in group VI ensuring that $R_i^{\mathcal{A}}cd$. An analogous argument shows that if $i \in V$ then $\neg R_i^{\mathcal{A}}cd$. In other words, $\{i : R_i^{\mathcal{A}}(c, d)\}$ is a separating set for U and V, as desired.

We now give a more formal description of T by listing a set of axioms Γ for it, divided into seven axiom groups for clarity.

Ax I. The relation E is an equivalence relation and splits the universe into two equivalence classes. That is,

$$(\forall x)Exx,$$
$$(\forall x, y)[Exy \rightarrow Eyx],$$
$$(\forall x, y, z)[Exy \wedge Eyz \rightarrow Exz], \quad \text{and}$$
$$(\exists x, y)[\neg Exy \wedge (\forall z)(Exz \vee Eyz)].$$

Ax II. For each $\sigma \in 2^{<\omega}$ there are infinitely many y in each equivalence class such that $P^\sigma y$ holds. That is, for each $\sigma \in 2^{<\omega}$ and $n \in \omega$,

$$(\forall x)(\exists y_0, \ldots, y_n)[\bigwedge_{0 \leqslant i < j \leqslant n} (y_i \neq y_j \wedge Ey_i x) \wedge \bigwedge_{i=0,\ldots,n} P^\sigma y_i].$$

Ax III. No x, y in the same E-equivalence class are R_i related for any i. That is, for each $i \in \omega$,

$$(\forall x, y)[Exy \to \neg R_i xy].$$

Ax IV. Each R_i is symmetric. That is, for each $i \in \omega$,

$$(\forall x, y)[R_i xy \to R_i yx].$$

Ax V. The relation D holds of exactly two elements, one in each E-equivalence class. For any such element x, and any y, we have $\neg P_i x$ and $\neg R_i xy$ for every $i \in \omega$. That is,

$$(\exists x_0, x_1)[Dx_0 \wedge Dx_1 \wedge (\forall z)(Dz \to (z = x_0 \vee z = x_1)) \wedge \neg E x_0 x_1]$$

and for each $i \in \omega$,

$$(\forall x)[Dx \to \neg P_i x] \quad \text{and}$$
$$(\forall x, y)[Dx \to \neg R_i xy].$$

Ax VI. For each $\sigma \in 2^\omega$ and each i, if $i \in U_{|\sigma|}$ then we have the following axiom:

$$(\forall x, y)[P^\sigma x \wedge P^\sigma y \wedge \neg Exy \to R_i xy],$$

while if $i \in V_{|\sigma|}$ then we have the following axiom:

$$(\forall x, y)[P^\sigma x \wedge P^\sigma y \to \neg R_i xy].$$

Ax VII. This axiom group essentially says that anything not ruled out by the previous axioms and describable by a quantifier-free formula must hold of some tuple of elements. For $\sigma, \tau, \mu \in 2^{<\omega}$, we say that μ *is compatible with* σ, τ if for every $i < |\mu|$ and l equal to the length of agreement of σ and τ (that is, the length of the longest common initial segment of σ and τ),

$$i \in U_l \Rightarrow \mu(i) = 1$$

and

$$i \in V_l \Rightarrow \mu(i) = 0.$$

We write $\sigma \mid \tau$ to mean that there is an $i < |\sigma|, |\tau|$ such that $\sigma(i) \neq \tau(i)$.

Let $\sigma_0, \ldots, \sigma_n, \tau, \mu_0, \ldots, \mu_n \in 2^{<\omega}$ be such that $\tau \mid \sigma_k$ for all $k \leqslant n$ and μ_k is compatible with σ_k, τ for all $k \leqslant n$. Then we have axioms saying that if $x_0, \ldots, x_n$ are in the same E-equivalence class and D does not hold of any of them, then there are infinitely many y in the other equivalence class such that $P^\tau y$ and $R^{\mu_k} x_k y$ for all $k \leqslant n$. That is, for each m we have the following axiom:

$$(\forall x_0, \dots, x_n)[\bigwedge_{k=0,\dots,n}(Ex_0x_k \wedge \neg Dx_k \wedge P^{\sigma_k}x_k) \rightarrow$$
$$(\exists y_0, \dots, y_m)(\bigwedge_{j=0,\dots,m}(\neg Ex_0y_j \wedge P^\tau y_j) \wedge \bigwedge_{\substack{k=0,\dots,n \\ j=0,\dots,m}} R^{\mu_k}x_ky_j)].$$

We denote this axiom by AxVII$(\sigma_0, \dots, \sigma_n, \tau, \mu_0, \dots, \mu_n, m)$.

