FINITE COMPUTABLE DIMENSION AND DEGREES OF CATEGORICITY

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Abstract. We first give an example of a rigid structure of computable dimension 2 such that the unique isomorphism between two non-computably isomorphic computable copies has Turing degree strictly below $\mathbf{0}''$, and not above $\mathbf{0}'$. This gives a first example of a computable structure with a degree of categoricity that *does not* belong to an interval of the form $[\mathbf{0}^(\alpha), \mathbf{0}^(\alpha + \mathbf{1})]$ for any computable ordinal α. We then extend the technique to produce a rigid structure of computable dimension 3 such that if $\mathbf{d_0}$, $\mathbf{d_1}$, and $\mathbf{d_2}$ are the degrees of isomorphisms between distinct representatives of the three computable equivalence classes, then each $\mathbf{d_i} < \mathbf{d_0} \oplus \mathbf{d_1} \oplus \mathbf{d_2}$. The resulting structure is an example of a structure that has a degree of categoricity, but not strongly.

1. Introduction

In mathematics, we often identify structures up to isomorphism. We consider isomorphic copies of the same object to be the same. In computable structure theory, we need to be careful. Even if a structure has a computable presentation, it could be that not all computable copies are computably isomorphic. That is, computability theoretically, they are not the same. Indeed, in the case that they *are* the same, we have the following definition.

Definition 1.1. *A computable structure $\mathcal{A}$ is* computably categorical *if for all computable $\mathcal{B} \cong \mathcal{A}$ there exists a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

What if a computable structure is not computably categorical? There are a couple of interesting questions we might ask. One is, how many equivalence classes does the structure have up to computable isomorphism? This number, which is at most ω, is known as the *computable dimension* of the structure. The other is, does the structure have a natural Turing degree where it becomes computably categorical?

Goncharov was first to construct a structure of finite computable dimension [Gon80]. There has been much further work on constructing structures of finite computable dimension with various properties [Har93], [CGKS99], [Hir02].

Towards the second question, we first extend the definition of computably categorical to other degrees.

Definition 1.2. *A computable structure $\mathcal{A}$ is* $\mathbf{d}$-computably categorical *if for all computable $\mathcal{B} \cong \mathcal{A}$ there exists a $\mathbf{d}$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

Goncharov related these notions by showing that if a structure is $0'$-computably categorical, then it must have computable dimension 1 or ω [Gon82]. That is, the structures of finite computable dimension are necessarily unpleasant to build, in

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the sense that the fact that the various computable copies are indeed isomorphic cannot be verified by a limit computable isomorphism.

The various constructions of structures of finite computable dimension all use an infinite injury type construction, and so are all $0''$-categorical. No effort is made to control the complexity of the isomorphism(s).

The following definition intends to pin down the level of complexity required to compute isomorphisms of a given computable structure, and to say which Turing degrees can be realized as levels of complexity of isomorphisms of computable structures.

Definition 1.3. *A structure $\mathcal{A}$ has* degree of categoricity $\mathbf{d}$ *if $\mathcal{A}$ is $\mathbf{d}$-computably categorical, and for all $\mathbf{c}$ such that $\mathcal{A}$ is $\mathbf{c}$-computably categorical, $\mathbf{d} \leq \mathbf{c}$. We say a degree $\mathbf{d}$ is a* degree of categoricity *if there is some structure with degree of categoricity $\mathbf{d}$.*

Degrees of categoricity have been widely studied since they were first introduced by Fokina, Kalimullin and Miller in [FKM10]. So far, all examples constructed have been in intervals of the form $[\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$, and have had the following property, which is stronger than merely being a degree of categoricity.

Definition 1.4. *A degree of categoricity $\mathbf{d}$ is a* strong *degree of categoricity if there is a structure $\mathcal{A}$ with computable copies $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $\mathbf{d}$ is the degree of categoricity for $\mathcal{A}$, and every isomorphism $f : \mathcal{A}_0 \to \mathcal{A}_1$ satisfies $deg(f) \geq \mathbf{d}$.*

Fokina, Kalimullin and Miller [FKM10] showed that all strong degrees of categoricity are hyperarithmetical. Later, Csima, Franklin and Shore [CFS13] showed that in fact all degrees of categoricity are hyperarithmetical. Recently, Csima and Harrison-Trainor [CHTar] have shown that the only "natural" degrees of categoricity are those of the form $\mathbf{0}^{(\alpha)}$ for some computable α, and moreover that any computable structure has a strong degree of categoricity "on a cone".

In the first half of the paper we construct a rigid computable structure with computable dimension 2, and demonstrate a method for controlling the degree of the isomorphism between the two copies. Indeed, we show the existence of a degree of categoricity $\mathbf{d}$ which does not lie in an interval of the form $[\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$ for any computable ordinal α, answering an open question as stated in [FKM10],[Fra17], and [AC16].

Theorem 1.5. *There is a rigid computable structure with computable dimension 2 such that the isomorphism f between two computable copies satisfies $f \not\geq_T \emptyset'$ and $f \leq_T \emptyset''$, and therefore in particular $f <_T \emptyset''$.*

We obtain as a corollary.

Corollary 1.6. *There is a rigid computable structure which has a degree of categoricity that does not belong to an interval of the form $[\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$ for any computable ordinal α.*

One of the original aims of this paper was to show that there exists a structure that has a degree of categoricity but not strongly. We approached the problem by showing that we could control the degree of the isomorphism in the construction of a structure of finite computable dimension. In the mean time, we have received a preprint from Bazhenov, Kalimullin, and Yamaleev [BKYar], that shows as its main result the existence of a structure with a degree of categoricity but not strongly.

Their approach uses a new notion of "spectral dimension", and their structure is $\mathbf{0}'$-computably categorical, so in particular has computable dimension ω. We feel that our approach is of independent interest since it relates the natural notions of computable dimension and degrees of categoricity.

In the second half of the paper we apply the ideas of the first half to carry out a more involved construction: that of a rigid computable structure with computable dimension 3, and which has a degree of categoricity but no strong degree of categoricity. That is, we prove the following.

Theorem 1.7. *There exists a rigid structure of computable dimension 3 such that if* $\mathbf{d}_0$*,* $\mathbf{d}_1$*, and* $\mathbf{d}_2$ *are the degrees of isomorphisms between distinct representatives of the three computable equivalence classes, then each* $\mathbf{d}_i < \mathbf{d}_0 \oplus \mathbf{d}_1 \oplus \mathbf{d}_2 \leq \mathbf{0}''$.

We thus obtain the desired corollary.

Corollary 1.8. *There is a rigid computable structure with computable dimension 3 which has a degree of categoricity* $\mathbf{d} \leq \mathbf{0}''$*, but has no strong degree of categoricity.*

Neither our example nor the paper of [BKYar] answers the question of whether all degrees of categoricity are strong. That is, the degree of categoricity that our structure constructed in section 4 has is indeed a strong degree of categoricity as witnessed by a different structure.

2. Notation and Conventions

For general references, see Harizanov [Har98] for computable structure theory, and Soare [Soa16] for computability theory.

Our constructions will make use of Friedberg enumerations.

Definition 2.1. *Let* $\mathcal{S} \subseteq \mathcal{P}(\omega)$.

A c.e. Friedberg enumeration of $\mathcal{S}$ *is a c.e. binary relation* ν *such that for all* $i \neq j$*,* $\nu(i) \neq \nu(j)$*, where* $\nu(i) = \{x \mid (i,x) \in \nu\}$*, and* $\mathcal{S} = \{\nu(i) \mid i \in \omega\}$.

For each $\mathcal{S} \subseteq \omega$, we let $\mathcal{G}(\mathcal{S})$ be the rigid graph associated to $\mathcal{S}$. That is, for each $A \in \mathcal{S}$, $\mathcal{G}(\mathcal{S})$ has a connected component with a root node with an $(n+3)$-cycle attached to the root for each $n \in A$, and no other components. It is easy to see that a computable copy of $\mathcal{G}(\mathcal{S})$ gives a c.e. Friedberg enumeration of $\mathcal{S}$, and conversely. Under this association, isomorphisms between computable copies of $\mathcal{G}(\mathcal{S})$ correspond to bijections $f : \omega \to \omega$ such that $\nu(i) = \mu(f(i))$, where ν and μ are Friedberg enumerations of the same set. The effectiveness of the correspondence guarantees that the Turing degree of the isomorphism between two computable copies of $\mathcal{G}(\mathcal{S})$ is the same as that of the bijection f to which it corresponds. We opt to work directly with c.e. Friedberg enumerations instead of with the corresponding graphs. However, bearing in mind the strength of the correspondence, we will adopt much of the terminology associated with graphs. We note that our correspondence can be extended to associate arbitrary c.e. binary relations with (non-rigid) graphs. Because of this, we will refer to c.e. binary relations as *structures*, and fix an effective list $(\rho_e)_{e\in\omega}$ of all such relations. If ν and μ are c.e. binary relations, we let $\nu(i)$ and $\mu(i)$ be as in the definition of c.e. Friedberg enumerations. We refer to a bijection $f\colon \omega \to \omega$ for which $\nu(i) = f(\mu(i))$ for every i as an isomorphism from ν to μ. Note that if there is a unique isomorphism from ν to μ, then they are c.e. Friedberg enumerations of the same set. Furthermore, given a c.e. binary relation ν, we often refer to $i \in \omega$ as a *component* of ν, since i can be viewed as a name for the connected component associated to $\nu(i)$ in the corresponding graph.

3. Computable dimension 2

This section is devoted to the proof of our first theorem, which we repeat here.

Theorem 3.1. *There is a rigid computable structure with computable dimension 2 such that the isomorphism f between the two computable copies ν and μ satisfies $f \not\geq_T \emptyset'$ and $f \leq_T \emptyset''$, and therefore in particular $f <_T \emptyset''$.*

Before we begin with the proof, we show that we easily have the following corollary.

Corollary 3.2. *There is a rigid computable structure which has a degree of categoricity that does not belong to an interval of the form $[\mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}]$ for any computable ordinal α.*

Proof. Let $\mathcal{A}$ be a structure as guaranteed by Theorem 3.1, and let $\mathbf{d}$ be its degree of categoricity. Then as $\mathbf{d} \not\geq \mathbf{0}'$, the only eligible interval is $[\mathbf{0}, \mathbf{0}']$. However, $\mathcal{A}$ has computable dimension 2, so by the previously mentioned result of Goncharov [Gon82], $\mathbf{d} \not\leq \mathbf{0}'$. □

We build two c.e. Friedberg enumerations ν and μ of some $\mathcal{S} \subseteq \mathcal{P}(\omega)$.

We meet the requirements:

P_e: If φ_e is total then $\nu(i) \neq \mu(\varphi_e(i))$ for some $i \in \omega$.

N_e: If ρ_e is a Friedberg enumeration of $\mathcal{S}$, then ρ_e is computably isomorphic to ν or μ.

There is a $\mathbf{0}''$-computable bijection $f : \omega \to \omega$ such that for all i, $\nu(i) = \mu(f(i))$, and such that $\deg(f) \not\geq \mathbf{0}''$. To do this we build a c.e. set C and meet the requirements:

S_e: $\Phi_e^f \neq C$.

We give a stage-by-stage construction of ν, μ, and C. At each stage s, f_s will be the unique map satisfying $\nu(i) = \mu(f(i))[s]$ for all $i \in \omega$. We begin with $\nu(i)[0] = \mu(i)[0] = \{i\}$, so that f_0 is the identity function.

We split the requirements S_e into several subrequirements, each of which makes some guesses about the structure we are building. Meeting the subrequirement for which those guesses are correct will suffice to meet S_e. The S_e requirements will be able to injure eachother, and can also be injured by P_e requirements.

The P_e requirements are met via a finitary action, and will not be injured by any higher priority requirement. The N_e requirements will be met via an infinitary procedure. The strategy for meeting these requirements is to build the structures ν and μ in such a way that if ρ_e is is isomorphic to our two structures, it must be constructed by following the construction of one of our structures so closely that the two are computably isomorphic.

For $\eta = \nu$ or $\eta = \mu$, and for a sequence $n_0, \ldots, n_k \in \omega$, to *perform a right shuffle on the sequence $n_0, \ldots, n_k$ in η at stage $s+1$* is to set $\eta_{s+1}(n_i) = \eta_s(n_i) \cup \eta_s(n_{i+1})$ for $0 \leq i \leq k$, where $n_{k+1} := n_0$, and to set $\eta_{s+1}(j) = \eta_s(j)$ for $j \notin \{n_0, \ldots, n_k\}$. Similarly, to *perform a left shuffle on the sequence $n_0, \ldots, n_k$ in η at stage $s+1$* is to set $\eta_{s+1}(n_i) = \eta_s(n_i) \cup \eta_s(n_{i-1})$ for $0 \leq i \leq k$, where $n_{k+1} := n_0$, and to set $\eta_{s+1}(j) = \eta_s(j)$ for $j \notin \{n_0, \ldots, n_k\}$.

Remark 3.3. Note that if $n_0, \ldots, n_k, m_0, \ldots, m_k$ are such that $\nu(n_i) = \mu(m_i)$, then if we perform a right shuffle on $n_0, \ldots, n_k$ in ν and a left shuffle on $m_0, \ldots, m_k$ in μ, then after the shuffle we will have $\nu(n_i) = \mu(m_{i+1})$, where $m_{k+1} := m_0$.

At each stage of the construction we will perform a right shuffle in ν and a left shuffle in μ, and will always choose the components used in the shuffle to satisfy the condition laid out in Remark 3.3. This will ensure that $\nu[s]$ and $\mu[s]$ remain isomorphic at each stage s. By choosing which components we involve in the shuffle carefully, we will also ensure that ν and μ are still isomorphic at the end of the construction, and that we meet each requirement N_e.

We now introduce some ideas and notation which we will use in the construction.

Let a_e, b_e, c_e, d_e, x_e, y_e, and z_e, for $e \in \omega$, denote natural numbers distinct from one another.

For $\sigma \in 3^{<\omega}$ and $\eta \in \{\nu, \mu\}$, let $p^\eta_\sigma[0] = z_{\langle \sigma, 0 \rangle}$. These will be the "precious" components, used to meet the N_e requirements. N_0 will need only one precious component, but N_e will need one for each $\sigma \in 3^e$.

We will classify each stage of the construction according to what strategy we use at that stage. We will do so by assigning to each stage s a string α_s of length s, and say that s is a β-stage for each $\beta \preceq \alpha_s$. At each stage we will have a finite set of β-marked components in our structures ν and μ. To determine whether a β-stage s is a $\beta^\frown 0$-stage, a $\beta^\frown 1$-stage, or a $\beta^\frown 2$-stage, we will need to decide whether the structure ρ_e has recovered to resemble our structures ν and μ sufficiently closely. We do so by checking whether there is a well-behaved embedding from the β-marked components of η into ρ_e, where η is equal to each of ν and μ. A $\beta^\frown 2$-stage will be a β-stage at which there is no such embedding, whereas $\beta^\frown 0$- and $\beta^\frown 1$-stages will correspond to different kinds of recovery. At $\beta^\frown 1$-stages, we believe that ρ_e may be isomorphic to our structures, and work toward ensuring that if so, it is computably isomorphic to one of them. At $\beta^\frown 0$-stages, we work toward ensuring ρ_e is not isomorphic to our structures.

We let TP be the true path, defined by setting $TP \restriction n = \liminf_s \alpha_s \restriction n$.

If $\beta^\frown i \prec TP$ for some β of length e, the strategy corresponding to the string of form $\beta^\frown i$ indicated above succeeds. In the case that $i = 0$ or $i = 2$, ρ_e is not isomorphic to either of our structures, whereas if $i = 1$, ρ_e may be isomorphic to our structures. If so, it will be computably isomorphic to one of them.

We will now define α_{s+1}, using a recursive procedure specifying its initial segments.

Every stage is a λ-stage.

Stage $s = 0$**.**

We let $h^\eta_\beta[0]$ be empty for η equal to both ν and μ and for every β.

Stage $s + 1$**.**

Suppose that we know that $s + 1$ is a β-stage for some β with $|\beta| < s$. Let $e = |\beta|$. Let B^η be the set of components of η which are β-marked by the end of stage s, for $\eta = \mu$ and $\eta = \nu$.

We say that stage $s + 1$ is a recovery stage for N_e if, for $\eta = \nu$ and $\eta = \mu$, there exists a unique map $h^\eta_\beta[s] \colon B^\eta \to \rho_e$ such that $\rho_e(h^\eta_\beta(x))[s] \supseteq \eta(x)[s]$ for each $x \in B^\eta$, and furthermore that if $t + 1 < s + 1$ is the most recent β-stage which is a recovery stage for N_e, then $\text{range}(h^\eta_\beta)[t] \subseteq \text{range}(h^\eta_\beta)[s]$.

If $s+1$ is a β-stage which is a recovery stage for N_e, and β *was not* active at the end of stage s, say $s+1$ is an even ν-recovery stage, a β-initialization stage and a $\beta^\frown 0$-stage. Declare β to be active.

If $s+1$ is a β-stage which is a recovery stage for N_e, and β *was* active at the end of stage s, let $t+1 < s+1$ be the most recent β-stage that was a recovery stage for N_e. If t was an even/odd η-recovery stage for N_e, and if $h^\eta_\beta[s] \supseteq h^\eta_\beta[t]$, then we say $s+1$ is an odd/even η-recovery stage and a $\beta^\frown 1$-stage, whereas if $h^\eta_\beta[s] \not\supseteq h^\eta_\beta[t]$, we say that this is an even ι-recovery stage for N_e, where $\iota = \mu$ if $\eta = \nu$, and $\iota = \mu$ otherwise, and that stage $s+1$ is a $\beta^\frown 0$-stage.

If $s+1$ is not a recovery stage for N_e, then we say $s+1$ is a $\beta^\frown 2$-stage.

Let α_{s+1} be the unique string of length $s+1$ such that $s+1$ is an α_{s+1}-stage. Note that $s+1$ is a σ-stage iff $\sigma \preceq \alpha_{s+1}$.

At each stage of the construction the structures $\nu[s]$ and $\mu[s]$ will be isomorphic via a unique isomorphism f_s. To meet the requirement S_e, we define a modified map $f_{\tau,s}$ between μ and ν for each τ of length e. This map predicts which components of ν and μ should be infinite based on the assumption that $\tau \prec TP$, and is created by adjusting the current isomorphism f_s to match that prediction. The maps $f_{\tau,s}$ approximate f, in the sense that $\lim_n \lim_s f_{TP\restriction n,s} = f$.

For each σ such that $\sigma^\frown 1 \preceq \tau$, at each τ-stage s, $f_{\tau,s}$ will predict one component q of ν to be infinite, and predict its image in μ by specifying $f_{\tau,s}(q)$. If $\tau \prec TP$, then from some point onward, the components predicted to be infinite in this way will not change, nor will their predicted images. At each τ-stage thereafter, each component which is predicted to be infinite by τ will be involved in a shuffle, and its membership will increase. The shuffles are arranged so that at the end of the construction we have $f(q) = f_{\tau,s}(q)$ for sufficiently large s.

We now specify $f_{\tau,s}$ precisely for each τ, as follows.

If $\sigma^\frown 1 \preceq \tau$, let $t+1 < s+1$ be the most recent $\sigma^\frown 0$ stage.

Set $f_{\tau,s}(p^\nu_{\sigma,t+1}) = p^\mu_{\sigma,t+1}$. Say that τ s-predicts the components $p^\nu_{\sigma,t+1}$ of ν and $p^\mu_{\sigma,t+1}$ of μ to be precious. If $f_s(q) = p^\mu_{\sigma,t+1}$ for some other q, then set $f_{\tau,s}(q)$ to be undefined.

For all other components q of ν, set $f_{\tau,s}(q) = f_s(q)$.

We will need to refer to the functions $f_{\tau,s}$ when considering the requirements S_e. For this reason, we will now split S_e into requirements S_τ, where $\tau \in 3^{<\omega}$ has length e.

Towards meeting the S_τ requirements, let $w_{\langle\tau,n\rangle}$ for $\tau \in 3^{<\omega}$ and $n \in \omega$ denote natural numbers distinct from one another.

We will work to achieve the requirement S_τ by ensuring that there is some n such that for all sufficiently large s,

$$\Phi_e^{f_\tau}(w_{\langle\tau,n\rangle})[s] \downarrow \neq C(w_{\langle\tau,n\rangle}).$$

Provided that f_τ and f agree on the use of the computation, we will have met the requirement S_τ. We will build C by giving a computable enumeration $(C_s)_{s\in\omega}$.

To meet the overall requirement S_e, it will suffice to meet the subrequirement S_τ for $\tau = TP \restriction e$.

At each stage s we will define a restraint function $r : 3^{<\omega} \to \omega$ which we will use to protect computations that are being used to meet requirements of the form S_τ. We let $R_s(\tau) = \max\{r_s(\sigma) \mid \sigma \leq_L \tau\}$.

For each τ, at each stage of the construction we will have an active witness $w_{\langle \tau,n\rangle}$ for τ-diagonalization. Our strategy for meeting a requirement S_τ at stage $s+1$ will be to find a function $g\colon \{0,\dots,k-1\} \to \omega$ such that

$$\Phi^g_{e,s}(w_{\langle \tau,n\rangle}) \downarrow = 0,$$

enumerate $w_{\langle \tau,n\rangle}$ into C, and ensure that $g = f_{\tau,t} \restriction k$ for each $t \geq s+1$. At stage $s+1$ we search for $g \preceq f_{\theta,s}$, where we require $\tau \preceq \theta \preceq \alpha_{s+1}$. Because $f_{\tau,s} \restriction k \neq f_{\theta,s} \restriction k$, once we identify such a g, we will need to perform a shuffle which is designed to ensure that $g = f_{\tau,s+1} \restriction k$.

At stage $s+1$, we will say that a number q is fresh if $\nu(q)[s] = \mu(q)[s] = \{q\}$, and for each σ, q is not σ-reserved, and furthermore, for no $t < s+1$ do we have $q = p^\nu_{\sigma,t}$ or $q = p^\mu_{\sigma,t}$.

The Construction

Stage 0**:**

Set $r(\sigma)[0] = 0$ for all $\sigma \in 3^{<\omega}$. For $\eta \in \{\nu,\mu\}$, λ-mark the precious components p^η_λ. For the least components of form x_l and y_k, λ-mark them and say they are λ-reserved.

Let $C_0 = \emptyset$. Say that the active witness for τ-diagonalization is $w_{\langle \tau,0\rangle}$ for each τ.

Say that none of the requirements S_τ and P_e are currently satisfied.

Stage $s+1$**:**

We attempt to meet one requirement of the form S_τ or P_e.

We first check on the status of the requirements P_e for $e \leq s$.

Condition 3.4. Suppose P_e is not currently satisfied. Suppose there exists some $\langle e,j\rangle$ such that $\varphi_{e,s}(b_{\langle e,j\rangle}) \downarrow = b_{\langle e,j\rangle}$, that each of $a_{\langle e,j\rangle}, b_{\langle e,j\rangle}, c_{\langle e,j\rangle}$ is larger than $R(\alpha \restriction e)[s+1]$, is fresh, and if any of these three components is β-marked, then $\beta \geq_L \alpha_{s+1}$ or $\beta \prec \alpha_{s+1}$.

Then say that P_e requires attention.

We next check on the status of the requirements S_τ for each $\tau \preceq \alpha_{s+1}$.

Condition 3.5. Suppose $\tau \preceq \alpha_{s+1}$ is of length e and that S_τ is not currently satisfied.

If $\theta \succeq \tau$, say that a component q of ν is θ-unpredictable if there is some σ such that $\sigma^\frown i \preceq \theta$ for $i = 0$ or 1 for which q is σ-reserved, or such that $q = p^\nu_{\sigma,s}$ or $q = f_s^{-1}(p^\mu_{\sigma,s})$, but θ does not s-predict q to be precious.

Let $w_{\langle \tau,n\rangle}$ be the active witness for τ-diagonalization.

