

# Computability Results Used in Differential Geometry

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### Abstract

Topologists Nabutovsky and Weinberger discovered how to embed computably enumerable (c.e.) sets into the geometry of Riemannian metrics modulo diffeomorphisms. They used the complexity of the settling times of the c.e. sets to exhibit a much greater complexity of the depth and density of local minima for the diameter function than previously imagined. Their results depended on the existence of certain sequences of c.e. sets, constructed at their request by Csima and Soare, whose settling times had the necessary dominating properties. Although these computability results had been announced earlier, their proofs have been deferred until this paper.

Computably enumerable sets have long been used to prove *undecidability* of mathematical problems such as the word problem for groups and Hilbert’s Tenth Problem. However, this example by Nabutovsky and Weinberger is perhaps the first example of the use of c.e. sets to demonstrate specific *mathematical or geometric complexity* of a mathematical structure such as the depth and distribution of local minima.

## 1 Introduction

### 1.1 Computability and Differential Geometry

#### 1.1.1 Astonishing Richness of Riemannian Metrics

In the book from his Porter lectures, topologist and geometer Shmuel Weinberger [2005] explained the significance of the present interaction between computability and differential geometry on the space  $\text{Riem}(M)$  of Riemannian metrics (modulo diffeomorphisms) on certain smooth, compact manifolds  $M$ , a space which is of interest to a variety of mathematicians and physicists. Nabutovsky and Weinberger describe in their paper [2003] on fractals “*the astonishing richness of the space of Riemannian metrics on a smooth manifold, up to reparametrization.*” This “astonishing richness” of the space depends on two main parts: constructing a sequence of c.e. sets

with a certain complexity of the settling time functions; and embedding c.e. sets into the geometric space using diverse results of mathematics, group theory, differential topology, differential geometry, and other items.

### 1.1.2 Computably Enumerable Sets and Unsolvability

First, Nabutovsky and Weinberger expanded a long tradition of embedding c.e. sets into mathematical objects; unsolvability of the word problem and triviality problem for finitely presented groups; associating such a group with the fundamental group of a manifold; the unsolvability of the homeomorphism problem for manifolds, and much more. However, instead of merely obtaining unsolvable problems in a new area of mathematics, they linked sequences of computably enumerable (c.e.) sets  $W_n$ , and their settling times to the geometry of this space. Remarkably they related the halting time of the Turing machine enumerating  $W_n$  to the *depth and distribution of local minima* for certain functions on the space such as the diameter function as explained in §1.4. (Previously, embedding c.e. sets into a given mathematical structure had been used primarily to show *undecidability* of some associated theory, not the mathematical or geometric *complexity* of that structure.)

### 1.1.3 C. E. Sets and Geometric Complexity

Second, Weinberger asked Soare to prove a specific result about sequences of computably enumerable sets so that the complexity of halting times of the Turing machines could be transferred into the geometric complexity of the local minima. Soare constructed the required sequence of c.e. sets. Later Weinberger asked for a sequence with a stronger property in order to simplify the geometric part of the proof. Csima [2003] constructed this stronger sequence. In the present paper we prove the Main Theorem 5.1 which builds a sequence of c.e. sets  $\{A_n\}_{n \in \omega}$  which combines these two results and further generalizes them by making the sequence decrease in Turing degree as well. We give more details of the Nabutovsky-Weinberger results and their relation to c.e. sets and degrees in §1.4.

### 1.1.4 C.E. Turing Degrees and Depth of Minima

The main domination property requested by Weinberger of Soare and Csima was that the settling time of  $A_n$  dominates that of  $A_{n+1}$ , denoted by  $A_n >_{st} A_{n+1}$ , and defined below in Definition 1.5. However, the original results of Nabutovsky and Weinberger used the Sacks density theorem as

we explain in §1.4.1. This is because their full results, quoted in §1.4.2 and explained further in Soare [2004, §9], relate the Turing *degree* of a c.e. set (not merely the structure of the c.e. set itself) to the depth and distribution of the local minima of the diameter function on  $Al(M)$ . Therefore, it is of interest when constructing the sequence  $\{A_n\}_{n \in \omega}$  to consider *both* partial orderings  $A_n >_{st} A_{n+1}$  and  $A_n >_T A_{n+1}$  and this is accomplished in our Main Theorem 5.1 developed by Csima and Soare jointly.

In an upcoming paper, Csima and Shore [ta] show that any partial ordering embeds into the  $>_{st}$  ordering. This result also implies the required result for the Nabutovsky Weinberger work, though it does not control Turing degree.

### 1.1.5 Settling Time Reducibility and *ibT*-Reducibility

In addition, the partial ordering  $>_{st}$  and its underlying ordering, *identity bounded Turing reducibility* ( $>_{ibT}$ ), were of interest to Weinberger and had previously been studied in computability theory and in Kolmogorov complexity.<sup>1</sup> The special case of *identity bounded Turing reducibility* ( $B \leq_{ibT} A$ ) if  $A \leq_{bT} B$  with  $h(x)$  the identity, has been explicitly identified in Soare [2004], but the concept has often been used in the literature for decades. For example, the standard permitting method in Soare [1987, p. 85] to build a simple set  $B$  computable in a noncomputable c.e. set  $A$  arranges that  $x$  enters  $B$  only when some  $y \leq x$  enters  $A$ . This ensures not only that  $B \leq_T A$ , but in fact that  $B \leq_{ibT} A$ , although this is not always explicitly mentioned. (See Definition 2.1 for *bT* and *ibT*.)

### 1.1.6 Kolmogorov Complexity and *ibT*-reducibility

The *ibT*-reducibility is also used in Kolmogorov complexity. Lewis and Barmpalias begin their papers [ta] and [ta2] with a definition of *ibT* and in [ta1] they explain, “This gives a reducibility which is complexity sensitive and which, in particular, preserves most notions of randomness for binary strings.” They go on to relate *ibT* to the “computably Lipschitz” ( $B \leq_{cl} A$ ) condition where the bound is  $h(x) = x + c$  for some constant  $c$  called ( $B \leq_{sw} A$ ), a linear condition studied by Downey, Hirschfeldt, and LaForte, who wrote in *Randomness and Reducibility* [2004, p. 5],

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<sup>1</sup>In the literature *bT*-reducibility has also been called weak-truth-table(*wtt*) reducibility, viewed as a weakening of *tt*-reducibility which is already a strengthening of Turing reducibility, whereas *bT* reaches the same destination in one step from Turing reducibility.

“We begin with *sw*-reducibility, which has some nice features but also some shortcomings. It is related to a reducibility [*ibT*] recently studied by Soare [2004] and Csima [2003] in connection with computability-theoretic notions arising from the work of Nabutovsky and Weinberger [2003] in differential geometry. Informally, *sw*-reducibility says that there is a natural way, with little compression, to produce the bits of one real from another. It agrees with Solovay reducibility on *strongly* c.e. reals, but is in general different. Recently Yu and Ding have proven a number of interesting results about *sw*-reducibility, one of which is that there is no maximum *sw*-degree of c.e. reals. . . .”

The concept of *sw*-reducibility is natural and useful, as studied in Downey, Hirschfeldt, and LaForte [2004, §2], and as they explain on page 6 of §2.