This completes the description of our set of axioms Γ. It is easy to check that Γ is computable. Let T be the deductive closure of Γ. We now verify that T has the desired properties.

Lemma 5.1. *The set of sentences T is consistent, and hence is a theory.*

Proof. We build a model $\mathcal{M}$ of Γ as the union of a chain $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots$. Our construction will not be effective, so $\mathcal{M}$ will not be computable, but that is of course not relevant for this argument. By *closing* a partial structure we mean performing the following actions in order for each x, y, z in the structure:

1. If we have not declared that Dx, then declare that $\neg Dx$.
2. If we have declared that Dx, then declare that $\neg P_i x$ and $\neg R_i xw$ for all i and all w in the structure.
3. Declare that Exx.
4. If we have declared that $E^\epsilon xy$, then also declare that $E^\epsilon yx$.
5. If we have declared that Exy and Eyz, then also declare that Exz.
6. If we have declared that Exy, then also declare that $\neg R_i xy$ and $\neg R_i yx$ for all i.
7. If we have declared that $R_i^\epsilon xy$, then also declare that $R_i^\epsilon yx$.

It will be clear that every time we close a partial structure, we can do so consistently (that is, we never declare both φ and $\neg\varphi$ for an atomic formula φ).

Begin with $M_0 = \{a, b\} \cup \{c_\sigma^n, d_\sigma^n \mid \sigma \in 2^{<\omega},\ n \in \omega\}$. Declare that Da, Db, and $\neg Eab$. For each $\sigma \in 2^{<\omega}$ and $n \in \omega$, declare that Eac_σ^n, Ebd_σ^n, $P^\sigma c_\sigma^n$, and $P^\sigma d_\sigma^n$, and for each $i \geqslant |\sigma|$, that $P_i c_\sigma^n$ and $P_i d_\sigma^n$. For each $c \in \{c_\sigma^n \mid \sigma \in 2^{<\omega},\ n \in \omega\}$ and $d \in \{d_\sigma^n \mid \sigma \in 2^{<\omega},\ n \in \omega\}$, if $i \in U$ then declare that $R_i cd$, and otherwise declare that $\neg R_i cd$. Now close this partial structure to obtain $\mathcal{M}_0$.

Given $\mathcal{M}_i$, build $\mathcal{M}_{i+1} \supset \mathcal{M}_i$ as follows. Let $C = \{c \in M_i \mid E^{\mathcal{M}_i}ca \wedge c \neq a\}$ and $D = \{d \in M_i \mid E^{\mathcal{M}_i}db \wedge d \neq b\}$. Proceed as follows for each $p_0, \dots, p_n \in M_i \setminus D^{\mathcal{M}_i}$ and each $\sigma_0, \dots, \sigma_n, \tau, \mu_0, \dots, \mu_n \in 2^{<\omega}$ such that for all $k \leqslant n$,

1. $\tau \mid \sigma_k$,
2. μ_k is compatible with σ_k, τ,
3. $E^{\mathcal{M}_i} p_0 p_k$,
4. $P^{\sigma_k} p_k$.

Add infinitely many new elements $q_0, q_1, \dots$ to M_{i+1}. For each i, declare that $\neg E p_0 q_i$ and $P^\tau q_i$, and for each $j \geqslant |\tau|$, that $P_j q_i$. For each i and each $k \leqslant n$, declare that $R^{\mu_k} p_k q_i$. We say that we have declared $R^{\mu_k} p_k q_i$ for the sake of axiom group VII. If $p_0 \in C$ then add each q_i to D, and otherwise add each q_i to C.