Check whether there is some θ such that $\tau \preceq \theta \preceq \alpha_{s+1}$, $k < s$, and g, which meet the following conditions:

1. $g = f_{\theta,s} \restriction k$
2. if x is a component which is θ-unpredictable, $k \leq x$.
3. $\Phi^g_i(w_{\langle \tau,n\rangle})[s] \downarrow = 0$.

If such g and θ exist, say that S_τ requires attention.

Let e be the least number such that at least one of P_e and S_τ requires attention, where $\tau = \alpha_{s+1} \restriction e$. If P_e requires attention, we will meet it at this stage. Otherwise we will attempt to meet S_τ.

In any case, our action will consist of performing a right shuffle of some components of ν, and a left shuffle of the identical components in μ. The components shuffled in the two cases will be very similar.

We will specify which components should be part of the sequence of components used in the shuffle in each of ν and μ by considering each $\sigma \preceq \alpha_{s+1}$ in turn, and defining subsequences $\widehat{\nu}_n$ and $\widehat{\mu}_n$ of components corresponding to $\sigma = \alpha_{s+1} \restriction n$.

Case 1: Meeting P_e

Let $\langle e, j\rangle$ be as specified in Condition 3.4. Say that we attempt to meet P_e at this stage.

For each $n \leq s$, do as follows: Let $\sigma = \alpha_{s+1} \restriction n$.

If $s+1$ is an even ν-recovery stage for N_n, let $\widehat{\nu}_n = x_l, p^{\nu}_{\sigma,s}, y_k$, where x_l and y_k are σ-reserved components. Let $\widehat{\mu}_n = x_l, f_s(p^{\nu}_{\sigma,s}), y_k$.

If $s+1$ is an odd ν-recovery stage for N_n, let t be the most recent σ-stage which was a recovery stage for N_n. Let $\hat{l}$ be such that $x_{\hat{l}}$ was one of the σ-reserved components used in $\widehat{\nu}_n[t]$. Let $\widehat{\nu}_n = x_l, p^{\nu}_{\sigma,s}, x_{\hat{l}}$, where x_l is σ-reserved. Let $\widehat{\mu}_n = x_l, f(p^{\nu}_{\sigma})[s], p^{\mu}_{\sigma,t}$.

In both ν-recovery cases, let $p^{\nu}_{\sigma,s+1} = p^{\nu}_{\sigma,s}$ and $p^{\mu}_{\sigma,s+1} = f(p^{\nu}_{\sigma})[s]$.

Similarly, if $s+1$ is an even μ-recovery stage, let $\widehat{\mu}_n = x_l, p^{\mu}_{\sigma,s}, y_l$, where x_l and y_k are σ-reserved. Let $\widehat{\nu}_n = x_l, f^{-1}(p^{\mu}_{\sigma})[s], y_k$.

If $s+1$ is an odd μ-recovery stage, let t be the most recent σ-stage which was a recovery stage for N_n. Let $\hat{k}$ be such that $y_{\hat{k}}$ was one of the σ-reserved components used in $\widehat{\mu}_n[t]$. Let $\widehat{\mu}_n = y_{\hat{k}}, p^{\mu}_{\sigma,s}, y_k$, where y_k is σ-reserved. Let $\widehat{\nu}_n = p^{\nu}_{\sigma,t}, f^{-1}(p^{\mu}_{\sigma})[s], y_k$.

In both μ-recovery cases, let $p^{\mu}_{\sigma,s+1} = p^{\mu}_{\sigma,s}$ and $p^{\nu}_{\sigma,s+1} = f^{-1}(p^{\mu}_{\sigma})[s]$.

If $s+1$ is not an N_n recovery stage, let $\widehat{\nu}_n$ and $\widehat{\mu}_n$ be empty.

Perform a right shuffle on $a_{\langle e,j\rangle}, b_{\langle e,j\rangle}, c_{\langle e,j\rangle}, \widehat{\nu}_0, \dots, \widehat{\nu}_s$ in ν and perform a left shuffle on $a_{\langle e,j\rangle}, b_{\langle e,j\rangle}, c_{\langle e,j\rangle}, \widehat{\mu}_0, \dots, \widehat{\mu}_s$ in μ.

Say that P_e is satisfied.

Proceed to clean-up phase.

Case 2: Attempting to meet S_τ.

Let g and θ be as specified in Condition 3.5. Suppose that $|\tau| = e$ and $|\theta| = m$. Say that we attempt to meet S_τ at this stage.

Enumerate $w_{\langle \tau,n\rangle}$ into C_{s+1}.

For each $i < e$, let $\sigma = \alpha_{s+1} \restriction i$. Then let $\widehat{\nu}_i$ and $\widehat{\mu}_i$ be exactly as laid out above in the case where we are attempting to meet P_e.

For $e \leq i < m$, let $\sigma = \alpha_{s+1} \restriction i$.

Our choice of $\widehat{\nu}_i$ and $\widehat{\mu}_i$ will now be determined by the requirement that $f_{\tau,s+1}$ agrees with $f_{\theta,s}$ on every component $q < k$. We will say that i requires adjustment if $f_s(p^{\nu}_{\sigma,t_0+1}) \neq p^{\mu}_{\sigma,t_0+1}$, where $t_0 + 1 < s+1$ is the most recent $\sigma^\frown 0$-stage. We will later see that this occurs precisely when $s+1$ is an odd recovery stage for N_i.

If i requires adjustment, do as follows:

Suppose $t < s+1$ is the most recent σ-stage which is a recovery stage for N_i, and that t is an even ν-recovery stage. Let $\hat{l}$ be such that $x_{\hat{l}}$ was one of the σ-reserved components used in $\widehat{\nu}_i[t]$. Let $\widehat{\nu}_i = x_l, p^{\nu}_{\sigma,s}, x_{\hat{l}}$, where x_l is σ-reserved. Let $\widehat{\mu}_i = x_l, f(p^{\nu}_{\sigma})[s], p^{\mu}_{\sigma,t+1}$.

Otherwise if $t < s+1$ is the most recent σ-stage which is a recovery stage for N_i, then t is an even μ-recovery stage. Let $\hat{k}$ be such that $y_{\hat{k}}$ was one of the σ-reserved components used in $\widehat{\mu}_i[t]$. Let $\widehat{\mu}_i = y_{\hat{k}}, p^{\mu}_{\sigma,s}, y_k$, where y_k is σ-reserved. Let $\widehat{\nu}_i = p^{\nu}_{\sigma,t+1}, f^{-1}(p^{\mu}_{\sigma})[s], y_k$.

If $e \leq i < m$ and i does not require adjustment, let $\widehat{\nu}_i$ and $\widehat{\mu}_i$ be empty.

Choose some fresh unmarked $d_j > \max(R(\alpha)[s], k)$.

Perform a right shuffle on $d_j, \widehat{\nu}_0, \ldots, \widehat{\nu}_{m-1}$ in ν and perform a left shuffle on $d_j, \widehat{\mu}_0, \ldots, \widehat{\mu}_{m-1}$ in μ.

Define $r_{s+1}(\tau) = \max(k, r_s(\tau))$, where the domain of g is $\{0, \ldots, k-1\}$. Say that S_τ is satisfied.

Proceed to clean-up phase.

Clean-up phase. Declare that any component involved in the shuffle is no longer σ-reserved for any σ.

For each $\sigma >_L \tau$, do as follows: Declare any σ-reserved components to no longer be σ-reserved. Declare σ inactive. Declare that S_σ is no longer satisfied. Choose the least number $w_{\langle \sigma, n\rangle} > s$ such that $w_{\langle \sigma, n\rangle} \notin C_s$ as the active witness for σ-diagonalization. Let $r_{s+1}(\sigma) = 0$.

For $\sigma \leq_L \tau$, define $r_{s+1}(\sigma) = r_s(\sigma)$ if not already defined.

For each $\sigma \geq_L \tau$, find the least fresh $z_{\langle \sigma, n\rangle} > R(\alpha)[s+1]$ which is not β-marked for any β, and set $p^\nu_\sigma[s+1] = p^\mu_\sigma[s+1] = z_{\langle \sigma, n\rangle}$.

We begin by describing the marking of components in ν.

For each $\sigma \preceq \alpha_{s+1}$, σ-mark the component $p^\nu_{\sigma, s+1}$.

For each $i < e$, let $\alpha_{s+1} \upharpoonright i = \sigma$. If $s+1$ is a recovery stage for N_i, do as follows: σ-mark each component q of ν such that q is part of the right shuffle at this stage or $q < s$, except those for which $q = p^\nu_{\beta, s+1}$ or $q = f^{-1}(p^\mu_\beta)[s+1]$, or q is β-reserved, for some $\beta \prec \sigma$. If any of a_j, b_j, and c_j becomes σ-marked for some j, σ-mark all three.

For all $\sigma \in 3^{<\omega}$ with $|\sigma| \leq s+1$, if there are no σ-reserved components in ν, choose fresh unmarked components x_l and y_k of ν which are larger than $R(\alpha)[s+1]$, σ-mark those components and say that they are σ-reserved. For each component σ-marked in this manner, if $\beta^\frown 0 \preceq \sigma$ or $\beta^\frown 1 \preceq \sigma$, β-mark that component.

Finally, for each component q of ν which has been σ-marked at any point for any σ, σ-mark the component $f_{s+1}(q)$ of μ.

This completes the construction. We now pause prior to verifying the construction to give some intuition about the construction.

We focus in particular on how the components of ν and μ behave under the shuffling processes we use, to give some intuition as to how the construction proceeds. We will need to check that ν and μ are isomorphic and rigid, which we achieve by making ν and μ rigid and isomorphic at each stage of the construction, and arranging that the infinite components of ν and μ are isomorphic. This is achieved by splitting the construction into odd/even recovery stages.

We first consider what happens if $\sigma^\frown 1 \prec TP$ for some σ of length n. In that case we must focus on the last $\sigma^\frown 0$-stage of the construction, and what happens at the $\sigma^\frown 1$-stages that follow it.

So suppose that that $t_0 + 1$ is a $\sigma^\frown 0$-stage, and that $t_1 + 1 < t_1 + 2$ are the next two σ-stages which are recovery stages for N_n, and that each is a $\sigma^\frown 1$-stage. For simplicity, assume $t_0 + 1$ is the final σ-initialization stage of the construction, and hence an even ν-recovery stage for N_n. In addition, suppose $p^\nu_{\sigma, t_0} = z$ so that $\nu(z)[t_0] = \{z\}$. Note that $t_0 + 1$ and $t_1 + 1$ are odd and even ν-recovery stages for N_n, respectively.

The following diagram indicates the shuffles which will be performed in ν at stages $t_0 + 1, t_1 + 1$ and $t_2 + 1$. We include only the members of $\widehat{\nu}_n[s]$ for $|\sigma| = \nu$ in the diagram.

The symbols $\star, \star\star$, and $\star\star\star$ are used to indicate the elements enumerated into a member of $\widehat{\nu}_n$ from a component which is part of the shuffle, but which is not part of $\widehat{\nu}_n$.

Each arrow in the diagram is from the set $\nu(q)[t]$ to the set $\nu(q)[s]$ at a later stage of the construction.

The solid arrows are from $\nu(q)[t]$ to $\nu(q)[t+1]$, where $t+1$ is a stage at which q is part of a shuffle. In the first two rows, the components are listed in the order in which they occur in $\widehat{\nu}_n[t_0+1]$. In the first row, we indicate the members of the components at the end of stage t_0, and in the second, we indicate their members at the end of stage t_0+1. Likewise, the third and fourth rows indicate the same information for the components x_1, z, x_0 which make up $\widehat{\nu}_n[t_1+1]$, and the fifth and sixth row correspond to $\nu_n[t_2+1]$.

The dashed arrows are between $\nu(q)[t+1]$ and $\nu(q)[s]$, where $t+1$ and $s+1$ are consecutive stages at which q is part of a shuffle in ν.

$$\begin{tikzcd}
\textbf{stage } t_0 & \{x_0\}, \arrow[d] & \{z\}, \arrow[d] & \{y_0\} \arrow[d] \\
\textbf{stage } t_0+1 & \{x_0, z\}, \arrow[drr, dashed] & \{z, y_0\}, \arrow[d, dashed] & \{y_0\} \cup \star \\
\textbf{stage } t_1 & \{x_1\}, \arrow[d] & \{z, y_0\}, \arrow[d] & \{x_0, z\} \arrow[d] \\
\textbf{stage } t_1+1 & \{x_1, z, y_0\}, & \{z, y_0, x_0\}, \arrow[d, dashed] & \{x_0, z\} \cup \star\star \\
\textbf{stage } t_2 & \{x_2\}, \arrow[d] & \{z, y_0, x_0\}, \arrow[d] & \{y_2\} \arrow[d] \\
\textbf{stage } t_2+1 & \{x_2, z, y_0, x_0\}, & \{z, y_0, x_0, y_2\}, & \{y_2\} \cup \star\star\star
\end{tikzcd}$$

The following diagram shows how the corresponding shuffles carried out on μ proceed, with the same conventions as in the first diagram. Here we use $\dagger, \dagger\dagger, \dagger\dagger\dagger$ in the role occupied by $\star, \star\star, \star\star\star$ in the previous diagram, since the sets they represent are different. Note that each component used in the shuffle in μ have the same members as the corresponding component used in ν, as in Remark 3.3.

$$\begin{tikzcd}
\textbf{stage } t_0 & \{x_0\}, \arrow[d] & \{z\}, \arrow[d] & \{y_0\} \arrow[d] \\
\textbf{stage } t_0+1 & \dagger \cup \{x_0\}, & \{x_0, z\}, \arrow[dr, dashed] & \{z, y_0\} \arrow[dl, dashed] \\
\textbf{stage } t_1 & \{x_1\}, \arrow[d] & \{z, y_0\}, \arrow[d] & \{x_0, z\} \arrow[d] \\
\textbf{stage } t_1+1 & \dagger\,\dagger \cup \{x_1\}, & \{x_1, z, y_0\}, & \{z, y_0, x_0\}, \arrow[dl, dashed] \\
\textbf{stage } t_2 & \{x_2\}, \arrow[d] & \{z, y_0, x_0\}, \arrow[d] & \{y_2\} \arrow[d] \\
\textbf{stage } t_2+1 & \dagger\,\dagger\,\dagger \cup \{x_2\}, & \{x_2, z, y_0, x_0\}, & \{z, y_0, x_0, y_2\}
\end{tikzcd}$$

If this shuffling pattern is repeated, it will lead to $p^\nu_\sigma[t_0] = z$ being a member of $\widehat{\nu}_n$ at infinitely many stages. Likewise $f(p^\nu_\sigma)[t_0] = z$ will be a member of $\widehat{\mu}_n$ infinitely often. These two components will have the same members at the end of the construction, which is necessary to ensure ν and μ are isomorphic.

On the other hand, consider the case that $\sigma^\frown 0 \prec TP$. Suppose $t_0 + 1$ is a σ-initialization stage and that $p^\nu_{\sigma,t_0} = z$ as above, but that the next σ-stage $t_1 + 1$ which is a recovery stage for N_n is a $\sigma^\frown 0$-stage and hence an even μ-recovery stage for N_n. Then the components in $\widehat{\nu}_n$ at stages $t_0 + 1$ and $t_1 + 1$ are indicated by the following diagram with similar conventions to the previous diagram:

$$\begin{tikzcd}
\textbf{stage } t_0 & \{x_0\}, \arrow[d] & \{z\}, \arrow[d] & \{y_0\} \arrow[d] \\
\textbf{stage } t_0+1 & \{x_0, z\}, \arrow[dr, dashed] & \{z, y_0\}, & \{y_0\} \cup \star \\
\textbf{stage } t_1 & \{x_1\}, \arrow[d] & \{x_0, z\}, \arrow[d] & \{y_1\} \arrow[d] \\
\textbf{stage } t_1+1 & \{x_1, x_0, z\}, & \{x_0, z, y_1\}, & \{y_1\} \cup \star\star
\end{tikzcd}$$

Note that the component $p^\nu_\sigma[t_0] = z$ is not part of $\widehat{\nu}_n[t_1 + 1]$.

Indeed, if $s + 1$ is a $\sigma^\frown 0$-stage and η-recovery stage for N_n, we will see that no component which is part of $\widehat{\nu}_n[t]$ at a stage $t \leq s$ will ever again be used in a shuffle.

As before, we can give a similar diagram showing the components used in $\widehat{\mu}_n$ at stages $t_0 + 1$ and $t_1 + 1$.

In this case, since t_1+1 is a $\sigma^\frown 0$-stage, $h^\nu_\beta[t_0] \not\subseteq h^\nu_\beta[t_1]$. Because of the structure of the shuffles used, there are in fact only two possible ways in which $h^\nu_\beta[t_1]$ can extend $h^{nu}_\beta[t_0]$. The consequence of this which we will use is that if p_σ is the component of ρ_e for which $p_\sigma = h^\nu_\beta(p^\nu_\sigma)[t_0]$, then $p_\sigma = h^\mu_\beta(p^\mu_\sigma)[t_1]$. This argument can be repeated at future $\sigma^\frown 0$-stages to see that $\rho_e(p_\sigma)$ is in fact an infinite component, but that no component will be part of $\widehat{\nu}_n$ or $\widehat{\mu}_n$ infinitely often. This will suffice to show that ρ_e is not isomorphic to ν and μ. We will now verify that the construction succeeds.

$$\begin{tikzcd}
\textbf{stage } t_0 & \{x_0\}, \arrow[d] & \{z\}, \arrow[d] & \{y_0\} \arrow[d] \\
\textbf{stage } t_0+1 & \dagger\cup\{x_0\}, & \{x_0,z\}, \arrow[d, dashed] & \{z,y_0\} \\
\textbf{stage } t_1 & \{x_1\}, \arrow[d] & \{x_0,z\}, \arrow[d] & \{y_1\} \arrow[d] \\
\textbf{stage } t_1+1 & \dagger\dagger\cup\{x_1\}, & \{x_1,x_0,z\}, & \{x_1,z,y_1\},
\end{tikzcd}$$

It is not immediately clear that there is a unique isomorphism $f_s\colon \nu[s] \to \mu[s]$ at each stage s of the construction. That will be verified by induction on s, and we therefore impose the inductive hypothesis as an assumption for many of our lemmas until we are ready to complete the proof.

Lemma 3.6. *Fix some $\sigma \in 3^{<\omega}$ of length $n \leq s$. At the end of each stage t such that $n \leq t \leq s+1$ there are σ-reserved components x_l and y_k which have never been part of a shuffle, and which are σ-marked.*

Proof. At the beginning of the stage t clean-up phase, any component involved in the shuffle is declared no longer σ-reserved. At the end of the clean-up phase, fresh σ-reserved components are chosen.

□

Lemma 3.7. *Suppose $s+1$ is a σ-initialization stage, where $|\sigma| = n$, and that $\widehat{\nu}_n[s+1]$ and $\widehat{\nu}_n[s+1]$ are nonempty.*

If $\widehat{\nu}_n[s+1] = u_1, u_2, u_3$ and $\widehat{\mu}_n[s+1] = v_1, v_2, v_3$, then for each $i \leq 3$ we have $\nu(u_i)[s] = \mu(v_i)[s]$.

Proof. Because $s+1$ is a σ-initialization stage, none of the components which are part of $\widehat{\nu}_n[s+1]$ and $\widehat{\mu}_n[s+1]$ have been part of a shuffle before, so we have $f(p^\nu_\sigma)[s] = p^\mu_\sigma[s]$. Thus $\widehat{\nu}_n[s+1]$ is of the form x_l, p^ν_σ, y_k, whereas $\widehat{\mu}_n[s+1]$ is x_l, p^ν_σ, y_k, and $\nu(x_l)[s] = \mu(x_l)[s]$, $\nu(p^\nu_\sigma)[s] = \mu(p^\mu_\sigma)[s]$, $\nu(y_k)[s] = \mu(y_k)[s]$, as required. □

Lemma 3.8. *Let $s+1$ be some stage of the construction such that for each $t < s+1$, there is a unique isomorphism $f_t\colon \nu[t] \to \mu[t]$.*

Let σ be of length n. Suppose that $t < s+1$ is a σ-stage which is a recovery stage for N_n at which we attempt to meet a requirement of the form P_i for some $i > n$, or of form S_τ for some $\tau \succ \sigma$. Suppose that $s+1$ is the next σ-stage which is a recovery stage for N_n after stage t, and that $s+1$ is not a σ-initialization stage.

Then $p^\nu_\sigma[t] = p^\nu_\sigma[s]$, and $p^\mu_\sigma[t] = p^\mu_\sigma[s]$, and if q is a member of $\widehat{\nu}_n[t]$ and $\widehat{\nu}_n[s+1]$, then $f(q)[t] = f(q)[s]$.

Finally, if $\widehat{\nu}_n[s+1] = u_1, u_2, u_3$ and $\widehat{\mu}_n[s+1] = v_1, v_2, v_3$, then for each $i \leq 3$ we have $\nu(u_i)[s] = \mu(v_i)[s]$.

Proof. Let $s+1$ be a σ-stage which is a recovery stage for N_n, but not a σ-initialization stage.

We will work by induction on s. Consider the most recent σ-stage $t < s+1$. If t is not a σ-initialization stage, then we may apply the lemma. Otherwise Lemma 3.7 applies. In either case, if $\widehat{\nu}_n[t] = u_1, u_2, u_3$ and $\widehat{\mu}_n[t] = v_1, v_2, v_3$, then for each

$i \leq 3$ we have $\nu(u_i)[t-1] = \mu(v_i)[t-1]$. So we have $f_t(u_i) = v_{i+1}$ for $i = 1$ and $i = 2$, by Remark 3.3.

We get one of the following cases depending on what kind of recovery stage t is.

If t is an even ν-recovery stage for N_n, suppose $\widehat{\nu}_n[t] = x_l, p^\nu_\sigma[t-1], y_k$. Then $p^\nu_\sigma[t-1] = p^\nu_\sigma[t]$, and $p^\mu_\sigma[t] = f_t(x_l)$.

If t is an even μ-recovery stage for N_n, suppose $\widehat{\mu}_n[t] = x_l, p^\mu_\sigma[t-1], y_k$. Then $p^\mu_\sigma[t-1] = p^\mu_\sigma[t]$, and $p^\nu_\sigma[t] = f_t^{-1}(y_k)$.

If t is an odd ν-recovery stage for N_n, suppose $\widehat{\nu}_n[t] = x_l, p^\nu_\sigma[t-1], x_{\hat{l}}$. Then $p^\nu_\sigma[t-1] = p^\nu_\sigma[t]$, and $p^\mu_\sigma[t] = f_t(p^\mu_{\sigma,t-1}) = x_l$.

If t is an odd μ-recovery stage for N_n, suppose $\widehat{\mu}_n[t] = y_{\hat{k}}, p^\mu_\sigma[t-1], y_k$. Then $p^\mu_\sigma[t] = p^\mu_\sigma[t]$ and $p^\nu_\sigma[t] = f_t^{-1}(p^\mu_{\sigma,t-1}) = y_k$.

Now we must consider what happens between stages t and $s+1$. Let $t < t_0 < s+1$. Then t_0 cannot be a θ-stage for $\theta = \sigma^\frown 1$ or $\sigma^\frown 0$.

Furthermore, t_0 cannot be a θ-stage if $\theta <_L \sigma$ and $|\theta| = n$, because $s+1$ is not a σ-initialization stage.

If t_0 is a θ-stage for some $\theta >_L \sigma$ such that $|\theta| = n$, or for $\theta = \sigma^\frown 2$, then $p^\nu_{\sigma,t_0} = p^\nu_{\sigma,t_0-1}$ $p^\mu_{\sigma,t_0} = p^\mu_{\sigma,t_0-1}$, and no elements are enumerated into $\nu(p^\nu_{\sigma,t_0})$ at stage t_0, so $f_{t_0}(p^\nu_{\sigma,t_0}) = f_{t_0-1}(p^\nu_{\sigma,t_0})$.