### 1.1.7 Other Papers on Settling Time and *ibT*

Csima [2003] developed new results about these reducibilities and related them to earlier work in the subject by R.W. Robinson and others. Some of these results are given in §2. Further properties of the  $>_{st}$  ordering will be given in Csima and Shore [ta].

## 1.2 Background Sources

All the computability results related to differential geometry and announced previously in Soare [2004], Csima [2003], Nabutovsky and Weinberger [2003], and Weinberger [2005] are contained in this paper. The notation and background results on computability theory can be found in Soare [1987] and the forthcoming book Soare [cta] on computability theory and applications.

Soare [2004] gives an account of the Nabutovsky-Weinberger results in [NW, 2000] and [NW, 2003] and provides a background for logicians of the topology, number theory, and differential geometry needed to understand these results. There in §9 the Nabutovsky-Weinberger results are stated and very brief sketches of some of the main ideas of the proofs are given. Weinberger [2005] gives a very interesting description of the main mathematical areas needed to understand the proof including logic. This provides much more explanation and intuition into the geometry and topology than the original papers by Nabutovsky and Weinberger [2000] and [2003]. However, the present paper can be read without knowledge of the geometry because the results here are purely computability theoretic, and deal with:

(1) properties of the ordering  $>_{st}$  in §2; and (2) the Main Theorem 5.1 on the existence of dominating sequences.

### 1.3 Notation

We use the notation of Soare [cta] which is an update of that in Soare [1987], defining any new notation not in Soare [1987]. The following notation appears not only in Soare [cta], but also in Cooper [2004] and is becoming standard.

**Definition 1.1** (i) Let  $P_e$  be the  $e^{\text{th}}$  Turing program and let  $\varphi_e$  denote the  $e^{\text{th}}$  partial computable (p.c.) function, that computed by  $P_e$ .

(ii) We let  $\widehat{P}_e$  be the  $e^{\text{th}}$  oracle Turing program and  $\Phi_e^A(x) = y$  the partial function computed by  $\widehat{P}_e$  with  $A$  on the oracle tape on input  $x$  if it halts with output  $y$ . We call  $\Phi_e$  the *Turing reduction* (*Turing functional*) defined by oracle Turing program  $\widehat{P}_e$ .

(iii) Corresponding to  $\Phi_e^A(x) = y$  is the *use function*  $\varphi_e^A(x) = u$ , where  $u$  is the maximum argument in the characteristic function of  $A$  scanned (used) during the computation.

**Convention 1.2** The p.c. function  $\varphi_e(x)$  in part (i) should *never* be confused with the  $A$ -computable use function  $\varphi_e^A(x)$  in part (iii). The latter *always* has a superscript of the corresponding oracle set  $A$ , but the former *never* does.

#### 1.3.1 Quantifiers, Domination, and Setting Time

**Definition 1.3** (i)  $(\exists^\infty x) R(x)$  abbreviates  $(\forall y)(\exists z > y) R(z)$ , which is read, “there exist infinitely many  $x$  such that  $R(x)$ .”

(ii)  $(\forall^\infty x) R(x)$  abbreviates  $(\exists y)(\forall z > y) R(z)$ , which is read, “for almost every  $x$  we have  $R(x)$ ,” and is often written as “(a.e.  $x$ )” in text, while  $(\forall^\infty x)$  is used in displayed equations. (These quantifiers are dual to each other because  $(\forall^\infty x) R(x)$  holds iff  $\neg(\exists^\infty x)\neg R(x)$ .)

(iii) For a set  $A$  (and similarly for a function  $f$ ) we define two restrictions,

$$A \upharpoonright x = \{ y \in A : y < x \} \quad \text{and} \quad A \upharpoonright\!\! \upharpoonright x = \{ y \in A : y \leq x \}.$$

(iv) For strings  $\sigma, \tau \in 2^\omega$  we write  $\sigma \subset \tau$  if  $\sigma$  is an proper initial segment of  $\tau$ , and write  $\sigma < \tau$  if  $\sigma \subset \tau$  or if

$$(\exists i) [ \sigma(i) < \tau(i) \quad \& \quad \sigma \upharpoonright i = \tau \upharpoonright i ].$$

**Definition 1.4** (i) Function  $g$  *dominates* function  $f$ , written  $f <^* g$ , if

$$(\forall^\infty x) [ f(x) < g(x) ].$$

(ii)  $f$  *escapes (domination by)  $g$*  if  $f \not<^* g$ , i.e.,

$$(\exists^\infty x) [ g(x) \leq f(x) ].$$

(iii) An infinite set  $A = \{a_0 < a_1 < \dots\}$  *dominates or escapes  $g$*  according as its principal function  $p_A$  does, where  $p_A(n) = a_n$ .

### 1.3.2 Modulus Function and Settling Time Ordering

**Definition 1.5** (i) For every computably enumerable c.e. set  $W_e$  define *modulus function*, as in Soare [cta], which is also called in this paper and in Soare [2004] the *settling function*,

$$(1) \quad m_e(x) = (\mu s) [ W_{e,s} \upharpoonright x = W_e \upharpoonright x ]$$

with respect to the canonical enumeration  $\{W_{e,s}\}_{e,s \in \omega}$ . If  $A$  is a c.e. set with enumeration  $\{A_s\}_{s \in \omega}$  understood, we write the modulus

$$m_A(x) = (\mu s) [ A_s \upharpoonright x = A \upharpoonright x ].$$

(ii) A c.e. set  $A$  with enumeration  $\{A_s\}_{s \in \omega}$  *settling-time dominates* a c.e. set  $B$  with enumeration  $\{B_s\}_{s \in \omega}$  (written  $A >_{st} B$ ) if

$$(2) \quad (\forall \text{ computable } f)(\forall^\infty x) [ m_A(x) > f(m_B(x)) ].$$

Andre Nies showed that (ii) is independent of the choice of enumerations, which we generalize in §2.

(iii) A uniformly c.e. sequence  $\{A_n\}_{n \in \omega}$  of c.e. sets is a *settling-time dominating sequence* if

$$(3) \quad (\forall n) [ A_n >_{st} A_{n+1} ].$$

(iv) If  $g$  is a computable function, then the sequence  $\{A_n\}_{n \in \omega}$  is a *settling-time  $g$ -dominating sequence* if place of (2) we have

$$(4) \quad (\forall n)(\forall \text{ computable } f)(\forall^\infty x) [ m_{A_n}(x) > f(m_{A_{n+1}}(g(x))) ].$$

## 1.4 The Nabutovsky-Weinberger Results

### 1.4.1 The Sacks Density Theorem and Infinite Injury

Before seeing the Soare result, Nabutovsky and Weinberger had used the Sacks density theorem as they explain in [2003] Theorem 11.1 page 25. They used the fact that if  $A$  and  $B$  are c.e. sets with enumerations which have modulus functions  $m_A(x)$  and  $m_B(x)$  and if  $A <_T B$ , then for every computable function  $f$  the modulus function  $m_B(x)$  escapes  $f \circ m_A(x)$ , i.e.,

$$(5) \quad (\exists^\infty x) [ f \circ m_A(x) \leq m_B(x) ],$$

because otherwise  $B \leq_T A$ . Using this and the Sacks density theorem which they cite in [2003] Theorem 11.1 page 25 they can get an infinite sequence  $\{A_n\}_{n \in \omega}$  with the weak escape ordering of (5). From this they concluded that the associated basins (local minima) corresponding to  $A_n$  were infinitely often much deeper than those corresponding to  $A_{n+1}$  even when the latter is composed with any arbitrary computable function  $f$ . This produces a settling time escape sequence in place of our stronger settling-time dominating sequence of Definition 1.5.