After the above has been done for all tuples of elements and strings, proceed as follows. For each $c \in C$ and $d \in D$, and each i such that we have so far declared neither that $R^\epsilon cd$ nor that $R^\epsilon dc$, if $i \in U$ then declare that $R_i cd$, and otherwise declare that $\neg R_i cd$. Finally, close the resulting partial structure to obtain $\mathcal{M}_{i+1}$.

Now let $\mathcal{M} = \bigcup_i \mathcal{M}_i$. Because we begin by making D hold of exactly two elements, one in each E-equivalence class, and we close each $\mathcal{M}_i$, it is easy to check that $\mathcal{M}$ satisfies axiom groups I, III, IV, and V. Because for each σ there are infinitely many elements of M_0 in each E-equivalence class satisfying P^σ, axiom group II is also satisfied by $\mathcal{M}$.

For each i and m, and each $\sigma_0, \dots, \sigma_n, \tau, \mu_0, \dots, \mu_n \in 2^{<\omega}$ such that $\tau \mid \sigma_k$ for all $k \leqslant n$ and μ_k is compatible with σ_k, τ for all $k \leqslant n$, we explicitly add elements to M_{i+1} that ensure that $\mathcal{M}_{i+1}$ satisfies $\mathrm{AxVII}(\sigma_0, \dots, \sigma_n, \tau, \mu_0, \dots, \mu_n, m)$ with the universal quantifiers ranging over M_i. This clearly implies that $\mathcal{M}$ satisfies $\mathrm{AxVII}(\sigma_0, \dots, \sigma_n, \tau, \mu_0, \dots, \mu_n, m)$ itself. Thus, $\mathcal{M}$ satisfies axiom group VII.

Finally, the definition of compatibility ensures that whenever we declare $R_i^\epsilon xy$ for the sake of axiom group VII, we do so in a way that is compatible with axiom group VI; and in all other cases, we declare $R_i xy$ if and only if $i \in U$, which again is compatible with axiom group VI. Thus, $\mathcal{M}$ satisfies axiom group VI. □

We now show that T has quantifier elimination. We begin with an auxiliary lemma, which essentially says that axiom group VII gives us enough freedom in choosing elements of models of T with particular atomic types to apply Theorem 2.1.

Lemma 5.2. *Let $\mathcal{A}, \mathcal{B} \vDash T$. Let $p_0, \dots, p_v \in A \setminus D^{\mathcal{A}}$ be elements of the same $E^{\mathcal{A}}$-equivalence class, and let $q_0, \dots, q_v \in B \setminus D^{\mathcal{B}}$ be elements of the same $E^{\mathcal{B}}$-equivalence class. Let $r \in A \setminus D^{\mathcal{A}}$ be in the opposite $E^{\mathcal{A}}$-equivalence class from the p_i. Suppose that for each $i \leqslant v$, if there is a k such that $\mathcal{A} \vDash P_k^\epsilon p_i \wedge P_k^{1-\epsilon} r$, then for the least such k we have $\mathcal{B} \vDash P_k^\epsilon q_i$. Then for each N there is an infinite set $S \subset B \setminus D^{\mathcal{B}}$ such that if $s \in S$ then s is in the opposite $E^{\mathcal{B}}$-equivalence class from the q_i, and for each $l < N$ and each $i \leqslant v$,*

$$\mathcal{A} \vDash P_l r \Leftrightarrow \mathcal{B} \vDash P_l s$$

and

$$\mathcal{A} \vDash R_l p_i r \Leftrightarrow \mathcal{B} \vDash R_l q_i s.$$

Proof. Fix N, and let $M \geqslant N$ be such that

1. for each $i \leqslant v$, if there is a k such that $\mathcal{A} \vDash P_k^{\epsilon} p_i \wedge P_k^{1-\epsilon} r$, then the least such k is less than M; and

2. for each $i < N$, if $i \in U$ then $i \in U_M$, and if $i \in V$ then $i \in V_M$.