It follows by induction on t_0 that $p^\nu_\sigma[t] = p^\nu_\sigma[s]$ and $p^\mu_\sigma[t] = p^\mu_\sigma[s]$. Furthermore, $\nu_s(p^\nu_{\sigma,s}) = \nu_t(p^\nu_{\sigma,t})$ and $\mu_s(p^\mu_{\sigma,s}) = \mu_t(p^\mu_{\sigma,t})$.

If $\widehat{\nu}_n[s+1]$ and $\widehat{\nu}_n[s+1]$ are empty, there is nothing to show. So assume that this is not the case.

If $s+1$ is an even ν-recovery stage for N_n, $\widehat{\nu}_n[s+1]$ is of the form $x_l, p^\nu_\sigma[s], y_k$, and $\widehat{\mu}_n[s+1] = x_l, f(p^\nu_\sigma)[s], y_k$, and since x_l and y_k are fresh components it follows that each component of $\widehat{\nu}_n[s+1]$ has the same members as the corresponding component of $\widehat{\mu}_n[s+1]$. The case for an even μ-recovery stage is essentially the same.

If $s+1$ is an odd ν-recovery stage for N_n, then $\widehat{\nu}_n[t]$ is of the form $x_{\hat{l}}, p^\nu_\sigma[t-1], y_{\hat{k}}$, and $\widehat{\mu}_n[t]$ is $x_{\hat{l}}, f(p^\nu_\sigma)[t-1], y_{\hat{k}}$. Thus $\nu(x_{\hat{l}})[t] = \{x_{\hat{l}}\} \cup \nu(p^\nu_\sigma)[t-1]$ and $\mu(f(p^\nu_\sigma)[t-1])[t] = \mu(f(p^\nu_\sigma))[t-1] \cup \{x_{\hat{l}}\}$. Since $p^\mu_\sigma[t] = f(p^\nu_\sigma)[t-1]$ and $\nu(p^\nu_\sigma)[t-1] = \mu(f(p^\nu_\sigma))[t-1]$, we have $\nu(x_{\hat{l}})[t] = \mu(p^\mu_\sigma)[t]$. That is, $f(x_{\hat{l}})[t] = p^\mu_\sigma[t]$. Thus from our above work, we see that $f(x_{\hat{l}})[s] = p^\mu_\sigma[s]$. Then $\widehat{\nu}_n[s+1] = x_l, p^\nu_\sigma, x_{\hat{l}}$ and $\widehat{\mu}_n[s+1] = x_l, f(p^\nu_\sigma)[s], p^\mu_\sigma[s]$. Since $f(x_{\hat{l}})[s] = p^\mu_\sigma[s]$ and x_l is a fresh component, it is clear that each of the components of $\widehat{\nu}_n[s+1]$ has the same members as the corresponding component of $\widehat{\mu}_n[s+1]$.

On the other hand, if $s+1$ is an odd μ-recovery stage for N_n, then $\widehat{\nu}_n[t]$ is of the form $x_{\hat{l}}, f^{-1}(p^\mu_\sigma)[t-1], y_{\hat{k}}$, and $\widehat{\mu}_n[t]$ is $x_{\hat{l}}, p^\mu_\sigma[t-1], y_{\hat{k}}$. But $p^\mu_\sigma[t] = p^\mu_\sigma[t-1]$, and thus $\nu(x_{\hat{l}})[t] = \mu(p^\mu_\sigma)[t]$, that is, $f^{-1}(p^\mu_\sigma)[t] = x_{\hat{l}}$. As in the ν-recovery case above, $\nu(x_{\hat{l}})[s] = \mu(p^\mu_\sigma)[s]$, and $p^\nu_\sigma[s] = x_{\hat{l}}$. Once again, each of the components of $\widehat{\nu}_n[s+1]$ has the same members as the corresponding component of $\widehat{\mu}_n[s+1]$. ☐

Corollary 3.9. *Let $s+1$ be some stage of the construction such that at each stage $t < s+1$ there is a unique isomorphism $f_t \colon \nu[t] \to \mu[t]$.*

Suppose $s+1$ is a σ-stage, where σ is of length n, and a recovery stage for N_n.

The only components which can be in $\widehat{\nu}_n[s+1]$ are the σ-reserved components, $p^\nu_{\sigma,s}$, and $f^{-1}(p^\mu_\sigma)[s]$.

Likewise the only components which can be in $\widehat{\mu}_n[s+1]$ are the σ-reserved components, $p^\mu_{\sigma,s}$, and $f(p^\nu_\sigma)[s]$.

Lemma 3.10. *Let $s+1$ be some stage of the construction such that at each stage $t < s+1$ there is a unique isomorphism $f_t \colon \nu[t] \to \mu[t]$.*

Then $\nu[s+1]$ and $\mu[s+1]$ are isomorphic.

Proof. For each n, $\widehat{\nu}_n[s+1]$ is nonempty if and only if $\widehat{\mu}_n[s+1]$ is, and if $\widehat{\nu}_n[s+1] = u_1, u_2, u_3$ and $\widehat{\mu}_n[s+1] = v_1, v_2, v_3$ then $\nu(u_i)[s] = \mu(v_i)[s]$ for $i \leq 3$. Furthermore, no component of the form $a_{\langle e,j\rangle}$, $b_{\langle e,j\rangle}$, $c_{\langle e,j\rangle}$, or d_j which appears in the right and left shuffles at this stage has previously been part of a shuffle, and therefore each of these components contains the same members in ν as in μ.

Thus the elements shuffled meet the condition of 3.3, and therefore $\nu[s+1]$ and $\mu[s+1]$ are isomorphic. □

Corollary 3.11. *Suppose that for each $t < s+1$ there is a unique isomorphism $f_t \colon \nu[t] \to \mu[t]$. Fix $n \leq s$, and let $\alpha_{s+1} \upharpoonright n = \sigma$. Assume $s+1$ is not a σ-initialization stage.*

If q is a component in $\widehat{\eta}_n[s+1]$, where η is either ν or μ, then either q is a σ-reserved component, or at the most recent σ-stage $t_0 < s+1$ which is a recovery stage for N_n, q was a component in $\widehat{\eta}_n[t_0]$.

If, in addition, q is part of $\widehat{\eta}_m[t]$ at some stage $t < s+1$, then $m = n$ and $\alpha_t \upharpoonright m = \sigma$.

Proof. The first claim follows from Corollary 3.9.

For the second claim, suppose that q is a component of $\widehat{\eta}_n[s+1]$, where η is either μ or ν. The only way that q can be a part of $\widehat{\eta}_n[s+1]$ if it has previously been part of a shuffle is if q is a member of $\widehat{\eta}_n[t_0]$ at the most recent σ-stage $t_0 < s+1$ which is a recovery stage for N_n. Let $t < s+1$ be the first stage at which q is part of a shuffle. By reverse induction on t_0, it follows that q is part of $\widehat{\eta}_n[t]$, and that t is a σ-stage. Furthermore, either q is a σ-reserved component, or t is a σ-initialization stage, in which case q is one of $p^{\nu}_{\sigma,t-1}$ or $f^{-1}(p^{\mu}_{\sigma})[t-1]$ if $\eta = \nu$, and one of $p^{\mu}_{\sigma,t-1}$ or $f(p^{\nu}_{\sigma})[t-1]$ if $\eta = \mu$. In either case, q is not part of $\widehat{\eta}_m[t]$ for any $m \neq n$, which suffices to prove the result. □

Remark 3.12. For each q, $\nu(q)[s+1] = \{q\}$ if and only if $\mu(q)[s+1] = \{q\}$. If this is the case, then $q \in \nu(x)[s+1]$ only for $x = q$.

Our next goal is to check that the shuffling action carried out during the construction preserves rigidity of ν (and μ) at each stage of the construction. We begin by establishing some limits on how elements are propagated between components of ν by the shuffling process.

Lemma 3.13. *Let $s+1$ be some stage of the construction such that for each $t < s+1$, there is a unique isomorphism $f_t \colon \nu[t] \to \mu[t]$.*

Suppose that p is a component of ν which is part of $\widehat{\nu}_n[t_0]$ at some σ-stage $t_0 \leq s+1$, where $\sigma = \alpha_{t_0} \upharpoonright n$.

Suppose that q is a component which is part of the right shuffle at stage $s+1$, and that $p \in \nu(q)[s+1]$.

Then $s+1$ is a σ-stage and either

(1) *q is a member of $\widehat{\nu}_n[s+1]$,*
(2) *or q is the rightmost member of $\widehat{\nu}_m[s+1]$ for the greatest $m < n$ such that $\widehat{\nu}_m[s+1]$ is nonempty, and furthermore $p \notin \nu(q)[s]$,*
(3) *or q is the component of form $c_{\langle e,j\rangle}$ or d_j involved in the right shuffle at this stage.*

Proof. Work by induction on s.

Let p be a component in $\widehat{\nu}_n[t_0]$ as stated, and assume t_0 is the first stage that p is ever part of the right shuffle. Suppose that for $t_0 \leq t < s+1$, if q is a component used in the right shuffle at stage t and $p \in \nu(q)[t]$, then t is a σ-stage, and q satisfies one of the conditions listed above. Note that for each m such that $\widehat{\nu}_m[t]$ is nonempty, the rightmost member of $\widehat{\nu}_m[t]$ can never again be part of a shuffle after this stage, and nor can the component of form $c_{\langle e,j\rangle}$ or d_j which is part of the right shuffle at stage t.

Thus if $p \in \nu(q)[s]$ and q is part of the right shuffle at stage $s+1$, q is a member of $\widehat{\nu}_n[s+1]$, and furthermore $\alpha_{s+1} \upharpoonright n = \sigma$.

Now, suppose that q is part of the right shuffle at stage $s+1$ and that $p \in \nu(q)[s+1]$. Let us suppose that q is not a member of $\widehat{\nu}_n[s+1]$. Then the position of q in the right shuffle must be immediately to the left of a component v such that $p \in \nu(v)[s]$. Now applying our inductive hypothesis to v, we see that v is a member of $\widehat{\nu}_n[s+1]$. It must be the leftmost, and q is the rightmost component of $\widehat{\nu}_m[s+1]$ for the greatest $m < n$ such that $\widehat{\nu}_m[s+1]$ is nonempty, or, if all are empty, q is whichever component of the form $c_{\langle e,j\rangle}$ or d_j is part of the shuffle at this stage.

This concludes the induction. □

Lemma 3.14. *Suppose $|\sigma| = n$, and that $s+1$ is some stage of the construction such that for each $t < s+1$, there is a unique isomorphism $f_t\colon \nu[t] \to \mu[t]$.*

Suppose $t_0 < s+1$ is the most recent σ-initialization stage, $p = p^{\nu}_{\sigma,t_0}$, and that $\tau \neq \sigma$.

Then $p \notin \nu(p^{\nu}_{\tau})[s+1]$, and $p \notin \mu(p^{\mu}_{\tau})[s+1]$.

However, if $s+1$ is not a σ-initialization stage then $p \in \nu(p^{\nu}_{\sigma})[s+1]$ and $p \in \mu(p^{\mu}_{\sigma})[s+1]$.

Proof. Suppose that $t_0 < t+1 \leq s+1$, and that $\tau \neq \sigma$. Suppose v is one of $p^{\nu}_{\tau,s+1}$ and $f^{-1}(p^{\mu}_{\tau})[s+1]$. We will check that $p \notin \nu_{s+1}(v)$.

Suppose not, and indeed that $p \in \nu_{t+1}(v)$ but $p \notin \nu_t(v)$ for some $t \leq s$. Then, by Lemma 3.13, v is part of the right shuffle at stage $t+1$, $\tau \prec \sigma$, and v is the rightmost member of $\widehat{\nu}_{|\tau|}[t+1]$. So v is never again part of the right shuffle. However, if $p^{\nu}_{\tau,s+1}$ and $f^{-1}(p^{\mu}_{\tau})[s+1]$ have been part of the right shuffle before the end of stage $s+1$, they are part of $\widehat{\nu}_m[t_0+1]$ at the most recent τ-stage $t_0+1 < s+1$ which is a recovery stage for N_m, and furthermore neither is the rightmost element of $\widehat{\nu}_m[s+1]$. So v cannot be either of these elements.

We now show that $p \in \nu(p^{\nu}_{\sigma})[s+1]$ and $p \in \mu(p^{\mu}_{\sigma})[s+1]$.

Clearly $p \in \nu(p^{\nu}_{\sigma})[t_0]$ and $p \in \mu(p^{\mu}_{\sigma})[t_0]$. Since there are no σ-initialization stages between t_0 and $s+1$ it suffices to note that at each stage t such that $t_0 < t \leq s+1$, either both of $p^{\nu}_{\sigma}[t]$ and $p^{\mu}_{\sigma}[t]$ are part of the shuffle, in which case $\nu(p^{\nu}_{\sigma})[s] \subseteq \nu(p^{\nu}_{\sigma})[s+1]$ and $\mu(p^{\mu}_{\sigma})[s] \subseteq \mu(p^{\mu}_{\sigma})[s+1]$, or neither of the precious components is part of the shuffle, in which case we have $p^{\nu}_{\sigma}[t] = p^{\nu}_{\sigma}[t+1]$ and $p^{\mu}_{\sigma}[t] = p^{\mu}_{\sigma}[t+1]$.

□

We are now ready to prove that there is a unique isomorphism $f_s\colon \nu[s] \to \mu[s]$ at each stage s of the construction. We will in fact prove a slightly stronger statements: that if x and y are distinct components of ν then $\nu(x)[s] \not\subseteq \nu(y)[s]$.

Lemma 3.15. *Suppose $s+1$ is some stage of the construction such that for each $t < s+1$, there is a unique isomorphism $f_t\colon \nu[t] \to \mu[t]$.*

Suppose that $\nu(x)[s] \not\subseteq \nu(y)[s]$ *for each* $x \neq y$.

Then $\nu(x)[s+1] \not\subseteq \nu(y)[s+1]$ *for each* $x \neq y$.

Thus by induction on s, *there is a unique isomorphism* $f_s \colon \nu[s] \to \mu[s]$ *at every stage* s *of the construction.*

Proof. Let x and y be components of ν.

If y is not a part of the right shuffle at stage $s+1$, then $\nu(y)[s] = \nu(y)[s+1]$ and by assumption $\nu(x)[s] \not\subseteq \nu(y)[s]$, so $\nu(x)[s+1] \supseteq \nu(x)[s] \not\subseteq \nu(y)[s+1]$. So we may assume y is part of the right shuffle.

Suppose that the components used in the right shuffle are $n_0, n_1, \ldots, n_d$. Let $y = n_i$ be a component in the right shuffle.

Firstly note that if n_i is being shuffled for the first time at this stage, then n_i will be a member of $\nu(n_i)[s+1]$ and $\nu(n_{i-1})[s+1]$, but not of $\nu(q)[s+1]$ for any other q. So we need only check that $\nu(n_i)[s+1] \not\subseteq \nu(n_{i-1})[s+1]$ and that $\nu(n_{i-1})[s+1] \not\subseteq \nu(n_i)[s+1]$ to conclude that neither of $\nu(n_{i-1})[s+1]$ and $\nu(n_i)[s+1]$ can be a subset of $\nu(q)[s+1]$ for any q. But $\nu(n_{i+1})[s] \not\subseteq \nu(n_{i-1})[s]$ and $\nu(n_{i-1})[s] \not\subseteq \nu(n_{i+1})[s]$ by hypothesis, and neither contains n_i. Thus it follows that we have $\nu(n_i)[s+1] \not\subseteq \nu(n_{i-1})[s+1]$ and $\nu(n_{i-1})[s+1] \not\subseteq \nu(n_i)[s+1]$.

On the other hand, if $q \neq n_i$ is a component of ν, and n_i has never been shuffled before stage $s+1$, then it cannot be the case that $\nu(q)[s+1] \subseteq \nu(n_i)[s+1]$ or that $\nu(q) \subseteq \nu(n_{i-1})[s+1]$, because $n_i \notin \nu(q)[s]$. In the former case we have $\nu(n_i)[s+1] = \nu(n_{i+1})[s] \cup \{n_i\}$, and in the latter case $\nu(n_{i-1})[s+1] = \nu(n_{i-1})[s] \cup \{n_i\}$ and hence $\nu(q)[s] \subseteq \nu(n_{i+1})[s]$ or $\nu(q)[s] \subseteq \nu(n_{i-1})[s]$, contrary to hypothesis.

Thus we need only check that $\nu(x)[s+1] \not\subseteq \nu(y)[s+1]$ in the case that x and y are components of ν such that:

1. x and y are both part of the right shuffle at stage $s+1$,
2. x and y have each been part of the right shuffle at a previous stage of the construction,
3. and x and y each occupy a position in the shuffle immediately to the left of a component which has been part of the right shuffle at a previous stage of the construction.

So suppose that for some i and j, we have components n_i and n_j satisfying these conditions, and that $\nu(n_j)[s+1] \subseteq \nu(n_i)[s+1]$.

Note that Lemma 3.13 shows that if $n_j \in \nu(n_i)[s+1]$, then n_j is a member of $\widehat{\nu}_m[t]$ for some m at some stage $t \leq s+1$. Furthermore, either n_i is either of the form $c_{\langle e,j \rangle}$ or d_j, and is not part of a shuffle before stage $s+1$, or is a member of $\widehat{\nu}_n[s+1]$ for some $n \leq m$. In this latter case let $\alpha_{s+1} \upharpoonright n = \sigma$ and $\alpha_t \upharpoonright m = \tau$. We have $\sigma \preceq \tau$.

The case $\sigma = \tau$ (i.e. $m = n$) cannot occur. This is because if $\widehat{\nu}_n[s+1]$ contains two nonfresh components, at least one must be immediately to the left of a fresh component in the shuffle, because the leftmost component of $\widehat{\nu}_n[s+1]$ is σ-reserved, and the component immediately to the right of the rightmost component of $\widehat{\nu}_n[s+1]$ is either a θ-reserved component for some θ, or is of the form $a_{\langle e,j \rangle}$ or d_j.

Likewise the case $\sigma \prec \tau$ (i.e. $n < m$) cannot occur. If it were, then by Lemma 3.13, n_i is the rightmost component in $\widehat{\nu}_n[s+1]$, and $n_j \notin \nu(n_i)[s]$. Now, suppose that $w \in \nu(n_j)[s+1]$, but that $w \neq n_j$. Thus w entered $\nu(n_j)$ during a shuffle at some earlier stage. Applying Lemma 3.13 to w we see that $w \in \nu(u)[t_0]$ for some component u which is part of $\widehat{\nu}_l[t_0]$ for some $l \geq m > n$ and $t_0 \leq s$. Applying Lemma 3.13 to n_i in turn implies $w \notin \nu(n_i)[s]$. Thus the only way that $\nu(n_j)[s+1] \subseteq$

$\nu(n_i)[s+1] = \nu(n_i)[s] \cup \nu(n_{i+1})[s]$ can possibly be true is if $\nu(n_j)[s] \subseteq \nu(n_{i+1})[s]$, contrary to hypothesis.

This is sufficient to complete the proof. $\square$

Lemma 3.16. *If the components involved in the right shuffle at stage* $s+1$ *are* $n_0, \ldots, n_d$, *and we define* $n_{-1} = n_d$, *then for* $0 \leq i \leq k$, $\nu(q)[s] \subseteq \nu(n_i)[s+1]$ *only for* $q = n_i$ *and* $q = n_{i-1}$.

Proof. Fix some component n_i involved in the right shuffle.

If n_i is part of the shuffle for the first time at stage $s+1$, then $n_i \in \nu(n_i)[s+1]$ and $n_i \in \nu(n_{i-1})[s+1]$, but this element belongs to no other components of ν. So $\nu(n_i)[s] \subseteq \nu(n_j)[s+1]$ only for $j = i$ and $j = i-1$.

If n_i is part of $\widehat{\nu}_n[s+1]$ for some n, and n_j is a component in the right shuffle which is not part of $\widehat{\nu}_n[s+1]$, but for which $\nu(n_i)[s] \subseteq \nu(n_j)[s+1]$, then Lemma 3.13 shows that n_j is either the rightmost member of $\widehat{\nu}_m[s+1]$ for the greatest $m < n$ such that $\widehat{\nu}_m[s+1]$ is nonempty, or in case there is no such m, n_j is the component of form $c_{\langle e,j \rangle}$ or d_e involved in the right shuffle at this stage. In either case, $\nu(n_j)[s] \cap \nu(n_i)[s] = \emptyset$ whereas $\nu(n_i)[s] \subseteq \nu(n_j)[s+1] = \nu(n_j)[s] \cup \nu(n_{j+1})[s]$. Thus $\nu(n_{j+1})[s] \subseteq \nu(n_i)[s]$, and by Lemma 3.15, $j+1 = i$.

So it suffices to assume that n_i has been used in the right shuffle at a stage prior to $s+1$, and to consider the case in which n_i and n_j both belong to $\widehat{\nu}_n[s+1]$ for the same n.

If $s+1$ is an even ν- or μ-recovery stage for N_n, $\widehat{\nu}_n[s+1]$ is of the form x_l, p, y_k, where x_l and y_k are $\alpha_{s+1} \restriction n$-reserved. In this case, we must have $n_i = p$. Let v be the component immediately to the right of y_k in the right shuffle. Then $v \in \nu(y_k)[s+1]$, whereas $v \notin \nu(p)[s+1]$ by Lemma 3.13. This is all that is necessary to check in this case.

Suppose $s+1$ is an odd ν-recovery stage for N_n. Then let $t+1 < s+1$ be the most recent $\alpha_{s+1} \restriction n$ stage which is a recovery stage for N_n, and suppose that $\widehat{\nu}_n[t+1]$ is $x_{\hat{l}}, p^{\nu}_{\sigma}[t], y_{\hat{k}}$. Then $\widehat{\nu}_n[s+1]$ is of the form $x_k, p^{\nu}_{\sigma}[s], x_{\hat{k}}$, where x_l is $\alpha_{s+1} \restriction n$-reserved, and $p^{\nu}_{\sigma}[t] = p^{\nu}_{\sigma}[s]$. As noted in the proof of Lemma 3.15, we have $\nu(p^{\nu}_{\sigma})[s+1] \not\subseteq \nu(x_{\hat{l}})[s]$. In addition, we may note that $x_{\hat{l}} \in \nu(x_{\hat{l}})[s]$, whereas $x_{\hat{l}} \notin \nu(x_l)[s] \cup \nu(p^{\nu}_{\sigma})[s] = \nu(x_l)[s+1]$. So $\nu(x_{\hat{l}})[s] \not\subseteq \nu(x_l)[s+1]$. This is sufficient to complete this case.

Suppose $s+1$ is an odd μ-recovery stage for N_n. Then let $t+1 < s+1$ be the most recent $\alpha_{s+1} \restriction n$ stage which is a recovery stage for N_n, and suppose that $\widehat{\nu}_n[t+1]$ is $x_{\hat{l}}, p^{\nu}_{\sigma}[t], y_{\hat{k}}$. Then $\widehat{\nu}_n[s+1]$ is of the form $p^{\nu}_{\sigma}[s], f^{-1}(p^{\mu}_{\sigma})[s], y_k$, where y_k is $\alpha_{s+1} \restriction n$ and $f^{-1}(p^{\mu}_{\sigma})[s] = x_{\hat{l}}$. Note that $x_{\hat{l}} \in \nu(p^{\nu}_{\sigma})[s]$ and $x_{\hat{l}} \in \nu(f^{-1}(p^{\mu}_{\sigma}))[s]$, whereas $x_{\hat{l}} \notin \nu(y_k)[s+1]$, so that $\nu(p^{\nu}_{\sigma})[s] \not\subseteq \nu(y_k)[s+1]$ and $\nu(f^{-1}(p^{\mu}_{\sigma}))[s] \not\subseteq \nu(y_k)[s+1]$. In addition, $y_k \in \nu(p^{\nu}_{\sigma})[s]$, but $y_k \notin \nu_{s+1}(f_s^{-1}(p^{\mu}_{\sigma,s})) = \nu(f^{-1}(p^{\mu}_{\sigma}))[s] \cup \{y_k\}$, establishing that $\nu(p^{\nu}_{\sigma})[s] \not\subseteq \nu_{s+1}(f_s^{-1}(p^{\mu}_{\sigma,s}))$, which completes the proof.