This surprising connection to the structure of local minima in differential geometry is probably the first application of an existing theorem in computability theory (the Sacks Density Theorem) which was applied to obtain structural results in differential geometry (as opposed to merely undecidability results). It was also the first application in differential geometry of any theorem on c.e. degrees proved with the infinite injury method. The advantage of the Sacks density theorem is that they did not have to explicitly construct the sequence  $\{A_n\}_{n \in \omega}$ , but rather they could apply the Sacks theorem infinitely many times to get a sequence off the shelf. The disadvantage is that it had each settling time exceed the next only infinitely often, not for almost all  $x$ .

### 1.4.2 Deeper Local Minima Almost Everywhere

In spite of partial success with the Sacks density theorem and local minima greater on *infinitely many* arguments, Nabutovsky and Weinberger wanted the sequence  $\{A_n\}_{n \in \omega}$  to have the property that the settling time of each was much more complex for *almost every* argument, not merely infinitely many, so they turned to Soare. Later to simplify their proof they asked the question which Csima answered, improving the Soare sequence. Nabutovsky and Weinberger cited the result of Soare on dominating settling-time sequences as they write in Nabutovsky-Weinberger [2003] “Section 11. C.E.



Sets,” on page 24. On page 26 of [2003] they write “Theorem (R. Soare)”, and they are even more explicit.

In [2003] “Section 12. First Fractal Properties of  $\text{Met}(M)$ ” in Theorem 0.1 (Rigorous version) on page 27 they again cite Soare’s c.e. sets and then they state the following remarkable theorem which relates the notions of “dense” and “deep” to c.e. degrees. Nabutovsky and Weinberger write the following.

“In order to use Soare’s c.e. sets  $\beta_i$  explained in the previous section we need the following stronger c.e. set version of our Theorem 0.1 (and which is, in fact, the version we proved):”

“THEOREM 0.1 (Rigorous version). Let  $M$  be a closed smooth manifold of dimension  $n > 4$ . Let  $S$  be any c.e. set. Let  $T$  denote the halting function of a Turing machine  $\tau$  enumerating  $S$ . There exist a constant  $c(n) > 0$ , depending only on  $n$ , and increasing unbounded computable functions  $f$  and  $g$ , ( $f < g$ ), such that for all sufficiently large  $x$ , the number of local minima of the diameter,  $D$ , on  $Al(M)$ , such that the value of the diameter does not exceed  $x$  and of depth between  $f(T([x]))$  and  $g(T([x]))$  is at least  $\lceil \exp(c(n)x^n) \rceil$ . Moreover, these  $f(T([x]))$ -deep local minima form a  $g(T([x]))$ -dense in the path metric subset of  $D^{-1}((0, x]) \subset Al(M)$ . These minima are  $C^{1,\alpha}$ -smooth Riemannian structures on  $M$  for any  $\alpha \in (0, 1)$ .”

This means that for every suitable  $n$ -manifold there are infinitely many local minima of the diameter functional on the subset  $Al(M)$  of  $\text{Met}(M)$ . Moreover, there is a constant  $c(n)$  depending only on  $n$  such that for every c.e. degree  $\alpha$  the local minima of depth at least  $\alpha$  are  $\alpha$ -dense in the path metric on  $Al(M)$ , and the number of  $\alpha$ -deep local minima where the diameter does not exceed  $d$  is not less than  $\exp(c(n)d^n)$ . For further explanation see Soare [2004].

### 1.4.3 Fractals

Using the results stated more precisely in §1.5 and explained in §1.4.4 Nabutovsky and Weinberger observed the fractal nature of the local minima. That is,  $A_n$  determines an infinite sequence of “basins” (local minima),  $A_{n+1}$  determines an infinite sequence of much smaller basins coming off of them, and the latter contain still smaller basins coming off the sides of them, and so

on, where the relative size of one set of basins to the next exceeds any computable function. The fact that the second set of basins comes off the first comes from the relatively close distribution of the local minima explained in Soare [2004, §9]. In the preface of [2003] Nabutovsky and Weinberger wrote,

“In particular, we will see that there are large ‘basins’ that have topology, and are repeated infinitely often within the space (and even, in some sense, ‘all over the space’). On the other hand, the structure is rather more complicated than what is usually associated with fractals. There seem to be infinitely many different sorts of basins with different geometries from each other.”

For a discussion of the fractal nature of the results see Weinberger [2005] Part IV, *The Large Scale Fractal Geometry of Riemannian Moduli Space*. The dependence of the depth and distribution of the local minima on the settling function for a c.e. set and its degree were described in §1.4.2 and is further described in Soare [2004].

#### 1.4.4 Computably Enumerable Degrees

The reason to consider c.e. *degrees* in Theorem 5.1 as well as modulus functions of c.e. sets in Corollary 1.7 is twofold. First Nabutovsky and Weinberger used the Sacks density theorem on c.e. degrees as a kind of first theorem to obtain the weaker sequence which is the settling-time *escaping sequence* (not dominating) as explained in §1.4.1.

Second Nabutovsky and Weinberger thought in terms of *degrees* not just sets in their applications to geometry. (See the  $\alpha$ -dense and  $\alpha$ -deep basins mentioned above for a c.e. degree  $\alpha$ .) In their final draft of [2003] the computability theorem which they quote and apply is the increasing c.e. *degree* version, proved by Csima and Soare as Theorem 5.1. Nabutovsky and Weinberger [2003] §11 p. 26 wrote,

“ Theorem. There exists an infinite strictly increasing of c.e. degrees  $d_i$  and an infinite sequence of c.e. sets  $\beta_i \in d_i$  with the following property: Let for any  $i = 1, 2, \dots$ ,  $T_i$  be an arbitrary Turing machine enumerating  $\beta_i$  and  $h_i$  be its halting function.  $\dots$ ”

Nabutovsky and Weinberger *never* cite or apply the nondegree version Corollary 1.7 in either [2000] or [2003]. For more on the relation of c.e. degrees to the geometric properties see Nabutovsky and Weinberger [2003] of Soare [2004].

## 1.5 A Precise Statement of Our Main Theorem

To apply their results in differential geometry [2000], [2003] Nabutovsky and Weinberger asked Soare to construct a dominating sequence which he did, as described in Soare [2004]. To simplify their proof Weinberger then asked for a  $g$ -dominating sequence which Csima [2003] constructed. The proofs of the results cited in Soare [2004, §8] and also cited and used by Nabutovsky and Weinberger [2000], [2003] have been deferred to this paper. Combining these methods with those to ensure strictly decreasing Turing degree ( $A_n >_T A_{n+1}$ ). Csima and Soare obtained the Main Theorem 5.1 proved in §5 and restated here as Theorem 1.6.