For each $i \leqslant v$, let $\alpha_i \in 2^M$ be such that $\mathcal{A} \vDash P^{\alpha_i} p_i$.

Let $\tau' \in 2^M$ be such that $\mathcal{A} \vDash P^{\tau'} r$. For each $i \leqslant v$, let $\sigma_i \in 2^{M+v+1}$ be such that $\mathcal{B} \vDash P^{\sigma_i} q_i$. We can clearly find $\tau \in 2^{M+v+1}$ extending τ' such that $\tau \mid \sigma_i$ for all $i \leqslant v$.

For $i \leqslant v$, let $\mu_i \in 2^N$ be such that $\mathcal{A} \vDash R^{\mu_i} p_i r$. By axiom group VI, μ_i is compatible with α_i, τ'. If there is a k such that $\mathcal{A} \vDash P_k^{\epsilon} p_i \wedge P_k^{1-\epsilon} r$ then by hypothesis the length of agreement of σ_i and τ is no greater than the length of agreement of α_i and τ', so μ_i is compatible with σ_i, τ. Otherwise, $\alpha_i = \tau'$, so, by condition 2 in the choice of M and axiom group VI, for all $j < N = |\mu|$,

$$j \in U \Rightarrow j \in U_M \Rightarrow \mu(j) = 1$$

and

$$j \in V \Rightarrow j \in V_M \Rightarrow \mu(j) = 0,$$

and hence μ_i is compatible with any pair of strings; in particular, μ_i is compatible with σ_i, τ.

Thus, we see that $\sigma_0, \dots, \sigma_v, \tau, \mu_0, \dots, \mu_v$ satisfy the hypotheses of axiom group VII, so there are infinitely many $s \in B \setminus D^{\mathcal{B}}$ in the opposite $E^{\mathcal{B}}$-equivalence class from the q_i such that $\mathcal{B} \vDash P^{\tau} s \wedge \bigwedge_{i \leqslant v} R^{\mu_i} q_i s$. By the choice of τ and $\mu_0, \dots, \mu_v$, the set S of such s has the required properties. □

Lemma 5.3. *The theory T has quantifier elimination.*

Proof. We apply Theorem 2.1. Let $\mathcal{A}, \mathcal{B} \vDash T$ and let $\mathcal{C}$ be a common substructure of both $\mathcal{A}$ and $\mathcal{B}$. We will show that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same existential sentences with constants from C. We can assume without loss of generality that $D^{\mathcal{A}} = D^{\mathcal{B}} = D^{\mathcal{C}}$. Let d_0 and d_1 be the elements of $D^{\mathcal{C}}$. For $i = 0, 1$, let A_i be the elements of A that are in the same $E^{\mathcal{A}}$-equivalence class as d_i, and let B_i be the elements of B that are in the same $E^{\mathcal{B}}$-equivalence class as d_i.

Suppose that $\psi(\overline{x}, \overline{y})$ is a quantifier-free formula in L such that $\mathcal{A} \vDash (\exists \overline{y}) \psi(\overline{c}, \overline{y})$ for some $\overline{c} \in C^{<\omega}$. We can assume that $\psi(\overline{c}, \overline{y})$ is in disjunctive normal form and choose a disjunct $\delta(\overline{c}, \overline{y})$ of $\psi(\overline{c}, \overline{y})$ such that $\mathcal{A} \vDash \delta(\overline{c}, \overline{a})$ for some $\overline{a} \in A^{<\omega}$. We need to find a $\overline{b} \in \mathcal{B}^{<\omega}$ such that $\mathcal{B} \vDash \delta(\overline{c}, \overline{b})$. This $\overline{b}$ will then witness the fact that $\mathcal{B} \vDash (\exists \overline{y}) \psi(\overline{c}, \overline{y})$, as desired.