$\square$

Lemma 3.17. *Fix* σ, *and suppose that* $|\sigma| = n$, *and that* q *is a component in* $\widehat{\nu}_n[s+1]$ *or* $\widehat{\mu}_n[s+1]$ *at infinitely many* σ*-stages* $s+1$.

Then $\sigma^\frown 1 \prec TP$.

Proof. Let q be as above and assume q is a component in $\widehat{\eta}_n[s+1]$ at infinitely many σ-stages $s+1$.

It cannot be the case that $\sigma <_L TP \restriction n$ because then there are only finitely many σ-stages.

If $\sigma >_L TP \upharpoonright n$ and q is part of $\widehat{\eta}_n[s+1]$, then there is a stage $t > s+1$ which is a τ-stage for some $\tau <_L \sigma$ such that $|\tau| = n$. After stage t, the next σ-stage which is a recovery for N_n is a σ-initialization stage, and no component in $\widehat{\eta}_n[s+1]$ will be part of the shuffle at that stage or any later stage.

If $\sigma^\frown 2 \prec TP$ then for sufficiently large t, every σ-stage is a $\sigma^\frown 2$ stage, and so $\widehat{\eta}_n[t]$ is empty.

If $\sigma^\frown 0 \prec TP$, let $t_0 + 1 < t_1 + 1 < t_2 + 1$ be three successive $\sigma^\frown 0$-stages such that for $t \geq t_0$, $\alpha_t \upharpoonright n \geq_L \sigma$, and such that we do not attempt to meet S_τ for any $\tau \leq_L \sigma$ at any stage $t > t_0$.

If η is ν, suppose that $t_0 + 1$ is an even μ-recovery stage for N_n, so that $t_1 + 1$ is the first ν-recovery stage for N_n after $t_0 + 1$, and that $t_2 + 1$ is the first μ-recovery stage for N_n after $t_1 + 1$. Suppose that q is in $\widehat{\nu}_n[t_0 + 1]$, but has been shuffled at a stage prior to stage $t_0 + 1$. Then $q = f^{-1}(p^\mu_\sigma)[t_0]$, since this is the only such component in $\widehat{\nu}_n[t_0 + 1]$. Note that for q to also be in $\widehat{\nu}_n[t_1 + 1]$ it must be the case that $q = p^\nu_{\sigma,t_1}$ and hence $q = p^\nu_{\sigma,t_2}$. Note that $f(p^\nu_\sigma)[t_2] \neq p^\mu_{\sigma,t_2}$. The only component in $\widehat{\nu}_n[t_2 + 1]$ which has been part of a shuffle by the end of stage t_2 is $f^{-1}(p^\mu_\sigma)[t_2]$. Thus it follows that q cannot be a member of $\nu_n[t_2 + 1]$.

If η is μ, the argument is similar. Suppose $t_0 + 1$ is an even ν-recovery stage for N_n, so that $t_1 + 1$ is the first μ-recovery stage for N_n after $t_0 + 1$, and that $t_2 + 1$ is the first ν-recovery stage for N_n after $t_1 + 1$. If q is in $\widehat{\mu}_n[t_0 + 1]$, but has been part of a shuffle at an earlier stage, then $q = f(p^\nu_\sigma)[t_0]$. In order for $q \in \widehat{\mu}_n[t_1 + 1]$ to be true, it must be the case that $q = p^\mu_{\sigma,t_1} = p^\mu_{\sigma,t_2} \neq f(p^\nu_\sigma)[t_2]$. But the only component of $\widehat{\mu}_n[t_2 + 1]$ which has been shuffled by the end of stage t_2 is $f(p^\nu_\sigma)[t_2]$.

Thus a component can only be a member of $\widehat{\eta}_n[t]$ for infinitely many t if $\sigma^\frown 1 \prec TP$. □

Lemma 3.18. *Suppose that* $|\sigma| = n$ *and that* $t + 1$ *is a* $\sigma^\frown 0$*-stage, and hence an even* η*-recovery stage, where* η *is either* ν *or* μ*.*

Suppose that $s + 1 > t + 1$ *is a* $\sigma^\frown 1$*- or* $\sigma^\frown 2$*-stage, and that for* $t + 1 < t_0 < s + 1$*,* $\alpha_{t_0} \geq_L \sigma^\frown 1$*.*

If $t + 1$ *is a* ν*-recovery stage for* N_n *then* $p^\nu_{\sigma,t} = p^\nu_{\sigma,s+1}$*. If, in addition,* $s + 1$ *is an even* ν*-recovery stage for* N_n*, then* $p^\mu_{\sigma,t+1} = f(p^\nu_\sigma)[t] = f(p^\nu_\sigma)[s] = p^\mu_{\sigma,s+1}$*.*

If $t + 1$ *is a* μ*-recovery stage for* N_n *then* $p^\mu_{\sigma,t} = p^\mu_{\sigma,s+1}$*. If, in addition,* $s + 1$ *is an even* μ*-recovery stage for* N_n*, then* $p^\nu_{\sigma,t+1} = f^{-1}(p^\mu_\sigma)[t] = f^{-1}(p^\mu_\sigma)[s] = p^\nu_{\sigma,s+1}$*.*

Proof. Suppose that $t + 1$ is an η-recovery stage for N_n, where η is either μ or ν. Then each σ-stage t_0 such that $t + 1 < t_0 \leq s + 1$ which is a recovery stage for N_n is also an η-recovery stage for N_n. There are no stages t_0 such that $t + 1 < t_0 \leq s + 1$ at which we attempt to meet S_τ for any $\tau \leq_L \sigma$, because after such a stage the next σ-stage which is a recovery stage for N_n is a $\sigma^\frown 0$-stage. Thus for $t + 1 < t_0 \leq s + 1$, $p^\eta_{\sigma,t_0} = p^\eta_{\sigma,t_0-1}$. So $p^\eta_{\sigma,t} = p^\eta_{\sigma,s+1}$.

If $s + 1$ is an odd ν-recovery stage for N_n, let $s_1 + 1 < s_2 + 1 < s_3 + 1 \leq s + 1$ be consecutive among σ-stages which are ν-recovery stages for N_n after stage $t+1$, and assume that s_1+1 is an even ν-recovery stage. Suppose that $\widehat{\nu}_n[s_1+1] = x_{\hat{l}}, p^\nu_{\sigma,s_1}, y_{\hat{l}}$. Then $\widehat{\mu}_n[s_1 + 1] = x_{\hat{l}}, f(p^\nu_\sigma)[s_1], y_{\hat{l}}$. We have $f_{s_1+1}(x_{\hat{l}}) = f(p^\nu_\sigma)[s_1] = p^\mu_{\sigma,s_1+1}$. In addition, $f_{s_1+1}(x_{\hat{l}}) = f_{s_2}(x_{\hat{l}})$, and $p^\mu_{\sigma,s_1+1} = p^\mu_{\sigma,s_2}$. Note that $\widehat{\nu}_n[s_2 + 1]$ is of the form $x_l, p^\nu_{\sigma,s_2}, x_{\hat{l}}$, and $\widehat{\mu}_n[s_2 + 1] = x_l, f(p^\nu_\sigma)[s_2], p^\mu_{\sigma,s_2}$ which is the same as $x_l, f(p^\nu_\sigma)[s_2], f(p^\nu_\sigma)[s_1]$. Then

$$p^\mu_{\sigma,s_3+1} = f(p^\nu_\sigma)[s_3] = f(p^\nu_\sigma)[s_2 + 1] = f_{s_2+1}(p^\nu_{\sigma,s_2}) = f(p^\nu_\sigma)[s_1]$$

with the last equality following from the form of $\widehat{\nu}_n[s_2+1]$ and $\widehat{\mu}_n[s_2+1]$. So $p^{\mu}_{\sigma,s_3+1} = p^{\mu}_{\sigma,s_1+1}$, and therefore it follows by induction that $p^{\mu}_{\sigma,t+1} = f(p^{\nu}_{\sigma})[t] = f(p^{\nu}_{\sigma})[s] = p^{\mu}_{\sigma,s+1}$.

The case for μ-recovery is very similar. If $s+1$ is an odd μ-recovery stage for N_n, let $\leq s_1+1 < s_2+1 < s_3+1 \leq s+1$ be consecutive among σ-stages which are μ-recovery stages for N_n after stage $t+1$, where s_1+1 is an even μ-recovery stage. Suppose that $\widehat{\mu}_n[s_1+1] = x_{\hat{l}}, p^{\mu}_{\sigma,s_1}, y_{\hat{l}}$. Then $\widehat{\nu}_n[s_1+1] = x_{\hat{l}}, f^{-1}_{s_1}(p^{\nu}_{\sigma,s_1}), y_{\hat{l}}$. Then $f^{-1}_{s_1+1}(y_{\hat{k}}) = f^{-1}(p^{\mu}_{\sigma})[s_1] = p^{\nu}_{\sigma,s_1+1}$. In addition, $f^{-1}_{s_1+1}(y_{\hat{k}}) = f^{-1}_{s_2}(y_{\hat{k}})$, and $p^{\nu}_{\sigma,s_1+1} = p^{\nu}_{\sigma,s_2}$. So $\widehat{\mu}_n[s_2+1]$ is of the form $y_{\hat{k}}, p^{\nu}_{\sigma,s_2}, y_k$, and $\widehat{\nu}_n[s_2+1] = p^{\nu}_{\sigma,s_2}, f^{-1}(p^{\mu}_{\sigma})[s_2], y_k = f^{-1}(p^{\mu}_{\sigma})[s_1], f^{-1}(p^{\mu}_{\sigma})[s_2], y_k$. Then

$$p^{\nu}_{\sigma,s_3+1} = f^{-1}(p^{\mu}_{\sigma})[s_3] = f^{-1}(p^{\mu}_{\sigma})[s_2+1] = f^{-1}_{s_2+1}(p^{\mu}_{\sigma,s_2}) = f^{-1}(p^{\mu}_{\sigma})[s_1]$$

and it follows by induction that $p^{\nu}_{\sigma,t+1} = f^{-1}(p^{\mu}_{\sigma})[t] = f^{-1}(p^{\mu}_{\sigma})[s] = p^{\nu}_{\sigma,s+1}$.

□

Lemma 3.19. *Suppose that $|\sigma| = n$ and that $\sigma\widehat{\ }1 \prec TP$. Let $t+1$ be the final $\sigma\widehat{\ }0$-stage of the construction.*

If $t+1$ is a ν-recovery stage for N_n then $p^{\nu}_{\sigma}[t]$ is the only component which is in $\widehat{\nu}_n[s+1]$ at infinitely many stages $s+1$, and $f(p^{\nu}_{\sigma})[t]$ is the only component in $\widehat{\mu}_n[s+1]$ at infinitely many stages. In addition $\nu(p^{\nu}_{\sigma,t}) = \mu(f_t(p^{\nu}_{\sigma,t}))$.

If $t+1$ is a μ-recovery stage for N_n then $p^{\mu}_{\sigma}[t]$ is the only component in $\widehat{\mu}_n[s+1]$ at infinitely many stages $s+1$ and $f^{-1}(p^{\mu}_{\sigma})[t]$ is the only component which is in $\widehat{\nu}_n[s+1]$ at infinitely many stages. In addition $\mu(p^{\mu}_{\sigma,t}) = \nu(f^{-1}_t(p^{\mu}_{\sigma,t}))$.

Proof. Suppose that $t+1$ is an η-recovery stage for N_n where η is either μ or ν. Then each σ-stage after t which is a recovery stage for N_n is also an η-recovery stage for N_n. There are no stages $s > t+1$ at which we attempt to meet S_τ for any $\tau \leq_L \sigma$, nor at which $\alpha_s <_L \sigma$, because after such a stage the next σ-stage which is a recovery stage for N_n is a $\sigma\widehat{\ }0$-stage.

If $t+1 < s+1$ are both even η-recovery stages for N_e, the only component that $\widehat{\eta}_n[t+1]$ and $\widehat{\eta}_n[s+1]$ both contain is $p^{\eta}_{\sigma}[t] = p^{\eta}_{\sigma}[s]$. So this is the only component which is included in $\widehat{\eta}_n$ at infinitely many stages of the construction.

If $\eta = \nu$ and $s+1 > t+1$ is a $\sigma\widehat{\ }1$-stage which is an even ν-recovery stage for N_n, then $p^{\mu}_{\sigma,s+1} = f(p^{\nu}_{\sigma})[s] = f(p^{\nu}_{\sigma})[t]$ which is therefore part of $\widehat{\mu}_n[s+1]$ at each such stage. Therefore we have $\nu(p^{\nu}_{\sigma,t})[s] = \mu(p^{\mu}_{\sigma,s+1})[s]$ at such stages and hence $\nu(p^{\nu}_{\sigma,t}) = \mu(f_t(p^{\nu}_{\sigma,t}))$.

On the other hand, at even ν-recovery stages $s+1$, $f_s(p^{\nu}_{\sigma,s})$ is the only non-fresh component in $\widehat{\mu}_n[s+1]$, and therefore that it is the only component which is part of $\widehat{\mu}_n$ infinitely often.

Likewise, if $\eta = \mu$, then at even μ-recovery stages $s+1 > t+1$ for N_n, we have $p^{\nu}_{\sigma,s+1} = f^{-1}(p^{\mu}_{\sigma})[s] = f^{-1}(p^{\mu}_{\sigma})[t]$, which is the only component that is part of $\widehat{\nu}_n$ often, and that $\mu(p^{\mu}_{\sigma,t}) = \nu(f^{-1}_t(p^{\mu}_{\sigma,t}))$. □

Note that because a component of ν or of μ can only grow by participating in a shuffle, the above lemma characterizes the infinite components.

Lemma 3.20. *The structures ν and μ are Friedberg enumerations of the same set, and furthermore for components $x \neq y$ we have $\nu(x) \not\subseteq \nu(y)$, and hence there is a unique homomorphic embedding $f\colon \nu \to \mu$, which is an isomorphism.*

Proof. At each stage s of the construction, $\nu[s]$ and $\mu[s]$ are isomorphic, and if x and y are distinct components of ν, then $\nu(x)[s] \not\subseteq \nu(y)[s]$.

This suffices to show that if x is a finite component of ν, then $\nu(x) \not\subseteq \nu(y)$ for every other component y, and furthermore that there is exactly one component z of μ for which $\mu(z) = \nu(x)$.

So we need only consider the infinite components. Let η and ι each be one of ν and μ (allow that they might be equal), and let x be an infinite component of η. Then x is part of a shuffle at infinitely many stages, so there is a $\sigma\widehat{\ }1 \prec TP$ such that if $|\sigma| = n$, then x is part of $\widehat{\nu}_n[s]$ at infinitely many σ-stages s. Let t be the last σ-initialization stage of the construction. Let $p = p^{\eta}_{\sigma}[t]$. By Lemma 3.13 and Lemma 3.17, $p \in \eta(x)$, and x is the only infinite component of η for which this is true. If $\iota \neq \eta$, then by Lemma 3.19 there is exactly one infinite component y of ι such that y is in $\widehat{\iota}_n[s]$ at infinitely many σ-stages s, and furthermore $\eta(x) = \iota(y)$. In addition, Lemma 3.13 shows that y is the only infinite component of ι for which $p \in \iota(y)$. □

Lemma 3.21. *For each e there is at most one stage at which we attempt to meet P_e, which thereafter remains satisfied, and φ_e is not an isomorphism from ν to μ.*

Proof. Suppose that at some σ-stage $t+1$ we attempt to meet P_e. At the end of stage $t+1$ we declare that P_e is satisfied and will never attempt to meet P_e again. At stage $t+1$ we have some j such that $\varphi_e(b_{\langle e,j\rangle}) = b_{\langle e,j\rangle}$, where $\nu(b_{\langle e,j\rangle})[t] = \mu(b_{\langle e,j\rangle})[t] = \{b_{\langle e,j\rangle}\}$, and include $a_{\langle e,j\rangle}, b_{\langle e,j\rangle}$, and $c_{\langle e,j\rangle}$ in the shuffle at this stage to ensure that $\nu(b_{\langle e,j\rangle})[t+1] \neq \mu(b_{\langle e,j\rangle})[t+1]$. The component $b_{\langle e,j\rangle}$ will never be used in a right shuffle again, so $f_t(b_{\langle e,j\rangle}) = f(b_{\langle e,j\rangle}) \neq \varphi_e(b_{\langle e,j\rangle})$. Thus φ_e is not an isomorphism from ν to μ. □

Lemma 3.22. *For $\sigma <_L TP$, $R(\sigma) < \infty$, and we only attempt to meet S_σ finitely often.*

Proof. If $\sigma <_L TP$ but $\sigma \not\prec TP$, then there are finitely many σ-stages, and hence only finitely many stages at which we can attempt to meet S_σ.

Now suppose that $\sigma \prec TP$, and that $|\sigma| = e$. Assume that for each $\tau \prec \sigma$ we attempt to meet S_τ at most finitely often.

Let t_0 be a stage of the construction such that no P_i for $i \leq e$ or S_τ for for $\tau <_L \sigma$ requires attention after stage t_0, and such that for $s \geq t_0$, $\alpha_s \geq_L \sigma$.

Suppose that at some stage $t_1 > t_0$ we attempt to meet S_σ. Then we never declare S_σ to no longer be satisfied, since that happens only at τ-stages where $\tau <_L \sigma$ and we are acting to meet S_τ or $P_{|\tau|}$. It is also immediate from the construction that if t_1 exists, then $\lim_s R_s(\sigma) = R_{t_1}(\sigma)$, and that otherwise $\lim_s R_s(\sigma) = R_{t_0}(\sigma)$, so that in either case $\lim_s R_s(\sigma) = R(\sigma)$ exists and is finite.

□

Lemma 3.23. *Each requirement P_e is satisfied.*

Proof. We have seen in Lemma 3.21 that if we ever act to meet P_e then it remains satisfied throughout the construction. Suppose we never act to meet P_e.

Let $\sigma = TP \upharpoonright e$, and assume s to be a stage such that after stage s we never attempt to meet a requirement P_i with $i < e$ or S_τ with $|\tau| \leq e$, and such that $\alpha_t \geq_L \sigma$ for all $t \geq s$. Let $\langle e,n\rangle$ be such that $a_{\langle e,n\rangle}$, $b_{\langle e,n\rangle}$, and $c_{\langle e,n\rangle}$ are all fresh numbers greater than $R(\sigma)$, and are not β-marked for any $\beta <_L \sigma$ at stage s. Since P_e never acts, $\nu(b_{\langle e,n\rangle}) = \{b_{\langle e,n\rangle}\} = \mu(b_{\langle e,n\rangle})$. It suffices to show that $\varphi_e(b_{\langle e,n\rangle}) \neq$

$b_{\langle e,n\rangle}$ to show that P_e is met. Assume for a contradiction that $\varphi_e(b_{\langle e,n\rangle}) = b_{\langle e,n\rangle}$. Let $t > s$ be a σ-stage such that $\varphi_{e,t}(b_{\langle e,n\rangle}) = b_{\langle e,n\rangle}$. Then P_e is the highest priority requirement which requires attention at stage t and so according to our construction, we attempt to meet P_e at stage t, contrary to our assumption. □

Lemma 3.24. *Let $\sigma = TP \restriction e$. Suppose that $s+1$ is a stage of the construction after which we never attempt to meet a requirement of the form P_j for any $j \leq e$, nor any requirement of the form S_τ for $\tau <_L \sigma$. Assume that for $t \geq s$ we have $\alpha_t \geq_L \sigma$. Suppose that $s+1$ is a stage at which S_σ requires attention. Let g be the function as given in Condition 3.5 which we use when attempting to meet S_σ.*

Then $g \preceq f$, where f is the isomorphism between μ and ν.

Proof. Note that at stage $s+1$, S_σ is the highest priority requirement which requires attention, and so we we attempt to meet it at this stage.

Let g be the function of Condition 3.5, let $w_{\langle\sigma,n\rangle}$ be the active witness for σ-diagonalization, and let $\sigma \preceq \theta \preceq \alpha_{s+1}$ be such that $g = f_{\theta,s} \restriction k$. So $\Phi_e^{f_{\theta,s}\restriction k}(w_{\langle\sigma,n\rangle}) = 0$.

We will show that $f_{\sigma,s+1} \restriction k = f_{\theta,s} \restriction k = g$.

By our choice of k, the only components $q < k$ of ν which are involved in the shuffle at stage $s+1$ are those which are s-predicted to be precious by θ.

Suppose that $q < k$ is involved in the shuffle.

If q is s-predicted to be precious by σ we have $f_{\sigma,s+1}(q) = f_{\sigma,s}(q)$, since if $\theta_0{}^\frown 1 \preceq \sigma$, then s is not a $\theta_0{}^\frown 0$-stage.

If $q < k$ is s-predicted to be precious by θ but not σ, let $q = p^\nu_{\theta_0}[t]$, where $\sigma \prec \theta_0{}^\frown 1 \preceq \theta$ and $t+1$ is the most recent $\theta_0{}^\frown 0$-stage, or $t = s$ if $p^\nu_{\theta_0}[s]$ and $p^\mu_{\theta_0}[s]$ have never been shuffled. Note that $|\theta_0|$ requires adjustment at this stage precisely if $f_s(p^\nu_{\theta_0,t}) \neq p^\mu_{\theta_0}[t]$. By the reasoning used in Lemma 3.18, we see that after applying the shuffle at this stage we will have $f_{s+1}(p^\nu_{\theta_0,t}) = p^\mu_{\theta_0,s+1} = p^\mu_{\theta_0}[t]$. Indeed, we have $f_{\theta,s} \restriction k = f_{\sigma,s+1} \restriction k$, and thus we get $\Phi_e^{f_{\sigma,s+1}\restriction r_{s+1}(\sigma)}(w_{\langle\sigma,n\rangle}) = 0 \neq C_{s+1}(w_{\langle\sigma,n\rangle}) = 1$, as desired.

Note that components $q < R_s(\sigma)$ cannot be unpredictable (Condition 3.5 forbids this), and that components $q < R_{s+1}(\sigma)$ will not be reserved or chosen to be fresh precious components at any future stage. Therefore at any stage $s_0 > s+1$, the only components $q < R_{s+1}(\sigma)$ of ν/μ which can be part of the right/left shuffle are those which are s_0-predicted to be precious by σ. Those are the same components which are s-predicted to be precious, since for $\tau{}^\frown 1 \prec \sigma$ there are no $\theta{}^\frown 0$-stages after s. Furthermore, for $\tau <_L \sigma$ we do not τ-mark or τ-reserve any component $q < R_{s+1}(\sigma)$ at any stage after stage s. So at every stage $s_0 > s$ we have $f_{\sigma,s_0} \restriction R_{s+1}(\sigma) = f_{\sigma,s+1} \restriction R_{s+1}(\sigma)$.

Finally, note that by Lemma 3.19, we see that any component q which is s_0-predicted to be precious by σ at each stage $s_0 > s$ satisfies $f_{\sigma,s+1}(q) = f(q)$. Since these are the only components smaller than $R_{s+1}(\sigma)$ which are ever shuffled after stage $s+1$, we have $g \preceq f$, as desired. □

Lemma 3.25. *We have $f <_T \emptyset''$, that is, every requirement S_e is met.*

Proof. Firstly, note that $f \leq_T \emptyset''$, since $f(x) = y$ is true precisely when $f_s(x) = y$ for infinitely many s, i.e. when $(\forall t)(\exists s > t)[f_s(x) = y]$, which is a Π^2_0 condition. Let $C = \{e \mid (\exists t)(\forall s > t) e \in C_s(x)\}$. Note that C is Σ^0_1 and hence $C \leq_T \emptyset' < \emptyset''$. We will show $f \not\geq_T C$.