**Theorem 1.6 (Main Theorem)** *For every computable function  $g$  there is a  $g$ -dominating sequence  $\{A_n\}_{n \in \omega}$  such that  $A_n >_T A_{n+1}$  for every  $n$ .*

**Corollary 1.7** *For every computable function  $g$  there is a  $g$ -dominating sequence  $\{A_n\}_{n \in \omega}$ .*

Because the proof for the general case of the theorem and even the corollary can become quite cumbersome in notation, we develop the main ideas through a series of special cases. In §3 we consider a Lachlan game between two players which captures the dynamic relationship  $A >_{st} B$ . We prove that if  $B$  is infinite and  $A >_{st} B$ , then  $A$  is high, but not necessarily complete. In §4 we build three c.e. sets  $A, B, C$  such that  $A >_{st} B >_{st} C$  and  $A >_T B >_T C$ , and in §5 we extend this to an infinite sequence to prove the Main Theorem 5.1.

## 2 Examining the Ordering

We first examine basic properties of the ordering  $<_{st}$  on c.e. sets.

**Definition 2.1** (i) A set  $A$  is *bounded Turing reducible* to a set  $B$  ( $A \leq_{bT} B$ ) if  $A \leq_T B$  and there is a computable function  $h(x)$  and a Turing reduction  $A = \Phi_e^B$  with use function  $\varphi_e^B(x) \leq h(x)$ .

(ii) Set  $A$  is *identity bounded Turing reducible* to  $B$  ( $A \leq_{ibT} B$ ) if  $A \leq_{bT} B$  with  $h(x) = x$ , namely  $A = \Phi_e^B$  with  $\varphi_e^B(x) \leq x$  for all  $x$ .

Properties of the reducibilities  $bT$  and  $ibT$  are developed in Chapter 5 of Soare [cta], and here in §1.1.5 and §1.1.6. The following generalizes Nies's observation that the notion of  $A >_{st} B$  does not depend on the particular enumerations of  $A$  and  $B$ .

**Theorem 2.2** *Let  $A$  and  $B$  be c.e. sets,  $B$  infinite, with enumerations  $\{A_s\}_{s \in \omega}$  and  $\{B_s\}_{s \in \omega}$  respectively.*

(i) *Suppose  $A \leq_{bT} B$ , with use bounded by  $h$ . Then there is a strictly increasing computable function  $f$  such that*

$$(\forall x)[m_A(x) \leq f \circ m_B(h(x))].$$

(ii) *Suppose  $A \leq_{iT} B$ . Then there is a strictly increasing computable function  $f$  such that*

$$(\forall x)[m_A(x) \leq f \circ m_B(x)].$$

*Proof.* Note that (ii) follows immediately from (i), so we prove (i). Say  $A = \Phi_j^B$ , with  $\varphi_j^B(x) \leq h(x)$  for all  $x$ . We assume without loss of generality that  $h$  is non-decreasing and that  $h(x) \geq x$  for all  $x$ . Let  $f(0) = 0$ . Let  $f(n)$  be the max of  $f(n-1) + 1$  and  $t_n$ , where  $t_n$  is obtained as follows.

Find  $s_n > n$  such that some  $z_n$  enters  $W_i$  at  $s_n$ . Let  $t_n > s_n$  be the least  $t$  such that

$$(6) \quad A_t \parallel z_n = \{y \leq z_n : \Phi_{j,t}^{B_i}(y) = 1\}$$

Note that  $t_n$  must exist since  $A = \Phi_j^B$ .

*Verification:* Suppose  $n = m_B(h(x))$ . Then since  $s_n > n$ , we have  $z_n > h(x)$ . Also, since  $t_n > n$ , we have  $B_{t_n} \parallel h(x) = B \parallel h(x)$ .

So by (6) and since  $x \leq h(x) < z_n$ , we have

$$(7) \quad A_{t_n} \parallel x = \{y \leq x : \Phi_{j,t_n}^{B_{t_n}}(y) = 1\}$$

since  $B_{t_n} \parallel h(x) = B \parallel h(x)$  and  $\varphi_j^B(y) \leq h(y)$ ,

$$(8) \quad A_{t_n} \parallel x = \{y \leq x : \Phi_j^B(y) = 1\}$$

$$(9) \quad = A \parallel x$$

That is,  $m_A(x) \leq t_n \leq f(n) = f(m_B(h(x)))$ . ■

**Theorem 2.3** *Let  $A$ ,  $B$ , and  $C$  be c.e. sets with enumerations  $\{A_s\}_{s \in \omega}$ ,  $\{B_s\}_{s \in \omega}$ , and  $\{C_s\}_{s \in \omega}$ , respectively. If  $A \leq_{iT} B$  and  $B <_{st} C$  then  $A <_{st} C$ .*

*Proof.* We wish to show that for  $g$  computable, for a.e.  $x$ ,  $g(m_A(x)) < m_C(x)$ . Note that it suffices to show this for  $g$  strictly increasing, so we assume this w.l.o.g. By Theorem 2.2, there is a  $f$  strictly increasing function such that for all  $x$ ,  $m_A(x) \leq f(m_B(x))$ . So for all  $x$ ,  $g(m_A(x)) \leq g(f(m_B(x)))$ . Since  $B <_{st} C$ , we have  $g(f(m_B(x))) < m_C(x)$  for a.e.  $x$ . So for a.e.  $x$ ,  $g(m_A(x)) < m_C(x)$ , as desired. ■

**Theorem 2.4** *Let  $A$ ,  $B$ , and  $C$  be c.e. sets with enumerations  $\{A_s\}_{s \in \omega}$ ,  $\{B_s\}_{s \in \omega}$ , and  $\{C_s\}_{s \in \omega}$ , respectively. If  $A <_{st} B$  and  $B \leq_{ibT} C$  then  $A <_{st} C$ .*

*Proof.* We wish to show for  $g$  computable, for a.e.  $x$ ,  $g(m_A(x)) < m_C(x)$ . By Theorem 2.2, there exists  $f$  strictly increasing such that for all  $x$ ,  $m_B(x) \leq f(m_C(x))$ . Since  $A <_{st} B$ ,  $f(g(m_A(x))) < m_B(x)$  for a.e.  $x$ . Now since  $f$  is strictly increasing, this gives  $g(m_A(x)) < m_C(x)$  for a.e.  $x$  as desired. ■

**Corollary 2.5**  *$<_{st}$  is well-defined on  $ibT$ -degrees.*

**Corollary 2.6 (Nies)**  *$<_{st}$  is well-defined on c.e. sets. That is, it is independent of the particular enumeration. So if  $\{A_s\}_{s \in \omega}$  and  $\{\tilde{A}_s\}_{s \in \omega}$  are enumerations of the same c.e. set  $A = \tilde{A}$ ,  $\{B_s\}_{s \in \omega}$  and  $\{\tilde{B}_s\}_{s \in \omega}$  are enumerations of the same c.e. set  $B = \tilde{B}$ , and  $A <_{st} B$ , then  $\tilde{A} <_{st} \tilde{B}$ .*

*Proof.* If  $A = \tilde{A}$  then  $A \equiv_{ibT} \tilde{A}$ , similarly for  $B$ . ■

The same does not hold for Turing degrees, as is a consequence of the next theorem.