So it is enough to show that for any conjunction of literals (that is, atomic formulas and negated atomic formulas) $\delta(\overline{x},\overline{y})$ and for every $\overline{c} \in C^{<\omega}$ such that $\mathcal{A} \vDash \delta(\overline{c},\overline{a})$ for some $\overline{a} \in A^{<\omega}$, there is a $\overline{b} \in \mathcal{B}^{<\omega}$ such that $\mathcal{B} \vDash \delta(\overline{c},\overline{b})$. We proceed by induction on the length n of $\overline{a}$. The case $n = 0$ is obvious, so assume that we have proved the statement for n and fix a formula $\delta(x_0,\dots,x_m,y_0,\dots,y_n)$ and tuples $\overline{c} = c_0,\dots,c_m$ and $\overline{a} = a_0,\dots,a_n$ as above. Assume that $a_n \in A_0$, the other case being symmetric.

We can assume that no a_i is in C, since otherwise we could add it to $\overline{c}$, so in particular none of the a_i are in $D^{\mathcal{A}}$. We can also assume that $c_0 = d_0$ and $c_1 = d_1$ and that none of the c_i for $i \geqslant 2$ are equal to either d_0 or d_1.

It is also clearly harmless to add to δ any literals satisfied by $(\overline{c},\overline{a})$, so we can make the following assumptions. First, for each variable z in δ, either Ex_0z or Ex_1z appears in δ. Second, for each $2 \leqslant i \leqslant m$, if there is a k such that $\mathcal{A} \vDash P_k^{\epsilon}c_i \wedge P_k^{1-\epsilon}a_n$, then $P_k^{\epsilon}x_i$ is in δ. Finally, for each $i < n$, if there is a k such that $\mathcal{A} \vDash P_k^{\epsilon}a_i \wedge P_k^{1-\epsilon}a_n$, then $P_k^{\epsilon}y_i$ is in δ.

Let $\delta'(\overline{x},\overline{y})$ be the conjunction of the literals in $\delta(\overline{x},\overline{y})$ not involving y_n. By induction, there are $b_0,\dots,b_{n-1} \in B$ such that $\mathcal{B} \vDash \delta'(c_0,\dots,c_m,b_0,\dots,b_{n-1})$.

If δ contains a literal of the form $y_n = y_i$ for $i < n$, then we can take $b_n = b_i$, so we can suppose otherwise.

Let $(p_0,\dots,p_v)$ be the subsequence of $(c_2,\dots,c_m,a_0,\dots,a_{n-1})$ consisting of those elements that are in A_1, let $r = a_n$, and let $(q_0,\dots,q_v)$ be the subsequence of $(c_2,\dots,c_m,b_0,\dots,b_{n-1})$ consisting of those elements that are in B_1. It follows easily from the assumptions on δ that, with these definitions, the hypotheses of Lemma 5.2 are satisfied. Let N be the maximum of all k such that the symbols P_k or R_k appear in δ, and let S be the infinite set given by Lemma 5.2 for this value of N. Let $s \in S$ be different from all c_i and all b_i for $i < n$, and define $b_n = s$.

Since b_n is different from all c_i and all other b_i, any negated equality involving y_n in δ is satisfied by $(c_0,\dots,c_m,b_0,\dots,b_n)$ (and we have assumed that there are no equalities involving y_n in δ). Since $b_n \in B_0$, any literal in δ involving E and y_n is satisfied by $(c_0,\dots,c_m,b_0,\dots,b_n)$. If P_iy_n is in δ, then $\mathcal{A} \vDash P_ia_n$, so, by our choice of b_n, we have $\mathcal{B} \vDash P_ib_n$, and similarly if $\neg P_iy_n$ is in δ.

If $R_ix_ky_n$ is in δ then $c_k \in B_1$, so $\mathcal{B} \vDash R_ic_kb_n$ by the choice of b_n. If $\neg R_ix_ky_n$ is in δ then one of the following holds: $k < 2$, in which case $\mathcal{B} \vDash \neg R_ic_kb_n$ because $c_k = d_k$; or $c_k \in B_0$, in which case $\mathcal{B} \vDash \neg R_ic_kb_n$ because $b_n \in B_0$; or $k \geqslant 2$ and $c_k \in B_1$, in which case $\mathcal{B} \vDash \neg R_ic_kb_n$ by the choice of b_n.