If $\sigma = TP \upharpoonright e$, suppose as in Lemma 3.24 that t_0 is a stage after which we never attempt to meet P_j for any $j < i$ or S_τ for any $\tau <_L \sigma$, and that for $s > t_0$ we have $\alpha_s \geq \sigma$. Then after stage t_0 the active witness $w_{\langle\sigma,n\rangle}$ for σ-diagonalization does not change.

If S_σ ever requires attention at a σ-stage $s+1 > t_0$, we attempt to meet it. Then by Lemma 3.24, $g \prec f$, where as usual g is the function of Condition 3.5, and $\Phi^f_e(w_{\langle\sigma,n\rangle}) \neq C(w_{\langle\sigma,n\rangle})$.

We must now show that $\Phi^f_e \neq C$, even if we never succeed at permanently satisfying S_σ. So suppose that Φ^f_e is total. Choose k and s_0 to be large enough that $\Phi^{f\upharpoonright k}_{e,s_0}(w_{\langle\sigma,n\rangle})\downarrow$. Choose $\theta \prec TP$ such that $\sigma \prec \theta$ and each infinite component $q < k$ of ν is $p^\nu_\tau[t]$ at infinitely many stages t, where $\tau^\frown 1 \preceq \theta$.

Let $s > s_0$ be a stage so large that for each $t > s$, $\alpha_t \geq_L \theta$, that no finite component $q < k$ of ν is part of a shuffle after stage s, and that after stage s, we never again attempt to meet a requirement P_i for $i \leq e$, nor S_τ for any $\tau <_L \sigma$.

At every θ-stage $t > s$, we have $f_{\theta,t} \upharpoonright k = f \upharpoonright k$, by choice of k, s, and θ. At the first such stage, $\Phi^{f_\theta\upharpoonright k}_e(w_{\langle\sigma,n\rangle})[t]\downarrow$, by our choice of s. By assumption, we do not attempt to meet S_σ at this stage. Since we can only only enumerate $w_{\langle\sigma,n\rangle}$ into C when attempting to meet S_σ, and change to a new active witness for σ-diagonalization whenever S_σ ceases to be satisfied, we have never enumerated $w_{\langle\sigma,n\rangle}$ into C. Since S_σ does not require attention, $\Phi^f_e(w_{\langle\sigma,n\rangle}) = \Phi^{f_\theta\upharpoonright k}_e(w_{\langle\sigma,n\rangle})[t]\downarrow\neq C_t(w_{\langle\sigma,n\rangle})$. We never change the active witness for σ-diagonalization, nor enumerate it into C. So $\Phi^f_e(w_{\langle\sigma,n\rangle}) \neq C(w_{\langle\sigma,n\rangle})$, as desired. Thus if Φ^f_e is total, $\Phi^f_e \neq C$. □

This concludes our discussion of the P_e and S_σ requirements. All that remains is to check the N_e requirements: that either ν and μ are not isomorphic to ρ_e, or that one of them is computably isomorphic to ρ_e. Which of these situations occurs depends on $TP(e)$. We first address the case in which $\sigma^\frown 2 \prec TP$ for some σ of length e, and eventually some finitary obstruction prevents ρ_e being isomorphic to our structures.

Lemma 3.26. *Suppose that σ is of length e and that $\sigma^\frown 2 \prec TP$. Then ρ_e is not isomorphic to ν (or to μ).*

Proof. Let s be such that for $t > s$, $\alpha_t >_L \sigma^\frown 2$ and such that after stage s we do not attempt to meet S_τ for any $\tau \leq_L \sigma$. Let $t_0 + 1 < s$ be the final σ-stage which is a recovery stage for N_e. Note that if q is a component of ν which is involved in the right shuffle at a stage after $t_0 + 1$, then q cannot be σ-marked. Thus, after stage s, no elements are ever enumerated into any σ-marked component of ν. Let M be the set of components of ν which are ever σ-marked.

Suppose there is some σ-stage $t+1 > s$ at which there are at least two embeddings of the components in M into $\rho_e[t]$. Then there is some $q \in M$ such that $\nu(q)[t] \subseteq \rho_e(x)[t]$ for two different components x of ρ_e. But $\nu(q) \not\subseteq \nu(y)$ for each component $y \neq q$ of ν, so ρ_e and ν cannot be isomorphic.

If at each σ-stage $t+1 > s$ there is no homomorphic injection of the components in M into $\rho_e[t]$, then there is no homomorphic injection of ν into ρ_e, either, and two structures are not isomorphic.

Otherwise, at some first σ-stage $t+1 > s$ there is a unique homomorphic injection $h^\eta_\beta[t]$ mapping the components in M to components of ρ_e such that $\rho_e(h^\eta_\beta(q))[t] \subseteq \eta(q)[t]$ for each $q \in M$. Since t is a $\sigma^\frown 2$-stage, $\text{range}(h^\nu_\sigma)[t_0] \not\subseteq \text{range}(h^\nu_\sigma)[t]$. Let $x \in \text{range}(h^\nu_\sigma)[t_0] - \text{range}(h^\nu_\sigma)[t]$. Say $x = h^\nu_\sigma(q)[t_0]$. Note that each component u of

ν such that $q \in \nu(u)[t_0+1]$ is σ-marked by the end of stage t_0+1, and therefore is in the domain of $h^\eta_\sigma[t]$. Thus q cannot become a member of any additional component of ν after stage t_0+1, since that would require a σ-marked component to be part of a shuffle. So there are finitely many components y of ν which have $q \in \nu(y)$. But $q \in \rho_e(x)[t]$, so there are more components y in ρ_e such that $q \in \rho_e(y)$ than there are components y in ν such that $q \in \nu(y)$, so ρ_e and ν are not isomorphic. □

In the case that $TP(e) = 0$ we will see that ρ_e and ν are not isomorphic, because there is an infinite component in ρ_e which is not present in ν. The following lemma will be used to demonstrate that.

Lemma 3.27. *Let σ be of length e, and suppose $\sigma \prec TP$. Assume that for $t \geq t_0$, $\alpha_t \geq_L \sigma$, that t_0 is a σ-stage which is a recovery stage for N_e, and that after stage t_0 we never attempt to meet S_θ for any $\theta \leq_L \sigma$.*

Suppose η is either ν or μ, and that $s+1 > t_0$ is a σ-stage of the construction which is an η-recovery stage for N_e.

Suppose that p_σ is the component of ρ_e for which $h^\eta_\sigma(p^\eta_\sigma)[s] = p_\sigma$.

Suppose $t+1 > s+1$ is the next σ-stage which is a recovery stage for N_e, and that $t+1$ is a $\sigma^\frown 0$ stage.

Let ι be μ if η is ν and ν if η is ν. Then $\iota(p^\iota_\sigma)[t] = \rho_e(p_\sigma)[t]$.

Proof. If $t+1$ is a $\sigma^\frown 0$ stage, then $h^\eta_\sigma[s] \not\subseteq h^\eta_\sigma[t]$. The disagreement between $h^\eta_\beta[t]$ and $h^\eta_\beta[s]$ occurs among the components which are part of the shuffle at stage s.

Suppose that at stage $s+1$ we perform a right shuffle at stage on the components $n_0, \ldots, n_d$ of ν, a left shuffle on the components $m_0, \ldots, m_d$ of μ. Interpret the indices $0, \ldots d$ modulo $d+1$. By Lemma 3.16, for $0 \leq i \leq d$ we have $\nu(n_i)[s] \subseteq \nu(n_i)[s+1]$ and $\nu(n_i)[s] \subseteq \nu(n_{i-1})[s+1]$, but $\nu(n_i)[s] \subseteq \nu(q)[s+1]$ for no other component q of ν. Likewise, for each i, $\nu(n_i)[s] \subseteq \mu(m_i)[s+1]$ and $\nu(n_i)[s] \subseteq \mu(m_{i+1})[s+1]$, but $\nu(n_i)[s] \subseteq \mu(q)$ for no other component q of μ.

For $i \geq n$ the components of $\widehat{\nu}_i[s+1]$ are all σ-marked by the end of stage s, whereas for $i < n$, none of the components of $\widehat{\nu}_i[s+1]$ are σ-marked by the end of stage s. The components of the form $a_{\langle e,j\rangle}, b_{\langle e,j\rangle}$, and $c_{\langle e,j\rangle}$ used in the right shuffle at this stage (if any), are either all σ-marked by the end of stage s, or none are. Thus the σ-marked components used in the right shuffle occur as a contiguous "block": they are of the form $n_a, n_{a+1}, \ldots, n_{b-1}, n_b$ for some a and b (numbering modulo $d+1$).

There are components $q_a, q_{a+1}, \ldots, q_b$ of ρ_e such that for $a \leq j \leq b$, $\rho_e(q_j)[s] \supseteq \nu(n_j)[s]$, i.e. $h^\nu_\sigma(n_j)[s] = q_j$. Because $t+1$ is a σ-stage and recovery stage for N_e, each such q_j is in the range of $h^\eta_\sigma[t]$. Each q_j must satisfy either $\rho_e(q_j)[t] \supseteq \nu(n_j)[t] = \mu(m_{j+1})[t]$, or $\rho_e(q_j)[t] \supseteq \nu(n_{j-1})[t] = \mu(m_j)[t]$. Furthermore, either the former is true for every j or the latter is true for every j, since h^η_σ includes every q_j in its range, and is one-to-one. But $h^\eta_\sigma[t] \not\supseteq h^\eta_\sigma[s]$.

Suppose that $\eta = \nu$. Let $n_j = p^\nu_\sigma[s]$. Then $h^\nu_\sigma(n_j) = n_{j+1}$. Then $p^\mu_\sigma[t] = p^\mu_\sigma[s+1] = f_s(n_j) = m_j$, whence $\rho_e(q_j)[t] = \mu(p^\mu_\sigma)[t]$.

Likewise if $\eta = \mu$, and $m_j = p^\mu_\sigma[s]$, then $p^\nu_\sigma[t] = p^\nu_\sigma[s+1] = f_s^{-1}(p^\mu_{\sigma,s}) = n_{j-1}$. So $\rho_e(q_j)[t] = \nu(p^\nu_\sigma)[t]$. □

Lemma 3.28. *If σ is of length e and $\sigma^\frown 0 \prec TP$ then ρ_e is not isomorphic to ν.*

Proof. Suppose that $\sigma^\frown 0 \prec TP$. Choose t to be large enough that at stages $s \geq t$, $\alpha_s \geq_L \sigma^\frown 0$, and that at stages $s \geq t$ we do not attempt to meet S_τ for any $\tau \leq_L \sigma$.

Let $t_0+1 > t$ be a $\sigma\widehat{\ }0$ stage. Then t_0+1 is an η-recovery stage for N_e, where η is either μ or ν.

Let p_σ be the component of ρ_e for which we have $h^\eta_\sigma(p^\eta_\sigma)[t_0] = p_\sigma[t_0]$. Let $t_1+1 > t_0+1$ be the next $\sigma\widehat{\ }0$ stage. Note that t_1+1 is an ι-recovery stage, where ι is μ if η is ν and ι is ν if η is μ. We have $h^\eta_\sigma(p^\mu_\sigma)[t_1] = p^\iota_\sigma[t_1]$, where ι is ν if η is μ and is μ if η is ν. Thus we have $\rho_e(p^\sigma)[t_1] \supseteq \iota(p^\iota_\sigma)[t_1]$, and $\eta(p^\eta_\sigma[t_0]$ is a proper subset of $\iota(p^\iota_\sigma)[t_1]$. Let t_i be the ith $\sigma\widehat{\ }0$ stage at which we perform a right shuffle after t_0. Then $\eta(p^\eta_\sigma)[t_0] \subset \iota(p^\iota_\sigma)[t_1] \subset \eta(p^\eta_\sigma)[t_2] \subset \iota(p^\iota_\sigma)[t_3] \cdots \subseteq \rho_e(p^\sigma)$, and $\rho_e(p^\sigma)$ is infinite.

There is no infinite component q of ν such that $\nu(p^\nu_\sigma)[t_i] \subseteq \eta(q)$ for any i, because such a component would by Lemma 3.13 need to be part of $\widehat{\nu}_e[s]$ at infinitely many σ-stages s, but by Lemma 3.17 that implies $\sigma\widehat{\ }1 \prec TP$, contrary to our assumption. □

Finally, we deal with the case in which $TP(e) = 1$.

Lemma 3.29. *If σ is of length e and $\sigma\widehat{\ }1 \prec TP$, then if ρ_e is isomorphic to ν and μ, it is computably isomorphic to one of them.*

Proof. Let t_0 be a stage such that for $t > t_0$, $\alpha_t \geq_L \sigma\widehat{\ }1$, and after which we never attempt to meet S_τ for any $\tau \leq_L \sigma$.

Suppose that ν, μ, and ρ_e are all isomorphic (or else there is nothing to show).

We will be able to computably determine an isomorphism $h^\eta\colon \eta \to \rho_e$, for η equal to one of ν and μ.

We will specify a finite set D^η of components of η, for both choices of η, and show that each component of η which is not in D^η is eventually σ-marked.

If $\tau <_L \sigma$ but $\tau \not\prec \sigma$, then after stage t_0 we do not τ-mark any additional components of ν or μ. We also have $p^\eta_\tau[t] = p^\eta_\tau[t_0]$. Put each component of η which is ever τ-marked into D. Put $p^\eta_\tau[t_0]$ into D^η.

If $\tau\widehat{\ }2 \preceq \sigma$, then only finitely many components of η are ever τ-marked. Put them into D^η.

If $\tau\widehat{\ }1 \preceq \sigma$, say $|\tau| = i$. There is a single component which is part of $\eta_i[t]$ at infinitely many stages t. Put that component into D^η.

Note that a component is in D^μ precisely if it is of the form $f(q)$ for some $q \in D^\nu$.

If $t_0 < t+1 < s+1$, and $t+1$ and $s+1$ are σ-stages which are recovery stages for N_e, both must be η-recovery stages for the same η, so we have $h^\eta_\sigma[t] \subseteq h^\eta_\sigma[s]$. In addition, each component $q < s$ of ν will be σ-marked by stage $s+1$ unless it is in D^ν, or if there is some $\tau \prec \sigma$ such that $\tau\widehat{\ }0 \preceq \sigma$ or $\tau\widehat{\ }1 \preceq \sigma$ and q is either τ-reserved, is $p^\nu_{\tau,s+1}$, or is $f^{-1}(p^\nu)[s+1]$. Note that each q can only be a τ-reserved component at one σ-stage, and that if $\tau\widehat{\ }0 \preceq \sigma$, then each component q can only be $p^\nu_{\tau,s+1}$ or $f^{-1}(p^\mu_\sigma)[s+1]$ at finitely many σ-stages $s+1$ of the construction.

So any component of ν other than those in D^ν will eventually be σ-marked at some stage.

We now check that every component of μ other than those in D^μ is σ-marked at some stage. It suffices to check that if q is a component of ν then if q is ever σ-marked, $f(q)$ is also σ-marked. In the case that $\nu(q)$ is finite, this is obvious. Otherwise q is infinite. In that case, since q is σ-marked at some stage, there is some τ such that $\sigma \preceq \tau\widehat{\ }1 \prec TP$ and q is part of $\widehat{\nu}_i[s+1]$ at infinitely many stages $s+1$, where $|\tau| = i$. But then $f(q)$ is part of $\widehat{\mu}_i[s+1]$ at infinitely many stages, and is σ-marked at the first such stage.

Now, let η be such that eventually every σ-stage which is a recovery stage for N_e is an η-recovery stage. Each component of η is either a member of D^η or is σ-marked at some stage.

Let $h^\eta_\sigma = \bigcup_{\sigma^\frown 1\text{-stages } s>t_0} h^\eta_\sigma[s]$. Then h^η_σ has domain $\omega \setminus D^\eta$, and furthermore is a computable homomorphic embedding of those components of η into ρ_e. In addition, if η and ρ_e are isomorphic, it is the only such homomorphic embedding, for if there were another, say h, then there must be some component $x \in \omega \setminus D^\eta$ such that $h(x) \neq h^\eta_\sigma(x)$, whence $\eta(x) \subset \rho_e(h^\eta_\sigma(x))$ and $\eta(x) \subset \rho_e(h(x))$, contrary to Lemma 3.20.

Now suppose that η and ρ_e are isomorphic. Rigidity of η implies there is only one isomorphism $h\colon \eta \to \rho_e$, but as the reasoning above shows, we must have $h \supseteq h^\eta_\sigma$. Since D^η is finite, and h^η_σ is computable, it follows that h is computable too. □

This concludes the verification that our construction succeeds in meeting the requirements of form N_n, S_e, and P_e. It follows that ν is a c.e. Friedberg enumeration of a set $\mathcal{S}$ for which $\mathcal{G}(\mathcal{S})$ has computable dimension 2, and that the isomorphism between the computable copies of $\mathcal{G}(\mathcal{S})$ corresponding to ν and μ has degree $\mathbf{d} \leq \mathbf{0}''$ and $\mathbf{d} \not\geq \mathbf{0}'$, as required by Theorem 3.1.

4. COMPUTABLE DIMENSION 3

We now devote our attention to the following theorem.

Theorem 4.1. *There exists a rigid structure of computable dimension 3 such that if* $\mathbf{d}_0$*,* $\mathbf{d}_1$*, and* $\mathbf{d}_2$ *are the degrees of isomorphisms between distinct representatives of the three computable equivalence classes, then each* $\mathbf{d}_i < \mathbf{d}_0 \oplus \mathbf{d}_1 \oplus \mathbf{d}_2 \leq \mathbf{0}''$.

We first note that Theorem 4.1 immediately yields the following corollary.

Corollary 4.2. *There is a rigid computable structure with computable dimension 3 which has a degree of categoricity* $\mathbf{d} \leq \mathbf{0}''$*, but has no strong degree of categoricity.*

Proof. Firstly, we note that $\mathbf{d} = \mathbf{d}_0 \oplus \mathbf{d}_1 \oplus \mathbf{d}_2$ is clearly able to compute an isomorphism between any two computable copies of our structure, since the degrees of isomorphisms between inequivalent copies are all of the form $\mathbf{d}_i$ for some i, whereas if $\mathbf{c} \not\geq \mathbf{d}$ then $\mathbf{c} \not\geq \mathbf{d}_i$ for some i. So $\mathbf{d}$ is clearly the degree of categoricity of our structure. On the other hand $\mathbf{d}_i < \mathbf{d}$ for each i, and therefore between any two computable copies there is an isomorphism with Turing degree strictly below $\mathbf{d}$, which is therefore not a strong degree of categoricity. □

We now proceed with the proof of Theorem 4.1.

Our strategy is based on that of the two-structure construction in the previous section, but we must make some significant changes to adapt the ideas to the new context.

The major change is that because we are building three structures, we will no longer be able to perform shuffles which are simultaneously in opposite directions in all three structures. We will, however, continue to use shuffles when building our structures, since they are an ideal tool for ensuring that our structures have finite computable dimension.

Our three isomorphic structures will be called ν^i, where $0 \leq i \leq 2$. As before, we will work with c.e. binary relations, and build Friedberg enumerations of a set $\mathcal{S}$, but note that one could equally well work with graphs and build computable copies of $\mathcal{G}(\mathcal{S})$ instead. Once again, we will ensure that no pair of our structures is computably isomorphic, and that the structure we are building has computable dimension 3. The isomorphism between the the structures ν^i and ν^j will be denoted by $f^{i,j}$. We will interpret our structures as having indices given modulo 3 under addition, so that for any i, ν^i, ν^{i+1}, and ν^{i+2} will refer to our three distinct structures.

We will have two kinds of requirement, as follows:

N_e: If ρ_e is a Friedberg enumeration of $\mathcal{S}$, then ρ_e is computably isomorphic to ν^i for some i.

In addition to these requirements, we also wish to ensure that if i and j are distinct, then $f^{i,j} <_T f^{0,1} \oplus f^{1,2} \oplus f^{2,0}$.

To achieve this, we will meet the requirements:

$Q_{e,i}$: $\Phi_e^{f^{i+1,i+2}} \neq f^{i,i+1}$, for $0 \leq i \leq 2$ and $e \in \omega$.

Note that meeting each of the requirements $Q_{e,i}$ guarantees that $f^{i+1,i+2} \not\geq_T f^{i,i+1}$ for each i. We will let $\mathbf{d}_i$ denote the Turing degree of $f^{i,i+1}$ for $0 \leq i \leq 2$. Since our structure is rigid it has degree of categoricity $\mathbf{d} = \mathbf{d}_0 \oplus \mathbf{d}_1 \oplus \mathbf{d}_2$.

Furthermore, $\mathbf{d} > \mathbf{d}_{i+1}$ for each i, because the requirements $Q_{e,i}$ collectively ensure that $\mathbf{d}_i \not\leq \mathbf{d}_{i+1}$. Thus the requirements are sufficient to guarantee that the three degrees $\mathbf{d}_0$, $\mathbf{d}_1$, $\mathbf{d}_2$ of the isomorphisms between distinct representatives of the three computable equivalence classes satisfy $\mathbf{d}_i < \mathbf{d}_0 \oplus \mathbf{d}_1 \oplus \mathbf{d}_2$, as required. Our method of construction will allow us to easily see that $\mathbf{d}_i \leq \mathbf{0}''$ for each i.

The requirements $(Q_{e,i})_{e\in\omega}$ together guarantee that $\mathbf{d}_i \neq \mathbf{0}$. Thus we no longer need to use requirements similar to the P_e from Section 3 to guarantee that none of the copies we build are computably isomorphic—that is already guaranteed by the other requirements.

We will build the structures ν^i stage-by-stage, as before, and at each stage s will let $f^{i,j}_s$ be the unique isomorphism from $\nu^i[s]$ to $\nu^j[s]$. Note that $f^{i,i}_s$ is always the identity function. We will let x_e, y_e, z_e, w_e for $e \in \omega$ be natural numbers, all chosen distinctly.

As before, each string σ of length e will correspond to a guess about the recovery patterns for the structures ρ_j for $j \leq e$. Each stage $s+1$ will be a associated to a string α_{s+1} of length $s+1$. We will interpret α_{s+1} as giving information about stage $s+1$ in much the same way as we did in the two-structure construction.

For each σ and i, we will specify a precious component $p^i_\sigma[s]$ of ν^i at each stage s of the construction. We begin by setting $p^i_\sigma[0] = z_{\langle\sigma,0\rangle}$ for each σ.

The strings α_{s+1} are defined as follows:

Every stage of the construction is a λ-stage.

Stage $s = 0$**.**

We say that $h^i_\beta[0]$ is the empty function for $0 \leq i \leq 2$ and for every β.

Stage $s + 1$**.**

Suppose that we know that $s+1$ is a β-stage for some β with $|\beta| < s$. Let $e = |\beta|$. For $0 \leq i \leq 2$ let B^i be the set of components of ν^i which are β-marked by the end of stage s.

We say that stage $s+1$ is a recovery stage for N_e if, for $0 \leq i \leq 2$, there exists a unique map $h^i_\beta[s] \colon B^i \to \rho_e$ such that $\rho_e(h^i_\beta(x))[s] \supseteq \nu^i(x)[s]$ for each $x \in B^i$, and furthermore that if $t+1 < s+1$ is the most recent β-stage which is a recovery stage for N_e, then $\text{range}(h^\eta_\beta)[t] \subseteq \text{range}(h^\eta_\beta)[s]$.

If $s+1$ is a β-stage which is a recovery stage for N_e, and β *was not* active at the end of stage s, say $s+1$ is a 0-recovery stage, a β-initialization stage, and a $\beta\widehat{\ }0$-stage. Declare β active.