**Theorem 2.7** *Given  $A >_{st} B$ ,  $B$  infinite, there exists  $C \equiv_T A$  such that  $C \not>_{st} B$ .*

*Proof.* We wish to construct a c.e. set  $C \equiv_T A$  and a computable function  $g$  such that  $g \circ m_B(x) \geq m_C(x)$  for infinitely many  $x$ . First we define a computable function  $f$  as follows:

$$f(x) = \max\{ \{0\} \cup \{s \leq x \mid (\exists y \leq x)[y \in B_s - B_{s-1}]\} \}.$$

Note that  $f$  is a nondecreasing, unbounded computable function with  $m_B(x) \geq f(x)$  for all  $x$ . Hence it suffices to build  $g$  computable, nondecreasing such that there are infinitely many  $x$  with  $g \circ f(x) \geq m_C(x)$ . We now construct  $g$  and  $C$ .

*Construction:*

*Stage 0:* For  $k \in \omega$ , set  $x_k^0 = k$ ,  $g_0 = \emptyset$ .

*Stage  $s + 1$ :* If there is some  $k \in A_{s+1} - A_s$ , enumerate  $x_k^s$  into  $C$ . Let  $x_k^{s+1} = \mu y [f(y) \notin \text{dom} g_s]$ . For  $l > k$  let  $x_l^{s+1} = x_k^{s+1} + (l - k)$ . For  $l \leq k$  let  $x_l^{s+1} = x_l^s$ . Let  $g_{s+1}(x) = g_s(x)$  for all  $x \in \text{dom} g_s$ , and let  $g_{s+1}(x) = s + 1$  for all  $0 \leq x \leq f(x_k^s)$  where  $g$  has not yet been defined.

If no element enters  $A$  at stage  $s + 1$ , set  $x_l^{s+1} = x_l^s$  for all  $l$ , and set  $g_{s+1} = g_s$ .

Let  $g = \cup_s g_s$ . Note that  $g$  is defined on progressively larger intervals.

*Verification:*

Let  $x_k = \lim_{s \rightarrow \infty} x_k^s$ . This limit is always finite since  $x_k^s$  is only redefined if some  $l \leq k$  enters  $A$ .

**Lemma 2.8**  $A \equiv_T C$ .

*Proof.* Note that  $k \in A_{s+1} - A_s$  iff  $x_k^s \in C_{s+1} - C_s$ .

Suppose we want to know whether  $y \in C$ .  $A$  computes the stage  $s$  by which  $A_s \parallel y = A \parallel y$ . Then

$$y \in C \iff (\exists t \leq s)(\exists k \leq y)[k \in A_t - A_{t-1} \wedge y = x_k^t].$$

Suppose we want to know whether  $k \in A$ . As  $x_k^s$  is only redefined if some  $x_l^s \leq x_k^s$  enters  $C$ ,  $C$  computes the stage  $s$  by which  $C_s \parallel x_k^s = C \parallel x_k$ . Then  $k \in A$  iff  $(\exists t \leq s)[x_k^t \in C_t - C_{t-1}]$ . ■

**Lemma 2.9**  $C \not\leq_{st} B$ .

*Proof.* Claim: For all  $k \in \omega$ ,  $m_C(x_k) \leq g \circ m_B(x_k)$ . Indeed, let  $s = m_C(x_k)$ . Note that  $s = m_A(k)$ . Suppose that  $l < k$  is such that  $l \in A_s - A_{s-1}$  (a unique such  $l$  exists). Then since  $k > l$ ,  $f(x_k^{s+1}) \notin \text{dom} g_s$  and so  $g(f(x_k^{s+1})) > s$ . As  $f$  and  $g$  are non-decreasing, and since  $x_k \geq x_k^{s+1}$ , we must have  $g(f(x_k)) \geq g(f(x_k^{s+1})) \geq s$ .

So  $m_C(x_k) = s \leq g \circ f(x_k) \leq g \circ m_B(x_k)$ , as desired. ■

■

**Corollary 2.10**  $<_{st}$  is not well-defined on Turing degrees.

In Csima and Shore [ta] it is shown that  $<_{st}$  is not well-defined on 1-1 degrees.

### 3 Building $A >_{st} B$

#### 3.1 The game for $A >_{st} B$

We wish to build two sets,  $A >_{st} B$ , in such a way that we can extend the method to build a whole chain of such. The idea is that we have control over both  $A$  and  $B$ , and we wish to ensure that  $A$  settles much more slowly than  $B$ . Indeed, we require that for each partial computable function  $\varphi_j$ , if  $\varphi_j$  is total then

$$(\forall^\infty x) [ m_A(x) > \varphi_j(m_B(x)) ].$$

Viewing this as a game, where we win if we can build  $A >_{st} B$ , the only tools the opponent can use against us are the partial computable functions  $\varphi_j$ . We win by enumerating members into  $B$  very sparsely, so that for every  $x$  that enters  $B$ , there are many  $x_j < x$  ready to enter  $A$  at a later stage, to guard against various possibly total functions  $\varphi_j$ . To illustrate the method, we prove the following theorem.

**Theorem 3.1** *For any computable function  $g$  there exist infinite c.e. sets  $A$  and  $B$  such that for any computable function  $f$*

$$(\forall^\infty x) [ m_A(x) > f(m_B(g(x))) ].$$

*Proof.* We first assume  $g$  is strictly increasing. For each  $j \in \omega$  we meet the requirements:

$$P_j : \varphi_j \text{ total} \implies (\forall^\infty x) [ m_A(x) > \varphi_j(m_B(g(x))) ].$$

We construct the sets  $A$  and  $B$  by stages as follows.

*Stage  $s$ :* Let  $\alpha_s$  be greater than any number mentioned so far in the construction. Enumerate  $g(\alpha_s + s)$  into  $B$  at stage  $s$ . If at a later stage  $t$ ,  $\varphi_{i,t}(s) \downarrow$  for some  $i \leq s$ , enumerate  $\alpha_s + i$  into  $A$  at stage  $t$ . We think of  $\alpha_s + i$  as the “guard” ready to enter  $A$  if  $\varphi_i$  converges on  $s$ .

*Verification:*

Suppose  $f$  is any computable function. Then  $f = \varphi_j$  for some  $j$ . Let  $t$  be the stage by which  $B \upharpoonright g(\alpha_j) = B_t \upharpoonright g(\alpha_j)$ .

Suppose  $x$  is such that  $m_B(g(x)) \geq t$ . This is true for a.e.  $x$ . To show that  $m_A(x) > f(m_B(g(x)))$ , it suffices to show that for each  $y \leq g(x)$  which

enters  $B$  at a stage  $s \geq t$ , there is some  $z \leq x$  that enters  $A$  at a stage greater than  $f(s)$ . Suppose  $y \leq g(x)$  enters  $B$  at a stage  $s \geq \max\{t, j\}$ . Then  $y = g(\alpha_s + s)$  by construction. Now  $\alpha_s + j$  is enumerated into  $A$  at the stage where  $\varphi_j(s)$  converges, which exists since  $\varphi_j$  is total. This stage is greater than  $f(s)$  since  $f(s) = \varphi_j(s)$  and  $\varphi_j(s)$  must be less than the stage at which it converged. Note that  $g(\alpha_s + s) = y \leq g(x)$ , so since  $g$  is increasing  $\alpha_s + s \leq x$ . So  $\alpha_s + j < x$ , and so a number less than  $x$  was enumerated into  $A$  at a stage greater than  $f(s)$ . Thus  $m_A(x) > \varphi_j(m_B(g(x))) = f(m_B(g(x)))$ . ■

To show that the theorem holds for arbitrary computable  $g$ , we make use of the following definition.