Similarly, if $R_iy_ky_n$ is in δ then $b_k \in B_1$, so $\mathcal{B} \vDash R_ib_kb_n$ by the choice of b_n. If $\neg R_iy_ky_n$ for $k < n$ is in δ then either $b_k \in B_0$, in which case $\mathcal{B} \vDash \neg R_ib_kb_n$ because $b_n \in B_0$; or $b_k \in B_1$, in which case $\mathcal{B} \vDash \neg R_ib_kb_n$ by the choice of b_n.

So every literal in δ involving y_n is satisfied by $(c_0,\dots,c_m,b_0,\dots,b_n)$. Together with the fact that $\mathcal{B} \vDash \delta'(c_0,\dots,c_m,b_0,\dots,b_{n-1})$, this shows that $\mathcal{B} \vDash \delta(c_0,\dots,c_m,b_0,\dots,b_n)$, as desired. □

Lemma 5.4. *The theory T is complete.*

Proof. Let φ be a sentence in L consistent with T. We need to show that $T \vdash \varphi$. By quantifier elimination, since there are no constants in L, there is a quantifier-free formula $\psi(x)$ with a single variable such that $T \vdash \varphi \leftrightarrow \psi(x)$, which implies that $T \vdash (\varphi \leftrightarrow (\exists x)\psi(x))$. We can write $\psi(x)$ as a disjunction of conjunctions of literals. Since φ is consistent with T, so is $(\exists x)\psi(x)$, and hence there is a disjunct $\theta(x)$ of $\psi(x)$ such that $(\exists x)\theta(x)$ is consistent with T. It is easy to check from the axioms that the conjuncts of $\theta(x)$ can only be of the forms $E(x,x)$, $D(x)$, $\neg D(x)$, $\neg R_i(x,x)$, $P_i(x)$, and $\neg P_i(x)$, where for each i, we cannot have both $P_i(x)$ and $\neg P_i(x)$ present, and if $D(x)$ is present then $P_i(x)$ cannot be present for any i. It is now easy to check that $T \vdash (\exists x)\theta(x)$. So $T \vdash (\exists x)\psi(x)$, and hence $T \vdash \varphi$. □

Lemma 5.5. *The theory T is decidable.*

Proof. Since T is computably axiomatizable and complete, it is decidable. □

Thus, T is a complete decidable theory. We now finish the proof of Theorem 1.1 with the following lemmas.

Lemma 5.6. *Let $\mathcal{A} \vDash T$ and let a and b be the two elements of $D^{\mathcal{A}}$. Then a and b have the same 1-type.*

Proof. By quantifier elimination, it is sufficient to show that a and b have the same atomic 1-type, that is, that they satisfy the same atomic formulas with one free variable. But the only such formulas are $x = x$, Exx, Dx, and P_ix and R_ixx for $i \in \omega$. The first three formulas hold of both a and b, while the last two families of formulas hold of neither a nor b. □

Lemma 5.7. *Let $\mathcal{A}$ be a countable homogeneous model of T. Then $\mathcal{A}$ has a PA degree.*

Proof. Let a and b be the two elements of $D^{\mathcal{A}}$. The elements a and b have the same 1-type, so, by homogeneity, for any c such that $E^{\mathcal{A}}ac$ there is a d with the same 1-type as c such that $E^{\mathcal{A}}bd$. In particular, $\mathcal{A}$ contains elements c and d such that $\neg E^{\mathcal{A}}cd$ and $(\forall i)[P_i^{\mathcal{A}}c \Leftrightarrow P_i^{\mathcal{A}}d]$. It now follows easily from axiom group VI that if $i \in U$ then $R_i^{\mathcal{A}}cd$, while if $i \in V$ then $\neg R_i^{\mathcal{A}}cd$. Thus, $\{i : R_i^{\mathcal{A}}cd\}$ separates U and V, and hence has a PA degree. However, we can compute this set from the atomic diagram of $\mathcal{A}$, so $\mathcal{A}$ has a PA degree. □

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