If $s+1$ is a β-stage which is a recovery stage for N_e, and β *was* active at the end of stage s, let $t+1 < s+1$ be the most recent β-stage which was a recovery stage for N_e. Suppose $t+1$ was an i-recovery stage for N_e. If $h^i_\beta[s] \supseteq h^i_\beta[t]$, then say $s+1$ is a $\beta\widehat{\ }1$-stage, and an i-recovery stage for N_e. If $h^i_\beta[s] \not\supseteq h^i_\beta[t]$, for each k such that the shuffles in ν^i and ν^k were in opposite directions at stage $t+1$, let t_k be the largest β-stage prior to $t+1$ which is a k-recovery stage for N_e (choosing $t_k = 0$ if there is none). Choose k to minimize t_k. Say $s+1$ is a k-recovery stage for N_e, and a $\beta\widehat{\ }0$-stage.

If $s+1$ is not a recovery stage for N_e, then we say stage $s+1$ is a $\beta\widehat{\ }2$-stage.

Let α_{s+1} be the unique string of length $s+1$ such that $s+1$ is an α_{s+1}-stage. Note that $s+1$ is a σ-stage iff $\sigma \preceq \alpha_{s+1}$.

We define TP to be the true path, given by $TP = \liminf_s \alpha_s$.

At each stage $s+1$ we define $f^{i,j}_{\sigma,s}$ as follows: for $\tau\widehat{\ }1 \preceq \sigma$, let $t+1 \leq s$ be the most recent $\tau\widehat{\ }0$-stage at which we performed a shuffle. Then for each i and j set $f^{i,j}_{\sigma,s}(p^i_{\tau,t+1}) = p^j_{\tau,t+1}$, and say that σ s-predicts $p^i_{\tau,t+1}$ to be precious. If q is a component of ν^i such that $f^{i,j}_s(q) = p^j_{\tau,t+1}$ for some j, let $f^{i,j}_{\sigma,s}(q)$ be undefined. For each other component q of ν^i, set $f^{i,j}_{\sigma,s}(q) = f^{i,j}_s(q)$.

At each stage we define restraint $r_s(\sigma)$ and set $R_s(\sigma) = \max\{r_s(\tau) \mid \tau \leq_L \sigma\}$.

Each requirement $Q_{e,i}$ will be split into different versions $Q_{\tau,i}$, where $\tau \in 3^{<\omega}$ is of length e. An attempt to meet a requirement $Q_{\tau,i}$ will be spread across two different τ-stages of the construction. When meeting the requirement $Q_{\tau,i}$ we will have a specified active witness $w_{\langle\tau,n\rangle}$ for τ-i-diagonalization. Our goal in meeting the requirement $Q_{\tau,i}$ will be to ensure that

$$\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle}) \downarrow = w_{\langle\tau,n\rangle} \neq f^{i,i+1}(w_{\langle\tau,n\rangle}).$$

At the first stage of each attempt to meet $Q_{\tau,i}$ we will *prepare* the requirement by attempting to find some partial function g such that that

$$\Phi_e^g(w_{\langle\tau,n\rangle}) = w_{\langle\tau,n\rangle}$$

and such that we are able to ensure that $g \prec f^{i+1,i+2}$. At the second stage we will *complete* the requirement by performing a second shuffle to ensure that $f^{i,i+1}(w_{\langle\tau,i\rangle}) \neq w_{\langle\tau,i\rangle}$ and thus that the computation given by g disagrees with $f^{i,i+1}$.

The procedure used at a stage $s+1$ at which we are preparing the requirement $Q_{\tau,i}$ will strongly resemble that used to meet the requirements S_τ of the previous sections. We will perform a right shuffle in ν^i and ν^{i+1}, and a left shuffle in ν^{i+2}. As in the two-structure construction, we will want to ensure $f^{i+1,i+2}_{\theta,s} \restriction k = f^{i+1,i+2}_{\tau,s+1} \restriction k$, for some appropriately chosen k and $\theta \succeq \tau$. We will then attempt to protect this initial segment.

At the second stage $t+1$ of our attempt to meet $Q_{\tau,i}$, we will complete the requirement, by diagonalizing against the previously-prepared computation. To do so we will perform a right shuffle in ν^i, and left shuffles in ν^{i+1} and ν^{i+2}. At this stage we will include the witness $w_{\langle\tau,n\rangle}$ in our shuffle, so that that $f^{i,i+1}_t(w_{\langle\tau,n\rangle}) \neq f^{i,i+1}_{t+1}(w_{\langle\tau,n\rangle})$ and in particular that $f^{i,i+1}_{t+1}(w_{\langle\tau,n\rangle}) \neq w_{\langle\tau,n\rangle}$. We will never include $w_{\langle\tau,n\rangle}$ in a shuffle for any other reason, so we will have $f^{i,i+1}(w_{\langle\tau,n\rangle}) \neq w_{\langle\tau,n\rangle}$.

The shuffles carried out in ν^{i+1} and ν^{i+2} while completing the requirement $Q_{\tau,i}$ will be in the same direction, to preserve the initial segment $g = f^{i+1,i+2}_{\tau,s+1} \upharpoonright k$ which is providing us a computation to diagonalize against.

If $\tau \prec TP$ and the requirement $Q_{\tau,i}$ is never injured after this stage we will have $g = f^{i+1,i+2} \upharpoonright k$, so that we meet the overall requirement $Q_{e,i}$.

If a requirement $Q_{\tau,i}$ is injured for any reason, we will abandon all progress which we have made towards meeting it, and choose a new active witness. If $\tau \prec TP$ then the requirements of the form $Q_{\tau,i}$ will only be injured finitely often.

The following schematics illustrate the shuffling strategy for preparing and then completing $Q_{\tau,i}$.

$$\begin{tikzcd}
\nu^i \arrow[r, "f^{i,i+1}_\tau \text{ preserved}"] & \nu^{i+1} \arrow[r, "f^{i+1,i+2}_\tau \text{ changes}"] & \nu^{i+2} \\
\text{right shuffle} & \text{right shuffle} & \text{left shuffle}
\end{tikzcd}$$

FIGURE 1. At stage $s+1$ we prepare $Q_{\tau,i}$. The shuffles in ν^{i+1} and ν^{i+2} are in opposite directions, which causes the isomorphism between them to change, and our shuffle is chosen so that $f^{i+1,i+2}_\tau[s+1] \upharpoonright k = g$.

$$\begin{tikzcd}
\nu^i \arrow[r, "f^{i,i+1}_\tau \text{ changes}"] & \nu^{i+1} \arrow[r, "f^{i+1,i+2}_\tau \text{ preserved}"] & \nu^{i+2} \\
\text{right shuffle} & \text{left shuffle} & \text{left shuffle}
\end{tikzcd}$$

FIGURE 2. At a later stage $t+1$ we complete $Q_{\tau,i}$. Because the shuffles in ν^{i+1} and ν^{i+2} are in the same direction, $f^{i+1,i+2}[t] = f^{i+1,i+2}[t+1]$. Thus we preserve the steps taken to prepare $Q_{\tau,i}$ and $f^{i+1,i+2}_\tau[t+1] \upharpoonright k = g$. By involving $w_{\langle \tau,n \rangle}$ in the shuffle, we get $f^{i,i+1}[t+1](w_{\langle \tau,n \rangle}) \neq f^{i,i+1}[t](w_{\langle \tau,n \rangle})$.

As before, we will meet the requirements N_e passively by building our structures using shuffles and keeping track of recovery stages. If τ is of length e and $s+1$ is a $\tau\widehat{\ }1$-stage and an i-recovery stage for N_e, then we believe $p^i_{\tau,s}$ will be an infinite component of ν^i. In that case we try to build an identical infinite component q in ν^j for each $j \neq i$. We do so using a method akin to the odd/even recovery stages in the two-structure construction, but which is complicated by the fact that we must now keep track of three structures. If $s+1$ is a $\tau\widehat{\ }1$-stage, and an i-recovery stage for N_e, and $q = f^{i,j}(p^i_\tau)[s]$, then if we involve $p^i_{\tau,s}$ in the shuffle in ν^i at stage $s+1$, we must also involve q in the shuffle in ν^j at this stage. If these shuffles are in opposite directions, then $q \neq f^{i,j}(p^i_\tau)[s+1]$. If we want $f^{i,j}(p^i_\tau) = q$ to be true, we wait for a later $\tau\widehat{\ }1$-stage $t+1$ at which ν^i and ν^j once again involve shuffles in opposite directions, at which point we can include both $f^{i,j}_t p^i_{\tau,t}$ and q in the shuffle in ν^j. This allows us to reverse the unwanted change to $f^{i,j}$, and get $q = f^{i,j}(p^i_\tau)[t+1]$. Determining whether we are allowed to include q in the shuffle at stage $t+1$ requires checking the status of the third structure at stage $t+1$. As an organizational tool, we will maintain a τ-j-queue for each j, which will be used to record which component q of ν^j (if any) we wish to involve in a shuffle at a future stage. Each queue will contain at most one component at a time. We will refer to each τ-j-queue as being a τ-queue.

As in the previous construction, at stage $s+1$, we will say that a number q is fresh if $\nu^i(q)[s]$ for each i, and for each σ, q is not σ-reserved, and furthermore, for no $t < s+1$ do we have $q = p^i_{\sigma,t}$ for any i.

We now give the construction.

The Construction

Stage 0**:**

We set $\nu^i(q) = \{q\}$ for each q and for $0 \leq i \leq 2$. Set $r(\sigma)[0] = 0$ for all $\sigma \in 3^{<\omega}$. For $0 \leq i \leq 2$, λ-mark the precious components $p^i_{\lambda,0}$ in ν^i. For the least components of form x_l and y_k, λ-mark them in each ν^i and say they are λ-reserved. For $0 \leq i \leq 2$, say that $w_{\langle\lambda,i\rangle}$ is the active witness for λ-i-diagonalization, and λ-mark the component $w_{\langle\lambda,i\rangle}$ in ν^j for $0 \leq j \leq 2$.

Say that none of the requirements $Q_{\tau,i}$ are currently satisfied, and that none have been prepared.

Say that the τ-i-queue is empty for each $\tau \in 3^{<\omega}$ and each i such that $0 \leq i \leq 2$.

Stage $s+1$**:**

We check on the status of the requirements $Q_{\tau,i}$ for each $\tau \preceq \alpha_{s+1}$ and $0 \leq i \leq 2$, so that we can decide which to meet.

Condition 4.3. Suppose $\tau \preceq \alpha_{s+1}$ is of length e and and $Q_{\tau,i}$ is not currently satisfied.

For each $\theta \succeq \tau$ say that a component q of ν^{i+1} is θ-unpredictable if there is some σ such that $\sigma^\frown l \preceq \theta$, where l is either 0 or 1, and such that q is either σ-reserved, or is of the form $f^{j,i+1}(p^j_\sigma)[s]$ for some j, and θ does not s-predict q to be precious.

Suppose $w_{\langle\tau,n\rangle}$ is the active witness for τ-i-diagonalization.

Check whether there is some θ such that $\tau \preceq \theta \preceq \alpha_{s+1}$, $k \leq s$, and g such that:

(1) $g = f^{i+1,i+2}_{\theta,s} \upharpoonright k$
(2) If x is a component of ν^{i+1} which is θ-unpredictable, then $k \leq x$
(3) $\Phi^g_e(w_{\langle\tau,n\rangle})[s]\downarrow= w_{\langle\tau,n\rangle}$.

If such θ, k, and g exist say that $Q_{\tau,i}$ requires attention.

We will say that $Q_{\tau,i}$ can be completed if it has previously been prepared, but the most recent preparation has not been canceled.

Preparing the requirement $Q_{\tau,i}$ will ensure that $\Phi_e^{f_\tau^{i+1,i+2}}(w_{\langle\tau,n\rangle})[s]\downarrow= w_{\langle\tau,n\rangle}$, where $w_{\langle\tau,n\rangle}$ is the active witness for τ-i-diagonalization. A requirement which has been prepared will still require attention at later stages.

Find the shortest $\tau \preceq \alpha_{s+1}$ such that for some least i, $Q_{\tau,i}$ requires attention.

At stage $s+1$ we will complete $Q_{\tau,i}$ if it can be completed. Otherwise we will prepare it.

Case 1: Preparing $Q_{\tau,i}$.

Let g and θ be as specified in Condition 4.3. Suppose that $|\tau| = e$ and $|\theta| = d$.

We will be performing a right shuffle in ν^i and in ν^{i+1}, and a left shuffle in ν^{i+2}.

We will define a finite sequence $\widehat{\nu}^j_n[s+1]$ of components in each ν^j for each j and for $n < d$.

For each $n < e$, do as follows: Let $\sigma = \alpha_{s+1} \upharpoonright n$. Let x_l and y_k be the σ-reserved components.

If $s+1$ is a $\sigma\widehat{\ }2$-stage, let $\widehat{\nu}^j_n$ be empty for each j, and do nothing to the σ-queues.

If $s+1$ is a $\sigma\widehat{\ }0$-stage and an h-recovery stage for N_n, empty the σ-queues, and for each j, let $\widehat{\nu}^j_n = x_l, f^{h,j}(p^h_\sigma)[s], y_k$.

If $s+1$ is a $\sigma\widehat{\ }1$-stage and an h-recovery stage for N_n, check whether there is some number q in the σ-$(i+2)$-queue.

If so, remove q from the σ-$(i+2)$-queue. Then, if we are performing a right shuffle in ν^h at this stage, for each j let $\widehat{\nu}^j_n = x_l, f^{h,j}(p^h_\sigma)[s], f^{i+2,j}_s(q)$. On the other hand, if we are performing a left shuffle in ν^h at this stage, for each j let $\widehat{\nu}^j_n = f^{i+2,j}_s(q), f^{h,j}(p^h_\sigma)[s], y_k$.

If there is no number q in the σ-$(i+2)$-queue, then for each j set $\widehat{\nu}^j_n = x_l, f^{h,j}(p^h_\sigma)[s], y_k$.

In each of the above cases, if $s+1$ is an h-recovery stage for N_n, let $p^h_{\sigma,s+1} = p^h_{\sigma,s}$, and for $j \neq h$ let $p^j_{\sigma,s+1} = f^{h,j}(p^h_\sigma)[s]$.

If this is not a recovery stage for N_n, then for each j let $p^j_{\sigma,s} = p^j_{\sigma,s+1}$.

For $e \leq n < s+1$, let $\sigma = \alpha_{s+1} \upharpoonright n$.

If $\sigma\widehat{\ }1 \not\preceq \theta$, let $\widehat{\nu}^j_n$ be empty for each j (this includes the case that $d \leq n \leq s$).

If $\sigma\widehat{\ }1 \preceq \theta$, let $t+1 < s+1$ be the most recent $\sigma\widehat{\ }0$-stage of the construction. Check whether $f^{i+1,i+2}_s(p^{i+1}_{\sigma,t+1}) = p^{i+2}_{\sigma,t+1}$. If so, let $\widehat{\nu}^j_n[s+1]$ be empty for each j. If not, then let $\widehat{\nu}^j_n[s+1] = x_l, f^{i+1,j}_s(p^{i+1}_{\sigma,t+1}), f^{i+2,j}_s(p^{i+2}_{\sigma,t+1})$ for each j.

Perform a right shuffle in ν^i of the components

$$\widehat{\nu}^i_0, \widehat{\nu}^i_1, \ldots, \widehat{\nu}^i_s.$$

Likewise perform a right shuffle in ν^{i+1} of the components

$$\widehat{\nu}^{i+1}_0, \widehat{\nu}^{i+1}_1, \ldots, \widehat{\nu}^{i+1}_s$$

and a left shuffle in ν^{i+2} of the components

$$\widehat{\nu}^{i+2}_0, \widehat{\nu}^{i+2}_1, \ldots, \widehat{\nu}^{i+2}_s.$$

This concludes the shuffling at this stage.

Let u be the largest number such that for some j such that $0 \leq j \leq 2$, and some $x < k$, $f^{j,i+1}_{s+1}(x) = u$. Now set $r_{s+1}(\tau) = \max(u, r_s(\tau))$ and $r_{s+1}(\sigma) = r_s(\sigma)$ for each $\sigma \neq \tau$.

Declare that $Q_{\tau,i}$ has been prepared.

Proceed to clean-up phase.

Case 2: Completing $Q_{\tau,i}$.

We will be performing a right shuffle in ν^i and a left shuffle in ν^{i+1} and ν^{i+2}. As before, let $|\tau| = e$.

We will define $\widehat{\nu}^j_n[s+1]$ for each j and for $n \leq s$.

For each $n \leq s$, do as follows: Let $\sigma = \alpha_{s+1} \upharpoonright n$. Let x_l and y_k be the σ-reserved components.

If $s+1$ is a $\sigma\widehat{\ }2$-stage, let $\widehat{\nu}^j_n$ be empty for each j.

If $s+1$ is a $\sigma\widehat{\ }0$-stage and an h-recovery stage for N_n, then for each l, let $\widehat{\nu}^l_n = x_l, f^{h,l}(p^h_\sigma)[s], y_k$. Empty the σ-i-queue.

If $s+1$ is a $\sigma\widehat{\ }1$-stage and an h-recovery stage for N_n, check whether there is a number q in the σ-i-queue. If so, remove q from the σ-i-queue.

If we are performing a right shuffle in ν^h at this stage then for each j let $\widehat{\nu}^j_n = x_l, f^{h,j}_s(p^h_\sigma), f^{i,j}_s(q)$.

If we are performing a left shuffle in ν^h at this stage then for each j let $\widehat{\nu}_n^j = f_s^{i,j}(q), f_s^{h,j}(p_\sigma^h), y_k$.

If there is no number q in the σ-i-queue, then for each j set $\widehat{\nu}_n^j = x_l, f_s^{h,j}(p_\sigma^h), y_k$.

In each of the above cases, if $s+1$ is an h-recovery stage for N_n, let $p_{\sigma,s+1}^h = p_{\sigma,s}^h$, and for $j \neq h$ let $p_{\sigma,s+1}^j = f^{j,m}(p_\sigma^h)[s]$. If this is not a recovery stage for N_n, then for each j let $p_{\sigma,s}^j = p_{\sigma,s+1}^j$.

Let the active witness for τ-i-diagonalization be $w_{\langle \tau,n \rangle}$, with companions $w_{\langle \tau,m_0 \rangle} < w_{\langle \tau,m_1 \rangle}$.

Perform a right shuffle in ν^i of the components

$$w_{\langle \tau,m_0 \rangle}, w_{\langle \tau,n \rangle}, w_{\langle \tau,m_1 \rangle}, \widehat{\nu}_0^i, \widehat{\nu}_1^i, \ldots, \widehat{\nu}_s^i.$$

Likewise perform a left shuffle in ν^{i+1} of the components

$$w_{\langle \tau,m_0 \rangle}, w_{\langle \tau,n \rangle}, w_{\langle \tau,m_1 \rangle}, \widehat{\nu}_0^{i+1}, \widehat{\nu}_1^{i+1}, \ldots, \widehat{\nu}_s^{i+1}$$

and a left shuffle in ν^{i+2} of the components

$$w_{\langle \tau,m_0 \rangle}, w_{\langle \tau,n \rangle}, w_{\langle \tau,m_1 \rangle}, \widehat{\nu}_0^{i+2}, \widehat{\nu}_1^{i+2}, \ldots, \widehat{\nu}_s^{i+2}.$$

Declare $Q_{\tau,i}$ to be currently satisfied. Proceed to clean-up phase.

Clean-up phase after preparing or completing $Q_{\tau,i}$

We begin the clean-up phase by updating the queues. Let m be such that the shuffle in ν^m at stage $s+1$ is in the opposite direction to that in the other two structures. For each $n \leq e$ do as follows. Let $\sigma = \alpha_{s+1} \upharpoonright n$. Let x_l and y_k be the σ-reserved components.

If $\sigma\widehat{\ }2 \preceq \alpha_{s+1}$ do nothing to the σ-queues.

If $\sigma\widehat{\ }0 \preceq \alpha_{s+1}$ or $\sigma\widehat{\ }1 \preceq \alpha_{s+1}$, suppose $s+1$ is an h-recovery stage for N_n. Let $t+1 < s+1$ be the most recent $\sigma\widehat{\ }0$-stage of the construction.

Suppose $h \neq m$. Let $f^{h,m}(p_\sigma^h)[t] = q^m$. If $f^{h,m}(p_\sigma^h)[s] = q^m$, add q^m to the σ-m-queue.

Suppose $h = m$. For each $j \neq m$ let $f^{h,j}(p_\sigma^h)[t] = q^j$. If $f^{h,j}(p_\sigma^h)[s] = q^j$ for each such j, then let $q = f_{s+1}^{j,h}(q^j)$ and add q to the σ-h-queue (note that q is independent of the choice of j here). If $f^{h,j}(p_\sigma^h)[s] \neq q^j$ for one $j \neq h$ then add q^j to the σ-j-queue for that j. If $f^{h,j}(p_\sigma^h)[s] \neq q^j$ for both $j \neq h$ then make no additions to any σ-queues.

Next, declare any component which is involved in the shuffle is no longer σ-reserved for any σ.

For each $\sigma >_L \tau$, do as follows: Declare any σ-reserved components to be no longer σ-reserved. Declare σ to be inactive. Declare each $Q_{\langle \sigma,i \rangle}$ to be unsatisfied, and cancel any preparation of $Q_{\langle \sigma,i \rangle}$. Declare that there is no active witness for τ-i-diagonalization. Set $r_{s+1}(\sigma) = 0$.

For each σ, set $r_{s+1}(\sigma) = r_s(\sigma)$ if we have not defined it yet.

For each $\sigma \geq_L \tau$, empty each σ-j-queue. Then choose some fresh unmarked component $z_{\langle \sigma,k \rangle} > R(\alpha)[s+1]$ and for each j set $p_\sigma^j[s+1] = z_{\langle \sigma,k \rangle}$.

For each $j > i$, if $Q_{\tau,j}$ has been prepared but the preparation has not been canceled, declare the preparation canceled. Declare $Q_{\tau,j}$ unsatisfied. Declare that there is currently no active witness for τ-j-diagonalization.

For each σ and j such that there is currently no active witness for σ-j-diagonalization, choose a fresh unmarked number $w_{\sigma,n} > R(\alpha)[s+1]$ as active witness for σ-j-diagonalization.

We now specify the marking in the structures.

If we prepared $Q_{\tau,i}$ at this stage, choose two fresh unmarked numbers $w_{\langle \tau,m_1\rangle} > w_{\langle \tau,m_2\rangle} > R(\alpha)[s+1]$ as companions to the active witness for τ-i-diagonalization. Then τ-mark them.

For each $\sigma \preceq \alpha_{s+1}$, each k, and each j, σ-mark the precious component $f^{j,k}_{s+1}(p^j_{\sigma,s+1})$, and each active witness for σ-k-diagonalization.

For $l < e$, let $\tau \restriction l = \sigma$. If $s+1$ is a recovery stage for N_l, do as follows: σ-mark each component q of ν^j which is part of the shuffle at this stage, except for those such that for some $\beta \prec \sigma$ and some k, $q = f^{k,j}(p^k_{\beta,s+1})$, q is the active witness for τ-k-diagonalization, or q is a companion to the active witness for τ-k-diagonalization, and those for which there is some $\beta \prec \sigma$ such that q is β-reserved.

For each β such that $|\beta| \leq s+1$, do as follows: if there are no β-reserved components, choose fresh unmarked components x_l and y_k which are larger than $R(\alpha)[s+1]$. Then β-mark those components in each structure and declare them β-reserved. Then β-mark the active witness for β-j-diagonalization. Finally, for each component of a structure ν^j which has been β-marked in the clean-up phase, σ-mark it for each $\sigma \prec \beta$ such that $\sigma^\frown 0 \preceq \beta$ or $\sigma^\frown 1 \preceq \beta$.

Now for each component q of each structure ν^j which is β-marked, β-mark the component $f^{j,k}_{s+1}(q)$ of ν^k for each k.

This completes the construction.

Our verification will be as similar as possible to that of the previous construction, and, when possible, we will refer the reader to the first construction for details.