**Definition 3.2** For any computable function  $h$ , define  $h^*$  as follows.  $h^*(0) = h(0)$ ,  $h^*(n+1) = \max\{h(n+1), h^*(n) + 1\}$ . Then  $h^*$  is computable, increasing, and  $(\forall x)[h^*(x) \geq h(x)]$ .

Now suppose  $g$  is any computable function. Then by the above, there exist c.e. sets  $A$  and  $B$  so that for any computable  $f$ , for a.e.  $x$ ,  $m_A(x) > f(m_B(g^*(x)))$ . In particular, for any computable  $f$ , for a.e.  $x$  we have

$$m_A(x) > f^*(m_B(g^*(x))) \geq f^*(m_B(g(x))) \geq f(m_B(g(x)))$$

So the result holds for arbitrary computable  $g$ . ■

This is just the basic game for the condition  $A >_{st} B$ , and in applications we shall use extensions of the game to control the Turing degree of  $A$ .

### 3.2 Dominant functions and $e$ -dominant sets

We use the following definitions from Soare [1987] pages 208 and 214.

**Definition 3.3** (i) A function  $f$  is *dominant* if it dominates every computable function  $g$ , namely  $f(x) > g(x)$  for all but finitely many  $x$ .

(ii) A c.e. set  $A$  is  *$e$ -dominant* if  $A = W_e$  for some  $e$  such that its settling function  $m_e(x)$  is dominant.

By Martin's theorem (see Soare [1987, p. 208]) any  $e$ -dominant set  $A$  is high, namely  $A' \equiv_T 0''$ . (See the results on  $e$ -dominant sets by R.W. Robinson in Soare [1987, p. 214].)

**Theorem 3.4** *If  $A$  and  $B$  are c.e. sets such that  $A >_{st} B$  and  $B$  is infinite then  $A$  is  $e$ -dominant, and hence high.*



*Proof.* Let  $\{A_s\}$  and  $\{B_s\}$  be enumerations of  $A$  and  $B$ . Since  $B$  is infinite, we may choose an infinite computable subset  $C \subseteq B$  such that  $C = \{c_0 < c_1 < \dots\}$ . For all  $k \in \omega$  let  $s_k$  be the stage at which  $c_k$  enters  $B$ . Note that the function  $h(k) = s_k$  is computable, and we may choose  $C$  so that  $h(k)$  is increasing. Let  $\{\varphi_e\}_{e \in \omega}$  be an effective listing of all partial computable functions. If  $\varphi_e(y) \downarrow$  for all  $y \leq c_{k+1}$ , then define  $\psi_e(s_k) = \max\{\varphi_e(y) \mid y \leq c_{k+1}\}$ . For  $s_{k-1} \leq s < s_k$ , define  $\psi_e(s) = \psi_e(s-1)$ . Suppose  $\varphi_e$  is total. Then so is  $\psi_e$ , so since  $A >_{st} B$  there is some  $N$  such that

$$(\forall x \geq N)[m_A(x) \geq \psi_e(m_B(x))].$$

Choose  $x \geq N$ . Let  $k$  be such that  $c_k \leq x < c_{k+1}$ . Note that  $m_B(c_k) \geq s_k$  and that  $\psi_e$  is non-decreasing. Then

$$m_A(x) \geq m_A(c_k) \geq \psi_e(m_B(c_k)) \geq \psi_e(s_k) \geq \varphi_e(x).$$

Thus  $m_A$  dominates  $\varphi_e$ . ■

This method is well suited to force an infinite computable set into  $A$  but has limitations. To make  $\psi_e$  of any use one must threaten to make it total, namely arrange that the values are defined in order. This is not so well suited to more delicate coding, such as trying to improve the preceding theorem by showing that  $A$  is complete. In fact, this is not possible.

**Theorem 3.5** *There are c.e. sets  $A$  and  $B$  such that  $A >_{st} B$ ,  $B$  is infinite and computable, and  $A$  is incomplete ( $A <_T K$ ).*

*Proof.* This is an immediate consequence of the Main Theorem 5.1. ■

## 4 Strictly increasing Turing degrees

**Theorem 4.1** *There are c.e. sets  $A$  and  $B$  such that  $A >_{st} B$ ,  $B$  is infinite, and  $A >_T B$ .*

*Proof.* In addition to meeting the requirements to make  $A >_{st} B$ , we must meet requirements to ensure  $A \not\leq_T B$ . That is we must meet the requirements

$$N_e : A \neq \Phi_e^B.$$

This is not difficult. Since we control  $B$ , we just spread out  $B$  more, so that there are enough guards ready to enter  $A$  both for the sake of the  $P_e$  as in

Theorem 3.1, and also to diagonalize against the possibly total  $\Phi_e^B$ . Again, since we control  $B$ , it is no problem to hold  $B$  on a segment in order to preserve a disagreement between  $A$  and  $\Phi_e^B$ . ■

The above case was not enough to give a proper feel for constructing an entire sequence as in the main theorem, since there was no pressure to enumerate into  $B$ . Suppose now that we want to build three sets,  $A >_{st} B >_{st} C$ , with strictly increasing Turing degrees. Not worrying about the Turing degrees for a moment, it is easy to see that to build  $A >_{st} B >_{st} C$ , we just need to spread out  $C$  sufficiently so that there are sparse enough guards ready to enter  $B$  for some  $P_e^{B >_{st} C}$ , so that when these guards enter  $B$  there are guards ready to enter  $A$  for some  $P_e^{A >_{st} B}$ . To have  $B \not\leq_T C$  is also easy, as before, since it is not a problem to spread out  $C$  more, and to hold  $C$  for the sake of keeping a computation. However, it is no longer so easy to ensure  $A \not\leq_T B$ . We can still spread out  $B$  (and so also  $C$ ) to have enough witnesses to enter  $A$  in order to diagonalize, however, we can no longer simply hold  $B$  to maintain a disagreement, since the requirements  $P_e^{B >_{st} C}$  may want us to enumerate into  $B$ . So now we must use a priority argument, noting that each  $\varphi_e$  is either total or is not.

**Theorem 4.2** *There are c.e. sets  $A$ ,  $B$ , and  $C$  such that  $A >_{st} B >_{st} C$ ,  $A >_T B >_T C$ , and  $C$  is infinite.*

*Proof.* For each  $e \in \omega$  we meet the requirements:

$$\begin{aligned} P_e^{A >_{st} B} : \quad \varphi_e \text{ total} &\implies (\forall^\infty x)[m_A(x) \geq \varphi_e(m_B(x))] \\ P_e^{B >_{st} C} : \quad \varphi_e \text{ total} &\implies (\forall^\infty x)[m_B(x) \geq \varphi_e(m_C(x))] \\ N_e^{A >_T B} : \quad A &\neq \Phi_e^B \\ N_e^{B >_T C} : \quad B &\neq \Phi_e^C \end{aligned}$$

At each stage, we will enumerate a large number into  $C$ . We will also appoint numbers that may enter  $A$  and  $B$ , in the future, should certain events come to pass. If a number is appointed as a guard or witness to enter a set, it will only have one possible reason for doing so. We will enumerate numbers into  $A$  and  $B$  at stage  $s + 1$  if they had been appointed at a prior stage and the event they were waiting for has come to pass.

At stage  $s + 1$ , we will enumerate into  $C$ , so as to make  $C$  infinite. We will choose a number  $c$  large enough so that there are  $s + 1$  many numbers less than  $c$  eligible to enter  $B$  for the sake of the  $P_e^{B >_{st} C}$  for  $e \leq s$ , labeled

$c_0^B, \dots, c_s^B$ . A number  $b$  will be “eligible to enter  $B$ ” if there are  $s + 1$  many numbers less than  $b$  ready to enter  $A$  for the sake of the  $P_e^{A > st B}$  for  $e \leq s$ , labeled  $b_0^A, \dots, b_s^A$ . We will also appoint  $s + 1$  many witnesses eligible to enter  $B$  for the sake of  $N_e^{B > TC}$  for  $e \leq s$ , labeled  $[B]_0^{s+1}, \dots, [B]_{s+1}^{s+1}$ , and  $2^{s+1}$  many witnesses ready to enter  $A$  for the sake of  $N_e^{A > TB}$ , labeled  $[A]_\tau^{s+1}$  where  $\tau$  is a string in  $2^{e-1}$ , and  $e \leq s$ .

At stage  $s$ ,  $\alpha_s$  will be a string of length  $s$  guessing at which functions are total. For  $\beta \in 2^\omega$  we call  $s$  a  $\beta$ -stage if  $\beta \upharpoonright s \subset \alpha_s$ . Let  $\alpha_0 = \emptyset$ , so  $0$  is a  $\beta$ -stage for all  $\beta$ . For  $s > 0$ , we define  $\alpha_s(i)$  by induction on  $i$  for  $0 \leq i < s$  as follows. Let  $\alpha_s(i) = 0$  iff  $\varphi_i$  increased its length of totality since  $t_s$  where  $t_s = \max\{t \mid t < s \ \& \ t \text{ is an } \alpha_s \upharpoonright i \text{-stage}\}$ .

For  $\beta \in 2^{<\omega}$ , let  $r_B(\beta, 0) = 0$ . Let  $R_B(\gamma, s) = \max\{r_B(\beta, s) \mid \beta \leq \gamma\}$ .

Stage  $s + 1$ : Enumerate  $c$  into  $C$ .

*Action for “ $P^{B > st C}$ ” requirements:* If  $\varphi_{e, s+1}(s_d^C) \downarrow$ , where  $d$  was enumerated into  $C$  at stage  $s_d^C$ , then if  $e \leq s_d^C$ , and  $d_e^B > R_B(\alpha_s \upharpoonright e, s)$ , then if not already done so, enumerate  $d_e^B$  into  $B$ , and set  $r_B(\beta, s + 1) = 0$  for those  $\beta > \alpha_s \upharpoonright e$  for which  $d_e^B < r_B(\beta, s)$ .

*Action for “ $P^{A > st B}$ ” requirements:* If  $\varphi_e(s_b^B) \downarrow$  at stage  $s + 1$ , where  $b$  was enumerated into  $B$  at stage  $s_b^B$ , then if there was some  $b_e^A$  assigned, enumerate  $b_e^A$  into  $A$ .

*Action for “ $N^{B > TC}$ ” requirements:* Choose the least  $e$  such that there is some  $[B]_e^k \notin B_s$ ,  $[B]_e^k > R_B(\alpha_s \upharpoonright e, s)$ , and  $(B \upharpoonright [B]_e^k = \Phi_e^C \upharpoonright [B]_e^k)[s]$ . For that  $e$ , choose the least such  $k$ , and enumerate  $[B]_e^k$  into  $B$ . Set  $r_B(\beta, s + 1) = 0$  for those  $\beta > \alpha_s \upharpoonright e$  for which  $[B]_e^k < r_B(\beta, s)$ .

*Action for “ $N^{A > TB}$ ” requirements:* Choose the least  $e \leq s$  such that  $r_B(\alpha_s \upharpoonright e, s) = 0$  and there is  $[A]_{\alpha_s \upharpoonright e}^k$  with  $(A \upharpoonright [A]_{\alpha_s \upharpoonright e}^k = \Phi_e^B \upharpoonright [A]_{\alpha_s \upharpoonright e}^k)[s]$ , and  $\varphi_j$  “spent” for all  $j < e$  with  $\alpha_s(j) = 0$ . By  $\varphi_j$  being “spent”, we mean that for every  $c_j^B \leq \varphi_e^B([A]_{\alpha_s \upharpoonright e}^k)$ , if  $c_j^B > r_B(\alpha_s \upharpoonright j, s)$ , then  $\varphi_j(s_c^C) \downarrow$  and  $c_j^B$  has already been enumerated into  $B$ . That is, requirement  $P_j^{B > st C}$  will not cause a change in  $B$  below the use. If such  $e$  exists, choose the least such  $k$  for that  $e$ , enumerate  $[A]_{\alpha_s \upharpoonright e}^k$  into  $A$ , set  $r_B(\alpha_s \upharpoonright e, s + 1) = \varphi_e^B([A]_{\alpha_s \upharpoonright e}^k)$ , and  $r_B(\beta, s + 1) = 0$  for those  $\beta > \alpha_s \upharpoonright e$  for which  $[A]_{\alpha_s \upharpoonright e}^k < r_B(\beta, s)$ .

Set  $r_B(\beta, s + 1) = r_B(\beta, s)$  for all  $\beta$  for which it has not been defined.

*Verification:*

The  $P_e^{A > st B}$  are met. For any  $b$  enumerated into  $B$  after stage  $e$ , a  $b_e^A$  was assigned. And so if  $\varphi_e(s_b^B) \downarrow$  at stage  $s$ ,  $b_e^A < b$  was enumerated into  $A$  at stage  $s$ .

**Lemma 4.3** *lim inf  $R_B(\alpha_s \upharpoonright e, s) < \infty$  for all  $e$ . Indeed, we will show that*

there is an infinite set  $T$  of true stages such that  $\lim_{t \in T} R_B(\alpha_t \upharpoonright e, t) < \infty$  for all  $e$ .

*Proof.* Let  $f \in 2^\omega$  be the “true path” on our tree of guesses at the total partial functions. That is,  $f(i) = 0$  iff  $\varphi_i$  is total. We’ll say a stage  $t$  is a *true stage* if the length of agreement between  $f$  and  $\alpha_t$  is longer than it has been at any previous stage.

We will show that for each  $e$ ,  $\lim_s r_B(f \upharpoonright e, s) < \infty$ . This will show that  $\lim_{t \in T} R_B(\alpha_t \upharpoonright e, t) < \infty$  since for  $\beta$  off the true path, either  $\beta$  will only be visited finitely often (there will be only finitely many  $\beta$ -stages), or  $r_B(\beta, t)$  will be reset to 0 at every true stage.