Lemma 4.4. *Let $s+1$ be some stage of the construction such that for each $t < s+1$, there are unique isomorphisms $f^{j,k}_t \colon \nu^j[t] \to \nu^k[t]$ for each j and k.*

Suppose $s+1$ is a σ-stage. Then at both the beginning and end of stage $s+1$ we have fresh σ-reserved components x_l and y_k which are σ-marked but not τ-marked for any $\tau >_L \sigma$.

Proof. See Lemma 3.6. □

Lemma 4.5. *Let $s+1$ be some stage of the construction such that for each $t < s+1$, and each j and k, there is a unique isomorphism $f^{j,k}_t \colon \nu^j[t] \to \nu^k[t]$.*

Suppose that for each j the components involved in the shuffle carried out in ν^j at stage $s+1$ are $u^j_0, \ldots, \nu^j_n$. Then for each j and k, and each $m \leq n$, $\nu^j(u^j_m)[s] = \nu^k(u^k_m)[s]$.

Thus $\nu^j[s+1]$ and $\nu^k[s+1]$ are isomorphic.

Proof. As in the previous construction, each component u^k_m used in the shuffle in $\widehat{\nu}^k_n$ is chosen to be the image of the corresponding component u^j_m under the map $f^{j,k}_s$. Thus by Remark 3.3, it follows that if the shuffles performed in ν^j and ν^k at stage $s+1$ are in opposite directions the structures $\nu^j[s+1]$ and $\nu^k[s+1]$ are isomorphic. Of course, if the shuffles performed in ν^j and ν^k are in the same direction, then $f^{j,k}_s$ is still an isomorphism between $\nu^j[s+1]$ and $\nu^k[s]$. □

We now work on proving that the isomorphisms between any pair of our structures are unique at each stage of the construction.

Lemma 4.6. *Suppose that for each $t \leq s$ and each i, j there is a unique isomorphism $f^{i,j}_t \colon \nu^i[t] \to \nu^j[t]$. Fix $n \leq s$, and let $\alpha_{s+1} \restriction n = \sigma$.*

Suppose $0 \leq i \leq 2$ and that q is part of $\widehat{\nu}^i_n[s+1]$. If q is part of $\widehat{\nu}^i_m[t]$ at some stage $t < s+1$ then $m = n$ and $\alpha_t \restriction n = \sigma$.

Proof. A non-fresh component which is used in $\widehat{\nu}^i[s+1]$ is either $f^{j,i}(p^j_\sigma)$ or $f^{j,i}(q)$ for some component q in a σ-j-queue. In either case it is clear that this component was most recently used in a shuffle at a σ-stage t and was part of $\widehat{\nu}^i[t]$ at that stage. $\square$

As before, our shuffling ensures that for each q, and each i and j, $\nu^i(q)[s+1] = \{q\}$ if and only if $\nu^j(q)[s+1] = \{q\}$. If this is the case, then $q \in \nu^i(x)[s+1]$ only for $q = x$.

Lemma 4.7. *Suppose that for each $t \leq s$ and each i, j there is a unique isomorphism $f^{i,j}_t : \nu^i[t] \to \nu^j[t]$.*

Suppose that $s+1$ is an h-recovery stage for N_n, and that $\alpha_{s+1} \restriction n = \sigma$. Suppose that $s+1$ is a $\sigma^\frown 1$-stage. Let $t_0 < s+1$ be the most recent $\sigma^\frown 0$-stage.

Then each σ-j-queue contains at most one element at the beginning of stage $s+1$.

For each j let $q^j = f^{h,j}(p^h_\sigma)[t_0]$.

There is an element q in the σ-h-queue at the beginning of stage $s+1$ precisely if $q \neq p^h_{\sigma,s}$ and for each $j \neq h$, $f^{j,h}_s(q^j) = q$. In this case q is the only component in any σ-queue.

Otherwise, if $j \neq h$, then the only element which can be in the σ-j-queue is q^j, which is in the σ-j-queue at the beginning of stage $s+1$ if and only if $f^{h,j}(p^h_\sigma)[s] \neq q^j$.

Proof. First note that there are no σ-initialization stages between t_0 and $s+1$, and each recovery stage for N_n between these stages is a $\sigma^\frown 1$-stage, $p^h_{\sigma,s+1} = p^h_{\sigma,t_0}$.

We work by induction on s. Let $t+1 < s+1$ be the most recent h-recovery stage for N_n such that $\alpha_{t+1} \restriction n = \sigma$.

For each i let $q^i = f^{i,h}(p^h_\sigma)[t_0]$. Let j be such that the shuffle in ν^j is in the opposite direction to the shuffles in the other two structures at stage $t+1$. We will check that the components in the σ-i-queues at the end of stage $t+1$ are the ones which the lemma claims to be present at the beginning of stage $s+1$. Our analysis of the situation depends on whether $j = h$.

Firstly, suppose $j = h$.

If at the start of stage $t+1$ the σ-queues are all empty, then by inductive hypothesis $f^{h,i}(p^h_\sigma)[t] = q^i$ for each i. Because the shuffle in ν^h is in the opposite direction to that in the other two structures, there is a component $q \neq p^h_{\sigma,t+1}$ of ν^h such that for each $i \neq h$, $f^{i,h}_{t+1}(q^i) = q$. We add q to the σ-h-queue. At the start of stage $s+1$ this is the only element in any σ-queue. Furthermore, we have $f^{i,h}_{t+1}(q^i) = f^{i,h}_s(q^i) = q \neq p^h_{\sigma,s}$, as required.

If at the start of stage $t+1$ the σ-i-queue is nonempty for one $i \neq h$, then by inductive hypothesis $f^{h,i}(p^h_\sigma)[t] \neq q^i$, and q^i is in the σ-i-queue. Let $0 \leq k \leq 2$, where k is equal to neither i nor h. Then $f^{h,k}_t(p^h_{\sigma,t_0}) = q^k$ by hypothesis, and because the shuffles in ν^h and ν^k are in opposite directions at stage $t+1$, $f^{h,k}_{t+1}(p^h_{\sigma,t_0}) \neq q^k$. We put q^k in the σ-k-queue at this stage. Furthermore, we do not use any elements from any of the σ-queues in our shuffles at stage $t+1$, so the only component which is part of $\widehat{\nu}^i_n[t+1]$ and which has previously been part of a shuffle is $f^{h,i}(p^h_\sigma)[t] \neq q^i$. Therefore $f^{i,h}_s(q^i), f^{k,h}_s(q^k)$, and $p^h_{\sigma,s}$ are all distinct, as required.

If at the start of stage $t+1$ the σ-i-queues are nonempty for both $i \neq h$, then the element in each of these σ-i-queues is q^i. For each such i, $f^{h,i}(p^h_\sigma)[t] \neq q^i$. Note that the two values of $f^{i,h}_t(q^i)$ are distinct by hypothesis. For each i, the element

q^i is not used in the shuffle in ν^i at stage $t+1$ and therefore the elements $f_s^{i,h}(q^i)$ are still distinct at the beginning of stage $s+1$, neither is equal to p^h_{σ,t_0}, and each q^i is still in the σ-i-queue at the start of stage $s+1$.

Finally, suppose that there is an element q in the σ-h-queue at the beginning of stage $t+1$. Then q is the only element in any σ-queue. Then $p^\sigma_{t_0} \neq q = f_t^{i,h}(q^i)$ for each $i \neq h$, and by including q in the shuffle in ν^h we ensure that for each $i \neq h$ we have $f_s^{i,h}(q^i) = f_{t+1}^{i,h}(q^i) = p^h_{\sigma,t_0}$. We remove q from the σ-h-queue, leaving all of the σ-queues empty at the beginning of stage $s+1$ as required.

Now suppose that $j \neq h$.

If at the start of stage $t+1$ the σ-j-queue and σ-h-queue are both empty, then $f^{h,j}(p^h_\sigma)[t] = f^{h,j}(p^h_\sigma)[t_0]$ and we add q^j to the σ-j-queue at stage $t+1$. Our shuffle at this stage ensures $f_{t+1}^{h,j}(p^h_{\sigma,t_0}) = f_s^{h,j}(p^h_{\sigma,t_0}) \neq q^j$.

If at the start of stage $t+1$, q^j is in the σ-j-queue, then we have $f_t^{j,h}(q^j) \neq p^h_{\sigma,t_0}$ and our choice of shuffle ensures that $f_s^{j,h}(q^j) = f_{t+1}^{j,h}(q^j) = p^h_{\sigma,t_0}$, and we remove q^j from the σ-j-queue.

Suppose that i is the index of the third structure, equal to neither j nor h. In each of the above cases, if q^i is in the σ-i-queue at the beginning of stage $t+1$, then $f_s^{h,i}(p^h_{\sigma,t_0}) = f^{i,h}(p^h_{\sigma,t_0}) \neq q^i$, and q^i is still in the σ-i-queue at the beginning of stage $s+1$, whereas if the σ-i-queue is empty, $f_s^{h,i}(p^h_{\sigma,t_0}) = f^{i,h}(p^h_{\sigma,t_0}) = q^i$, and the σ-i-queue is still empty at the beginning of stage $s+1$.

Finally, suppose that at the start of stage $t+1$ there is an element $q \neq p^h_{\sigma,t_0}$ in the σ-h-queue (and hence no other elements in any σ-queues). Then we have $f_t^{h,i}(q) = q^i$ for each $i \neq h$ and since q is not used in the shuffle in ν^i at this stage we ensure that $f_s^{h,i}(q) = f_{t+1}^{h,i}(q) = q^i$ for each such i. At the beginning of stage $s+1$, q is still in the σ-h-queue, as required.

□

Remark 4.8. Suppose that for each $t \leq s$ and each i,j there is a unique isomorphism $f_t^{i,j} : \nu^i[t] \to \nu^j[t]$. Suppose further that at some stage $t+1 < s+1$, $\widehat{\nu}^i_n[t+1]$ is nonempty. By observing which components can be the images of precious components and which components can be put into σ-queues, we may rule out the possibility of certain components ever being part of a shuffle at a later stage:

If we perform a right shuffle in ν^i at stage t then the rightmost component of $\widehat{\nu}^i_n[t+1]$ is not used in a shuffle at any later stage.

If we perform a left shuffle in ν^i at a stage t then the leftmost component of $\widehat{\nu}^i_n[t+1]$ is not used in a shuffle at any later stage.

Lemma 4.9. *Let $s+1$ be some stage of the construction such that for each $t < s+1$, there is a unique isomorphism $f_t^{i,j} : \nu^i[t] \to \nu^j[t]$ for each i and j.*

Suppose that p is a component of ν^i which is part of $\widehat{\nu}^i_n[t_0]$ at some first σ-stage $t_0 \leq s+1$.

Suppose that q is used in the shuffle in ν^i at stage $s+1$ and that $p \in \nu^i(q)[s+1]$. Suppose that $\alpha_{t_0} \upharpoonright n = \sigma$. Then $s+1$ is a σ-stage and either

1. *q is a member of $\widehat{\nu}^i_n[s+1]$,*
2. *or this is the last stage at which q is part of a shuffle.*

Proof. The proof is by induction on s.

If $s+1 = t_0$, suppose $p \in \nu^i(q)[t_0]$. If $q = p$ then q is certainly a member of $\widehat{\nu}^i_n[t_0]$. Otherwise, if the shuffle in ν^i at stage t_0 is a right shuffle q is directly to the left of p in the shuffle, and if the shuffle is a left shuffle, q is directly to the right of p in the shuffle.

Suppose q is not a member of $\widehat{\nu}^i_n[t_0]$. If the shuffle in ν^i at stage t_0 is a right shuffle q is either the rightmost component of $\nu^i_m[t_0]$ for for some m, or is the companion to the right of the active witness for τ-h-diagonalization, if this is a stage at which we are attempting to meet $Q_{\tau,h}$ for some h. If the shuffle in ν^i at stage t_0 is a left shuffle, q is either the leftmost component of $\widehat{\nu}^i_m[t_0]$ for for some m, or is the companion to the left of the active witness for τ-h-diagonalization. In either case, this is a component which is not part of a shuffle at any later stage.

Now suppose $s + 1 > t_0$.

Suppose q to be a component of ν^i such that $p \in \nu^i(q)[s + 1]$. Note that if $p \in \nu^i(q)[s]$ and $q \neq p$ then by hypothesis it must be the case that at some previous σ-stage $t < s + 1$, q is part of $\widehat{\nu}^i_n[t]$. Thus $s + 1$ is also a σ-stage and q is a member of $\widehat{\nu}^i_n[s + 1]$, by Lemma 4.6.

So we may assume that $s + 1$ is the first stage at which $p \in \nu^i(q)[s + 1]$. Then either we are performing a right shuffle in ν^i and q is directly to the left of a component u such that $p \in \nu^i(u)[s]$, or we are performing a left shuffle in ν^i and q is directly to the right of a component u such that $p \in \nu^i(u)[s]$ (here we view the shuffle as a cycle, so that the first component in the shuffle is immediately to the right of the last). Applying the inductive hypothesis to u, we see that $s + 1$ is a σ-stage, and that u is a member of $\widehat{\nu}^i_n[s + 1]$.

As in the base case, if q is not also a member of $\widehat{\nu}^i_n[s + 1]$, this is the last stage at which q is part of a shuffle in ν^i. □

Lemma 4.10. *Let $s + 1$ be some stage of the construction such that for each $t < s + 1$, there is a unique isomorphism $f^{i,j}_t : \nu^i[t] \to \nu^j[t]$ for each i and j.*

Suppose $t_0 < s + 1$ is the most recent σ-initialization stage, $p = p^i_\sigma[t_0]$, and that $\tau \neq \sigma$.

Then $p \notin p^j_\tau[s + 1]$ for any j.

However, if $s + 1$ is not a σ-initialization stage then $p \in p^j_\sigma[s + 1]$ for each j.

Proof. See the proof of Lemma 3.14. Lemma 4.9 should be used instead of Lemma 3.13. □

Lemma 4.11. *Let $s + 1$ be some stage of the construction such that for each $t < s + 1$, there is a unique isomorphism $f^{i,j}_t : \nu^i[t] \to \nu^j[t]$ for each i and j.*

Fix some i. Suppose that $\nu^i(x)[s] \not\subseteq \nu^i(y)[s]$ for each $x \neq y$.

Then $\nu^i(x)[s + 1] \not\subseteq \nu^i(y)[s + 1]$ for each $x \neq y$.

Thus by induction on s, $\nu^i[s]$ and $\nu^j[s]$ are rigid isomorphic structures for each s.

Proof. See Lemma 3.15. The itemized list of conditions (1)-(3) given in the proof of that lemma are still appropriate, although "left" and "right" may need to be swapped depending on the direction of the shuffle carried out in ν^i. Lemma 4.9 should be used in place of Lemma 3.13. □

Lemma 4.12. *Suppose that the components involved in the shuffle in ν^i at stage $s + 1$ are $n_0, \ldots, n_k$, and define $n_{-1} = n_k$, and $n_{k+1} = n_0$.*

If we perform a right shuffle in ν^i at stage $s + 1$, then for $0 \leq j \leq k$, $\nu^i(q)[s] \subseteq \nu^i(n_j)[s + 1]$ only for $q = n_j$ and $q = n_{j-1}$.

If we perform a left shuffle in ν^i *at stage* $s+1$*, then for* $0 \leq j \leq k$*,* $\nu^i(q)[s] \subseteq \nu^i(n_j)[s+1]$ *only for* $q = n_j$ *and* $q = n_{j+1}$*.*

Proof. See Lemma 3.16, once again using Lemma 4.9 instead of Lemma 3.13. □

Lemma 4.13. *Fix* σ*, and suppose that* $|\sigma| = e$*. For each* i*, if* q *is a component of* $\widehat{\nu}^i_e[s]$ *at infinitely many* σ*-stages of the construction, then* $\sigma^\frown 1 \prec TP$*.*

Proof. As in the two-structure construction, it is easy to check that if $\sigma <_L TP \upharpoonright e$, or $\sigma^\frown 2 \prec TP$, then there are only finitely many σ-stages s at which $\widehat{\nu}^i_e[s]$ is nonempty.

On the other hand, if $\alpha_s \upharpoonright e <_L \sigma$ then σ ceases to be active at stage s, and no component which is part of $\widehat{\nu}^i_e[t]$ at any σ-stage $t < s$ will ever be part of a shuffle after stage s, and so if $\sigma >_L TP \upharpoonright e$ each component of ν^i can be part of at most finitely many shuffles.

Now suppose $\sigma^\frown 0 \prec TP$, that for $s > t$ we have $\alpha_s \upharpoonright e \geq \sigma$, and that after stage t we never prepare or complete a requirement $Q_{\tau,i}$ for $\tau \leq_L \sigma$.

Let $t \leq t_0 + 1 < t_1 + 1$, where $t_0 + 1$ and $t_1 + 1$ are successive $\sigma^\frown 0$-stages of the construction. Suppose that $t_0 + 1$ is an h-recovery stage for N_e. Let $s + 1 < t_1 + 1$ be the most recent σ-stage which is a recovery stage for N_e. Then $t_1 + 1$ is a j-recovery stage for N_e, where j is a number such that the shuffles in ν^h and ν^j at stage $s+1$ are in opposite directions. If there are two such numbers j, then t_1+1 is a j-recovery stage for the j such that the most recent σ-stage which is a j-recovery stage for N_e is earliest. Thus $p^h_{\sigma,s+1} \neq f^{j,h}(p^j_\sigma)[s+1] = f^{j,h}(p^j_\sigma)[t_1]$. Because we empty the σ-queues at stage $t_1 + 1$, it is clear that no component can be part of $\widehat{\nu}^h_e[T]$ at a σ-stage $T < t_1 + 1$ and at a σ-stage $T \geq t_1 + 1$.

From this it follows that if there are infinitely many $\sigma^\frown 0$-stages which are h-recovery stages for N_e then no component will be part of $\widehat{\nu}^h_e[s]$ at infinitely many stages s of the construction.

Suppose then that ν^h, ν^i and ν^j are our three structures, than $\sigma^\frown 0 \prec TP$ and that there are only finitely many $\sigma^\frown 0$-stages which are i-recovery stages for N_e. Then from some point onward, all $\sigma^\frown 0$-stages alternate between h- and j-recovery stages for N_e.

Suppose as before that $t_0+1 < t_1+1$ are successive $\sigma^\frown 0$-stages of the construction after stage t, that $t_0 + 1$ is an h-recovery stage for N_e, $s + 1 < t_1 + 1$ be the most recent σ-stage which is a recovery stage for N_e. Assume that after stage t there are no more $\sigma^\frown 0$-stages which are i-recovery stages for N_e, and that the most recent $\sigma^\frown 0$-stage prior to $t_0 + 1$ is a j-recovery stage for N_e.

Then at at stage $s + 1$ the shuffles in ν^h and ν^i are in the same direction, since otherwise $t_1 + 1$ would be an i-recovery stage for N_e. Let $q^i = f^{h,i}(p^h_\sigma)[t_0]$.

If $q^i = f^{h,i}(p^h_\sigma)[s]$ then $q^i = f^{h,i}(p^h_\sigma)[s + 1]$ because the shuffles in ν^i and ν^h are in the same direction at stage $s + 1$. Then $q^i \neq f^{j,i}(p^j_\sigma)[t_1]$.

If $q^i \neq f^{h,i}(p^h_\sigma)[s]$, then because the shuffles in ν^i and ν^h are in the same direction at stage $s+1$, we do not include any elements from the σ-i-queue or σ-h-queue in any shuffles at this stage. Thus q^i cannot be part of $\nu^i_n[s + 1]$, and so $q^i \neq f^{j,i}(p^j_\sigma)[t_1]$, since that component is part of $\nu^i_n[s + 1]$.

In either case it follows that no component can be part of $\widehat{\nu}^i_n[T]$ at a σ-stage $T < t_1 + 1$ and a σ-stage $T \geq t_1 + 1$. Once again, we can use the fact that there are infinitely many $\sigma^\frown 0$-stages to see that there is no component of ν^i which is part of a shuffle at infinitely many stages of the construction. □

We now wish to check that our construction is guaranteed to produce the same infinite components in each of our structures, so that they really are isomorphic.

Before we do so, we note that any time we place a component into a σ-queue, it is because we wish to include it in another shuffle, and suspect it may be an infinite component. So we need to check that no component which should be infinite becomes permanently stuck in a σ-queue. To prove this, we will need to check that eventually there will be a σ-stage at which the shuffles in our structures are all in the appropriate directions to allow the component to be removed from its queue. Since the direction of shuffle is determined by which of the requirements we are attending to, we need to show that eventually we will attend to a suitable requirement. Thus we chose this point to show that the requirements $Q_{\tau,i}$ are only injured finitely often for $\tau \prec TP$.

Lemma 4.14. *Suppose that $\sigma <_L TP$. Fix some $i \leq 2$. Then there are finitely many stages at which we prepare or complete the requirement $Q_{\sigma,i}$, and $\lim_s r_s(\sigma)$ exists and is finite.*

Proof. We will show that for $\sigma <_L TP$, $\lim_s r_s(\sigma)$ exists and is finite.

If $\sigma <_L TP$ but $\sigma \not\prec TP$ then there are only finitely many σ-stages, we only attempt to meet $Q_{\sigma,i}$ finitely often, and $r_s(\sigma)$ approaches a finite limit.

Now we work by induction on the length of $\sigma \prec TP$. Given a pair σ, i such that $\sigma \prec TP$ and $i \leq 2$ suppose that t is a stage such that after stage t we never prepare or complete a requirement of the form $Q_{\tau,j}$ for $\tau <_L \sigma$ or for $\tau = \sigma$ and $j < i$, and that for $\tau <_L \sigma$, $r_t\tau = \lim_s r_s(\tau)$. Assume in addition that for $s \geq t$ we have $\alpha_s \restriction n \geq \sigma$. After stage t we we will always have the same active witness $w_{\langle\sigma,n\rangle}$ for σ-i-diagonalization.

Suppose that at some later stage $s+1 \geq t$, $Q_{\sigma,i}$ is not currently satisfied and has not been prepared (or the most recent preparation has been canceled), but that $Q_{\sigma,i}$ requires attention. Then at that stage we prepare $Q_{\sigma,i}$. At the next σ-stage we will complete the requirement $Q_{\sigma,i}$. This is the last stage at which we attend to this requirement. If $s+1$ is the last stage at which we attend to a requirement $Q_{\sigma,i}$ for any i, then for $t > s$ we have $r_{s+1}(\sigma) = r_t(\sigma)$. □

Lemma 4.15. *Let $\sigma \prec TP$. For each i there are infinitely many σ-stages of the construction at which the shuffle in ν^i is in the opposite direction to the shuffles in the other two structures.*

Proof. Let $\sigma \prec TP$. Fix some stage $s+1$. We will show that there is a σ-stage after stage $s+1$ at which the shuffle in ν^i is in the opposite direction to that in each of the other two structures.

Choose some large $n > e$ such that $\Phi_n^A(x) = x$ for every oracle A and every number x. Let $\beta = TP \restriction n$. Assume we have never prepared $Q_{\beta,i-1}$ before stage $s+1$. Since $\sigma \prec \beta$, if we ever prepare the requirement $Q_{\beta,i-1}$ at a stage after $s+1$, then that stage is a σ-stage at which the shuffle in ν^i is in the opposite direction to the shuffles in the other two structures. At some later β-stage $Q_{\beta,i-1}$ will require attention, and by Lemma 4.14 we may assume it is the highest priority requirement to do so. Thus at that stage we will prepare it. □

Lemma 4.16. *Suppose that $\sigma^\frown 1 \prec TP$, that $|\sigma| = e$, and that for $s+1 > t+1$, $\alpha_{s+1} \geq \sigma^\frown 1$ and that after stage $t+1$ we do not prepare or complete any requirement* *of the form* $Q_{\tau,i}$ *for* $\tau \preceq \sigma$. *Suppose furthermore that* $t+1$ *is a* $\sigma\frown 0$*-stage, and an* h*-recovery stage for* N_e.

Then for $s > t$, $p^h_\sigma[t] = p^h_\sigma[s]$, *and this is the only component which is part of* $\widehat{\nu}^h_n$ *at infinitely many* σ*-stages.*

For each $i \neq h$ *let* $q^i = f^{h,i}(p^h_\sigma)[t]$. *Then for each* $i \neq h$, q^i *is part of* $\widehat{\nu}^i_n[s+1]$ *at infinitely many* σ*-stages* $s+1 > t$, *and is the only component for which this is true. Furthermore, there are infinitely many* σ*-stages* $s+1 > t$ *at which* q^i *is part of the shuffle in* ν^i *and* $f^{h,i}_{s+1}(p^h_{\sigma,t}) = q^i$. *Thus* $\nu^i(q^i) = \nu^h(p^h_{\sigma,t})$.

Proof. Let σ, t and the q^i be as stated. It is clear that $p^h_{\sigma,t} = p^h_{\sigma,s}$ for $s \geq t$ and that this component is part of $\widehat{\nu}^h_n[s]$ at infinitely many σ-stages s.

Suppose that for some $i \neq h$, the component q^i is part of $\widehat{\nu}^i_n[s+1]$ at a σ-stage $s+1 \geq t$, and that $f^{h,i}_s(p^h_{\sigma,t}) = q^i$. We will show that q^i is also used in a shuffle at a later stage. If the shuffles in ν^h and ν^i are in the same direction at stage $s+1$, then $f^{h,i}_{s+1}(p^h_{\sigma,t}) = q^i$, and q^i will also be part of the shuffle in ν^i at the next σ-stage. So we assume that the shuffles in ν^h and ν^i are in opposite directions at stage $s+1$.

There are several cases to consider depending on whether the shuffle in the third structure is in the same direction as the shuffle in ν^i or the shuffle in ν^h at stage $s+1$.

If the shuffle in ν^i is in the opposite direction to both of the other shuffles at stage $s+1$, we add q^i to the σ-i-queue. If $s_0+1 > s+1$ is the next $\sigma\frown 1$-stage at which the shuffle in ν^i is in the opposite direction to the shuffles in the other two structures, then q^i is still in the σ-i-queue at the start of stage s_0+1, and is included in the shuffle in ν^i at stage s_0+1. Thus $f^{h,i}_{s_0+1}(p^h_{\sigma,t}) = q^i$.

Otherwise the shuffle in ν^h at stage $s+1$ is in the opposite direction to the shuffles in both of the other structures at that stage. Let ν^j be the third of our structures, and let $q^j = f^{h,j}_t(p^h_{\sigma,t})$.

If $f^{h,j}_s(p^h_{\sigma,t}) \neq q^j$ then there is already an element in the σ-j-queue at the beginning of stage $s+1$, and at this stage we add q^i to the σ-i-queue, and wait for a stage at which the shuffle in ν^i is in the opposite direction to the shuffles in the other two structures, at which point we include q^i in the shuffle in ν^i as above. If $s_0+1 > s+1$ is the next stage at which we include q^i in the shuffle in ν^i, then once again we have $f^{h,i}_{s_0+1}(q) = f^{h,i}_{s_0+1}(p^h_{\sigma,t}) = q^i$.

Finally, if $f^{h,j}_s(p^h_{\sigma,t}) = q^j$, then after the shuffle at stage $s+1$ we let q be the number such that $f^{i,h}_s(q^i) = f^{j,h}_s(q^j) = q$ and add q to the σ-h-queue. Once again, we are eventually guaranteed a stage s_0+1 at which the shuffle in ν^h is in the opposite direction to that in the other two structures, and will include q in the shuffle in ν^h and q^i in the shuffle in ν^i at stage s_0+1, and arranging that $f^{h,i}_{s_0}(q) = f^{h,i}_{s_0+1}(p^h_{\sigma,t}) = q^i$.

This exhausts all possible cases, and shows that that q^i is included in shuffles in ν^i at infinitely many $\sigma\frown 1$-stages s at which $f^{h,i}_s(p^h_{\sigma,t}) = q^i$, and so $\nu^i(q^i) = \nu^h(p^h_{\sigma,t})$, as required.

Now we check that $p^h_{\sigma,t}$ is the only component in $\widehat{\nu}^h_n[s+1]$ at infinitely many stages, and likewise that for each $i \neq h$, q^i is the only component which is part of $\widehat{\nu}^i_n[s+1]$ at infinitely many stages $s+1$ of the construction.

In the case of $\widehat{\nu}^h_n$, suppose that $q \neq p^h_{\sigma,t}$ is a component which is a member of $\widehat{\nu}^h_n[s+1]$ at some σ-stage $s+1 > t$. Note that the only way that this component can be part of $\widehat{\nu}_n^h[s_0+1]$ at some later σ-stage $s_0+1 > s+1$ is if at stage $s+1$ we add $f_{s+1}^{h,j}(q)$ to the σ-j-queue for some j. Let s_0+1 be the first such stage. Observe that if we are performing a right shuffle in ν^h, then q is the rightmost component in $\widehat{\nu}_n^h[s_0+1]$, whereas if we are performing a left shuffle, q is the leftmost component in $\widehat{\nu}_n^h[s_0+1]$. In either case q is never again part of a shuffle, as noted in Remark 4.8.

Now suppose that $i \neq h$, and that $q \neq q^i$ is part of $\widehat{\nu}_n^i[s+1]$ at some stage $s+1 > t$. We must show that q is only part of the shuffle in ν^i at finitely many stages, and therefore may assume that q has already been part of the shuffle in ν^i at some previous stage. Let ν^j be the third structure other than ν^i and ν^h. There are two cases to consider.

The first case we consider is that $q = f_s^{h,i}(p_{\sigma,t}^h)$. Then q is the middle element of $\widehat{\nu}_n^i[s+1]$. Let $s_0+1 \geq s+1$ be the first $\sigma^\frown 1$-stage at which the shuffles in ν^h and ν^i are in opposite directions. Then $q = f_{s_0}^{h,i}(p_{\sigma,t}^h)$ but $q \neq f_{s_0+1}^{h,i}(p_{\sigma,t}^h)$. The only reason that q can be included in a shuffle again at a future σ-stage $s_1+1 > s_0+1$ is if $q = f_{s_1}^{j,i}(q^j)$ and q^j is in the σ-j-queue at the beginning of stage s_1+1. We will treat this possibility as our second case.

The second case is that $q = f_s^{j,i}(q^j)$, and q^j is in the σ-j-queue at the beginning of stage $s+1$, and $s+1$ is a stage at which the shuffle in ν^j is in the opposite direction to that of the shuffles in the other two structures. But then if the shuffle performed in ν^i at stage $s+1$ is a right/left shuffle, then q is the rightmost/leftmost element in $\nu_n^i[s+1]$ and is therefore never part of a shuffle at any future stage.

Thus we see that q is included in a shuffle at most three times: after being shuffled once, it can be part of the shuffle at most once at a stage at which the first case applies, and then possibly at one more stage for the second case. □

Lemma 4.17. *For each i, if x and y are distinct components of ν^i then $\nu^i(x) \not\subseteq \nu^i(y)$, and for each i and j there is a unique homomorphic embedding $f\colon \nu^i \to \nu^j$, which is an isomorphism.*

Proof. It is clear that if either of x or y is finite then $\nu^i(x) \not\subseteq \nu^i(y)$, since no infinite component can embed into an finite component, and any embedding into a finite component is ruled out at the stage at which we enumerate the last element into it.

Lemma 4.16 precisely identifies the infinite components of our structures, and shows that we build identical infinite components in each structure. Furthermore, Lemma 4.10 and Lemma 4.11 together show that if x and y are distinct infinite components of ν^i then $\nu^i(x) \not\subseteq \nu^i(y)$. This suffices to give the result. □

Now that we have verified that our structures are all isomorphic, we return to the requirements $Q_{\tau,i}$: having already seen that we only attend to each one finitely often, we must check that we succeed in meeting them.

Lemma 4.18. *Let $\tau = TP \restriction e$, and suppose $0 \leq i \leq 2$. Suppose that $s+1$ is a stage of the construction such that for $t \geq s+1$, $\alpha_t \geq_L \tau$, and that after stage s we never prepare or complete a requirement of the form $Q_{\sigma,j}$ for any $\sigma \prec \tau$, nor any requirement of the form $Q_{\tau,j}$ for $j < i$.*

Suppose that $s+1$ is a τ-stage at which $Q_{\tau,i}$ requires attention, and at which any prior preparation of that requirement has been canceled. Let g be the function *of Condition 4.3 which we use when preparing $Q_{\tau,i}$ at stage $s+1$. Suppose $w_{\langle\tau,n\rangle}$ is the active witness for τ-i-diagonalization at stage $s+1$.*

Then $g \preceq f^{i+1,i+2}$, $f^{i,i+1}(w_{\langle\tau,n\rangle}) \neq w_{\langle\tau,n\rangle}$, and

$$\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle}) \downarrow = w_{\langle\tau,n\rangle} \neq f^{i,i+1}(w_{\langle\tau,n\rangle}).$$

Proof. Let θ, g and k be as in Condition 4.3. Suppose that $|\theta| = d$. We will first show that $f^{i+1}_{\tau,s+1} \upharpoonright k = f^{i+2}_{\theta} \upharpoonright k$.

Because of the restrictions placed on g by Condition 4.3, the only components $q < k$ of ν^{i+1} involved in the shuffle at stage $s+1$ are those s-predicted to be precious by θ.

Each component $q < k$ of ν^{i+1} which is s-predicted to be precious by τ has $f^{i+1,i+2}_{\theta,s}(q) = f^{i+1,i+2}_{\tau,s}(q) = f^{i+1,i+2}_{\tau,s+1}(q)$, since $\tau \preceq \theta$.

For $e \leq n < d$, $\widehat{\nu}^j_n[s+1]$ is nonempty for precisely those n such that $\sigma = \theta \upharpoonright n$ and $\sigma^\frown 1 \preceq \theta$ and for which $f^{i+1,i+2}_s(p^{i+1}_{\sigma,t+1}) \neq p^{i+2}_{\sigma,t+1}$, where $t+1 < s+1$ is the most recent $\sigma^\frown 0$-stage of the construction. Furthermore, in that case our choice of $\widehat{\nu}^{i+1}_n[s+1]$ and $\widehat{\nu}^{i+2}_n[s+1]$ ensures that $f^{i+1,i+2}_{s+1}(p^{i+1}_{\sigma,t+1}) = p^{i+2}_{\sigma,t+1}$. Thus if $q < k$ is a component of ν^{i+1} which is s-predicted to be precious by θ but not by τ, then $f^{i+1,i+2}_{s+1}(q) = f^{i+1,i+2}_{\tau,s+1}(q) = f^{i+1,i+2}_{\theta,s}(q)$.

Since these are the only components smaller than k which are used in the shuffle in ν^{i+1} at this stage, the only components on which the maps $f_{\tau,s} \upharpoonright k$ and $f_{\tau,s+1} \upharpoonright k$ can differ are those which are s-predicted to be precious by θ. So $f^{i+1,i+2}_{\tau,s+1} \upharpoonright k = f^{i+1,i+2}_{\theta,s} \upharpoonright k$.

We set $r_{s+1}(\tau) \geq u$, where $u = \max\limits_{x<k, 0\leq i,j\leq 2}(f^{i,j}(x))$. For each $\beta \geq \tau$ and every j, we define $p^j_\beta[s+1]$ to be a number larger than $R_{s+1}(\tau)$. In addition, we choose new active witnesses $w_{\langle\beta,n\rangle} > R_{s+1}(\tau)$ for β-j-diagonalization for each for $\beta >_L \tau$ as well as for for $\beta = \tau$ and $j > i$. We also β-reserve new elements x_l and y_k, both larger than $R_{s+1}(\tau)$, for each $\beta \preceq \tau$ and $\beta >_L \tau$.

But $\alpha_{s_0+1} \geq_L \tau$ for $s_0+1 > s+1$, and $s+1$ is a stage after which we never attempt to prepare or complete a requirement of higher priority than $Q_{\tau,i}$. So for each j, the only components $q < R_{s+1}(\tau)$ of ν^{i+1} which can be part of a shuffle at a stage $t > s+1$ are those which are s-predicted to be precious by τ, and the active witness $w_{\langle\tau,n\rangle}$ for τ-i-diagonalization.

When we first chose $w_{\langle\tau,n\rangle}$ as our active witness for τ-i-diagonalization, it had never been part of a shuffle. Since at stage $s+1$ we prepare $Q_{\tau,i}$, any witness for τ-i-diagonalization active during a previous canceled attempt to prepare $Q_{\tau,i}$ has since been replaced. So $w_{\langle\tau,n\rangle}$ has not been used in a shuffle before stage $s+1$.

Thus $f^{i,i+1}_s(w_{\langle\tau,n\rangle}) = f^{i,i+1}_{s+1}(w_{\langle\tau,n\rangle}) = w_{\langle\tau,n\rangle}$. Note that $w_{\langle\tau,n\rangle}$ is part of the right shuffle at the next τ-stage $s_1+1 > s+1$, since at that stage the requirement $Q_{\tau,i}$ is the highest priority requirement which requires attention, and is therefore completed at that stage. We never use the component $w_{\langle\tau,n\rangle}$ of ν^i in a shuffle after stage s_1+1, so $f^{i,i+1}(w_{\langle\tau,n\rangle}) = f^{i,i+1}_{s_1+1}(w_{\langle\tau,n\rangle}) \neq w_{\langle\tau,n\rangle}$.

Furthermore, at stage s_1+1 the shuffles in ν^{i+1} and ν^{i+2} are in the same direction, so that $f^{i+1,i+2}_{s+1} \upharpoonright k = f^{i+1,i+2}_{s_1} \upharpoonright k = f^{i+1,i+2}_{s_1+1} \upharpoonright k$ and $f^{i+1,i+2}_{\tau,s_1+1} \upharpoonright k = f^{i+1,i+2}_{\tau,s+1} \upharpoonright k$. After stage $s+1$, the only components $q < R_{s+1}(\tau)$ of ν^{i+1} which are ever involved in a shuffle are those which are s-predicted to be precious by τ. Each such component is of the form $p^{i+1}_{\sigma,t+1}$ for some σ such that $\sigma^\frown 1 \preceq \tau \prec TP$, where

$t+1 < s+1$ is the most recent $\sigma^\frown 0$-stage of the construction. As we saw in Lemma 4.16, for such components, $f^{i+1,i+2}(p^{i+1}_{\sigma,t+1}) = p^{i+2}_{\sigma,t+1}$.

This is sufficient to show $g \prec f^{i+1,i+2}$, since $g = f^{i+1,i+2}_{\tau,s_0+1} \upharpoonright k$ at all stages $s_0 + 1 > s + 1$, and we have $\sigma^\frown 1 \preceq \tau \prec TP$, so $f^{i+1,i+2}_{\tau,s_0+1}(p^{i+1}_{\sigma,t+1}) = p^{i+2}_{\sigma,t+1}$ for each such σ. So

$$\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle})\downarrow = w_{\langle\tau,n\rangle} \neq f^{i,i+1}(w_{\langle\tau,n\rangle})$$

and thus the requirement $Q_{\tau,i}$ has been satisfied by the preparation and completion performed at stages $s+1$ and s_1+1. □

Lemma 4.19. *For* $0 \leq i \leq 2$, $f^{i+1,i+2} \not\geq_T f^{i,i+1}$.

Proof. We must show that for each $\tau = TP \upharpoonright e$ we either prepare and complete the requirement $Q_{\tau,i}$ at a stage after which it is never canceled, or that we are able to passively meet the requirement.

So fix some i and e, let $\tau = TP \upharpoonright e$, and suppose that t_0 is a stage such that at stages $s+1 \geq t_0$ we never attempt to meet a requirement of the form $Q_{\beta,j}$ for any $\beta <_L \tau$ and any j or $Q_{\tau,j}$ for any $j < i$. Assume in addition that for $s+1 \geq t_0$ we have $\alpha_{s+1} \geq_L \tau$. We may suppose that t_0 is the least stage with this property, in which case any preparation or completion of the requirement $Q_{\tau,i}$ performed at a stage prior to t_0 is canceled at stage t_0. Let $w_{\langle\tau,n\rangle}$ be the active witness for τ-i-diagonalization at stage t_0. Note that this is the active witness for τ-i-diagonalization at every future stage $s+1 \geq t_0$ of the construction.

If we ever prepare the requirement $Q_{\tau,i}$ after stage $s+1$, we will then complete it at a later stage as in Lemma 4.18, in which case $\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle})\downarrow \neq f^{i,i+1}(w_{\langle\tau,n\rangle})$.

So suppose that we never prepare $Q_{\tau,i}$ after stage t_0, but that $\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle})\downarrow = f^{i,i+1}(w_{\langle\tau,n\rangle})$. If so, $f^{i,i+1}(w_{\langle\tau,n\rangle}) = w_{\langle\tau,n\rangle}$ because $w_{\langle\tau,n\rangle}$ is never used in a shuffle. Since $\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle})\downarrow = w_{\langle\tau,n\rangle}$, choose some k and s_0 such that $\Phi_{e,s_0}^{f^{i+1,i+2}\upharpoonright k}(w_{\langle\tau,n\rangle}) = w_{\langle\tau,n\rangle}$. Choose $\theta \prec TP$ such that $\tau \prec \theta$ and such that every infinite component $q < k$ of ν^{i+1} is of the form $p^{i+1}_{\sigma}[t]$ for infinitely many $t > t_0$, where $\sigma^\frown 1 \preceq \theta$. Let $s > s_0$ be a stage so large that for $t > s$, $\alpha_t \geq_L \theta$, and no finite component $q < k$ of ν^{i+1} is part of a shuffle after stage s. Then at every θ-stage $t > s$, we have $f^{i+1,i+2}_{\theta,t} \upharpoonright k = f^{i+1,i+2} \upharpoonright k$, by choice of k, s, and θ. At the first such stage t, $\Phi_e^{f_\theta \upharpoonright k}(w_{\langle\tau,n\rangle})[t]\downarrow = w_{\langle\tau,n\rangle}$. Thus at this stage $Q_{\tau,i}$ requires attention, and therefore we must prepare it, contrary to our assumption.

So either $\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle})\downarrow \neq f^{i,i+1}(w_{\langle\tau,n\rangle})$ or $\Phi_e^{f^{i+1,i+2}}(w_{\langle\tau,n\rangle})\uparrow$, and thus (considering every e) we have $f^{i+1,i+2} \not\geq_T f^{i,i+1}$. □

As in the two-structure construction, it is clear that for each i and j, $f^{i,j}(x) = y$ if and only if $f_s(x) = y$ for infinitely many s, and thus that $f^{i,j} \leq_T \emptyset''$. The requirements $Q_{\tau,i}$ also ensure that $f^{i,j} <_T \emptyset''$ for each i and j.

For each e we now wish to determine whether the structure ρ_e is isomorphic to our structures ν^i, and, if so, show that it is computably isomorphic to one of the ν^i. In this way we will meet the requirement N_e.

As in the two structure construction, it suffices to note that there is some string σ of length e and some number l such that $\sigma^\frown l \prec TP$, and then check the three cases.

Lemma 4.20. *If* σ *is of length* e *and* $\sigma^\frown 2 \prec TP$ *then for each* i, ρ_e *is not isomorphic to* ν^i.

Proof. See Lemma 3.26. ☐

Lemma 4.21. *If σ is of length e and $\sigma^\frown 1 \prec TP$, let i be such that after some stage, every $\sigma^\frown 1$-stage is an i-recovery stage for N_e. Then if ρ_e is isomorphic to ν^i, the isomorphism is computable.*

Proof. See Lemma 3.29. ☐

In the case that $\sigma^\frown 0 \prec TP$, there are slightly less trivial changes arising from the fact that there are three structures, and we also require a variant of Lemma 3.27. We now state this lemma and give a brief sketch to point out the (minor) changes to the proof.

Lemma 4.22. *Let σ be of length e, and suppose $\sigma \prec TP$. Assume that for $t \geq t_0$, $\alpha_t \geq_L \sigma$, that t_0 is a σ-stage which is a recovery stage for N_e, and that after stage t_0 we never attempt to meet $Q_{\theta,i}$ for any $\theta \leq_L \sigma$.*

Suppose $s+1 > t_0$ is a σ-stage of the construction which is an i-recovery stage for N_e.

Suppose that p_σ is the component of ρ_e for which $h^i_\sigma(p^i_\sigma)[s] = p_\sigma$.

Suppose $t+1 > s+1$ is the next σ-stage which is a recovery stage for N_e, and that $t+1$ is a $\sigma^\frown 0$ stage. Suppose that $t+1$ is a j-recovery stage for N_e.

Then $\nu^i(p^i_\sigma)[s] \subsetneq \nu^j(p^j_\sigma)[t] \subseteq \rho_e(p_\sigma)[t]$.

Proof. Let $p_\sigma = h^i_\sigma(p^i_\sigma)[t]$. As in the two-structure construction, we note that for $n \geq e$ the components of $\widehat{\nu}^i_n[s+1]$ are σ-marked at stage $s+1$. Using this together with Lemma 4.12, we see that $h^i_\sigma[s] \subseteq h^i_\sigma[t]$ if and only if $h^i_\sigma(p^i_\sigma)[t] = p_\sigma$. Since $h^i_\sigma[s] \not\subseteq h^i_\sigma[t]$, let j be such that the shuffles in ν^j and ν^i are in opposite directions at stage $s+1$. But p_σ is in the range of $h^i_\sigma[t]$, and must be the image of a component q of ν^j such that $\nu^i(p^i_\sigma)[s] \subseteq \nu^j(q)[t]$ but $\nu^j(q)[t] \neq p^i_\sigma[t]$. Since $p^j_{\sigma,t} = p^j_{\sigma,s+1} = f^{i,j}(p^i_\sigma)[s]$ and the shuffles in ν^i and ν^j are in opposite directions at stage $s+1$, $p^j_\sigma[s]$ is the unique component of ν^j satisfying this condition. So $\nu^i(p^i_\sigma)[s] \subsetneq \nu^j(p^j_\sigma)[t] \subseteq \rho_e(p_\sigma)[t]$. This is true in particular for that j for which $t+1$ is a j-recovery stage for N_e. ☐

Lemma 4.23. *If σ is of length e and $\sigma^\frown 0 \prec TP$ then for each i, ρ_e is not isomorphic to ν^i.*

Proof. Let t_0 be such that for $t \geq t_0$, $\alpha_t \geq_L \sigma$, that t_0+1 is a σ-stage which is an i_0-recovery stage for N_e, and that after stage t_0 we never attempt to meet $Q_{\theta,j}$ for any j and any $\theta \leq_L \sigma$.

Suppose that $t_0+1 < t_1+1 < \cdots$ are all of the $\sigma^\frown 0$-stages of the construction after t_0.

Let p_σ be the component of ρ_e such that $h^{i_0}_\sigma(p_\sigma)[t_0] = p^{i_0}_{\sigma,t_0}$.

It follows from Lemma 4.22 that we have $\rho_e(p_\sigma)[t_n] \supseteq \nu^{i_n}(p^{i_n}_\sigma)[t_n]$ for each n. So p_σ is an infinite component of ρ_e such that $\rho_e(p_\sigma) \supset \nu^i(p^{i_0}_\sigma)[t_0]$. However, there is no component q of our structure ν^i such that $\nu^i(q) \supset \nu^i(p^{i_0}_\sigma)[t_0]$, since by Lemma 4.13, no component is part of $\widehat{\nu}^i_n[t]$ at infinitely many stages t. ☐

As noted earlier, these three cases are exhaustive. Thus either our structures ν^i are not isomorphic to the computable structure ρ_e, or one of our structures is computably isomorphic to ρ_e. So we have met each requirement N_e.

This concludes the verification that we have met the requirements laid out in the construction. Theorem 4.1 and hence Corollary 4.2 now follow immediately from the correspondence between c.e. Friedberg enumerations and their associated graphs.

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