Let  $s_0$  be a true stage such that for all  $j < e$ , if  $\varphi_j$  is not total, then  $\varphi_j$  will never appear total after stage  $s$ . That is,  $\alpha_{s_0} \upharpoonright e = f \upharpoonright e$  and  $\alpha_s \upharpoonright e \geq f \upharpoonright e$  for all  $s \geq s_0$ . Let  $s_0$  also be such that for all  $j < e$ ,  $r_B(f \upharpoonright j, s_0) = \lim_s r_B(f \upharpoonright j, s)$ . Let  $s_1 > s_0$  be a stage such that  $r_B(f \upharpoonright e, s_1) \neq r_B(f \upharpoonright e, s_1 - 1) = 0$ . If no such  $s_1$  exists then  $\lim_s r_B(f \upharpoonright e, s) = r_B(f \upharpoonright e, s_0)$  or  $\lim_s r_B(f \upharpoonright e, s) = 0$  and we are done. So assume  $s_1$  exists, i.e. that  $N_e^{A > T^B}$  received attention at stage  $s_1$ . The only way this could be injured is if at some stage  $s > s_1$ ,  $\alpha_s \upharpoonright i < f \upharpoonright e$ , and either  $P_i^{B > st^C}$  or  $N_i^{A > T^B}$  received attention. We must have  $i < e$  by the assumption on  $s_0$ . So  $N_i^{A > T^B}$  cannot receive attention by induction hypothesis, and  $P_i^{B > st^C}$  will not desire to enumerate below  $r_B(f \upharpoonright e, s_1)$  as if  $\varphi_i$  were total it would have appeared so at  $s_1$  and there would be no  $c_i < r_B(f \upharpoonright e, s_1)$  that wasn’t already enumerated into  $B$ . Thus  $\lim_s r_B(\alpha_t \upharpoonright e, s) = r_B(f \upharpoonright e, s_1)$ . ■

Note that it follows that the  $N_e^{A > T^B}$  are satisfied. Indeed, this is clear if  $\lim_{t \in T} r_B(\alpha_t \upharpoonright e, t) \neq 0$ . Suppose  $\lim_{t \in T} r_B(\alpha_t \upharpoonright e, t) = 0$ . Assume for a contradiction that  $A = \Phi_e^B$ . Let  $[A]_{f \upharpoonright e}^k$  be least such that  $[A]_{f \upharpoonright e}^k \notin A$ . Let  $t$  be a true stage such that  $A$  has settled up to  $[A]_{f \upharpoonright e}^k$  and  $B$  has settled on  $\varphi_e^B([A]_{f \upharpoonright e}^k)$ , and  $\alpha_t \upharpoonright e = f \upharpoonright e$ . Now if  $j < e$  and  $\alpha_t(j) = 0$ , then  $\varphi_j$  really was total, and so any  $c_j^B > r_B(f \upharpoonright j, s)$  would be enumerated into  $B$ . If  $c_j^B \leq \varphi_e^B([A]_{f \upharpoonright e}^k)$  then this must have happened by stage  $t$  since we know that the stage  $t$  approximation is correct. So at stage  $t$ ,  $[A]_{f \upharpoonright e}^k$  would be enumerated into  $A$ , a contradiction.

To see that the  $P_e^{B > st^C}$  are met, note that any  $d_e^B > \liminf R_B(\alpha_s \upharpoonright e, s)$  that wants to be will eventually be enumerated, and that if something is enumerated later than it first wanted to be, that doesn’t hurt anything.

For the  $N_e^{B > T^C}$ , note that if any  $[B]_e^k$  is ever enumerated, the diagonalization will be preserved (we won’t change  $C$ ). So suppose for a contradiction

that  $B = \Phi_e^C$ . Let  $[B]_e^k > \liminf R(\alpha_s \upharpoonright e, s)$ . Then at the next true stage after  $\Phi_e^C$  converged up to  $[B]_e^k$ ,  $[B]_e^k$  would have been enumerated into  $B$ , a contradiction. ■

## 5 Proof of Main Theorem

We are now ready to prove the main theorem:

**Theorem 5.1 (Main Theorem)** *There is a dominating sequence  $\{A_n\}_{n \in \omega}$  such that  $A_n >_{\text{T}} A_{n+1}$  for all  $n$ . (Furthermore, for every computable function  $g$ , the sequence may be chosen to be  $g$ -dominating.)*

*Proof.* For each  $\langle e, n \rangle \in \omega$  we meet the requirements:

$$\begin{aligned} P_{\langle e, n \rangle} : & \quad \varphi_e \text{ total} \implies (a.e. x)[m_{A_n}(x) \geq \varphi_e(m_{A_{n+1}}(x))] \\ N_{\langle e, n \rangle} : & \quad A_n \neq \Phi_e^{A_{n+1}} \end{aligned}$$

At stage  $s$ ,  $\alpha_s$  will be a string of length  $s$  guessing at whether the requirements  $P_{\langle e, n \rangle}$  will require infinite action. As before, for  $\beta \in 2^\omega$  we call  $s$  a  $\beta$ -stage if  $\beta \upharpoonright s \subset \alpha_s$ . Let  $\alpha_0 = \emptyset$ , so  $0$  is a  $\beta$ -stage for all  $\beta$ . For  $s > 0$ , we define  $\alpha_s(i)$  by induction on  $i$  for  $0 \leq i < s$  as follows. Let  $\alpha_s(i) = 0$  iff  $i = \langle e, n \rangle$  and  $\varphi_e$  increased its length of totality since  $t_s$  where  $t_s = \max\{t \mid t < s \ \& \ t \text{ is an } \alpha_s \upharpoonright i\text{-stage}\}$ .

For  $\beta \in 2^{<\omega}$ , let  $r_n^i(\beta, 0) = 0$ . Let

$$R_n(\delta, s) = \max\{r_n^i(\beta, s) \mid \beta \leq \delta \ \& \ i \in \omega\}.$$

These will be the restraint functions for  $A_n$ .

*Stage  $s + 1$ :* We consider the sets  $A_0, \dots, A_s$ . For each  $0 \leq n \leq s$  we appoint witnesses  $[n]_\tau^{s+1}$  where  $e \leq s$  and  $\tau \in 2^{\langle e, n \rangle - 1}$ , which may later enter  $A_n$  for the sake of  $N_{\langle e, n \rangle}$ . Also, for each  $0 < n \leq s$  and for each number  $c$  we appoint to possibly enter  $A_{n+1}$ , we appoint guards  $c_0^n, \dots, c_s^n$ , all less than  $c$ , to possibly enter  $A_n$  for the sake of  $P_{\langle e, n \rangle}$ . All the numbers appointed at this stage should be larger than any numbers mentioned so far in the construction. Note that since we are considering only finitely many sets, we can certainly arrange to appoint the possible future entrants in this fashion. Also note that any possible entrant of  $A_n$  of the form  $c_i^n$  is linked to a possible entrant of  $A_m$  of the form  $[m]_\tau^{s+1}$  for some  $n < m \leq s$ .

*Action for "P" requirements:* If  $\varphi_{e,s+1}(s_d^{n+1}) \downarrow$ , where  $d$  was enumerated into  $A_{n+1}$  at stage  $s_d^{n+1}$ , then if  $d_e^n$  was appointed, and  $d_e^n > R_n(\alpha_s \upharpoonright \langle e, n \rangle, s)$ , then if not already done so, enumerate  $d_e^n$  into  $A_n$ , and set  $r_n^i(\beta, s+1) = 0$  for those  $\beta > m_s \upharpoonright \langle e, n \rangle$ ,  $i \in \omega$  for which  $d_e^n \leq r_n^i(\beta, s)$ .

*Action for "N" requirements:*

When we enumerate into  $A_n$  for the sake of some requirement  $N_{\langle e, n \rangle}$ , we can only restrain with priority  $\langle e, n \rangle$ . Hence we must be careful to put restraint onto the sets  $A_m$  with  $m \geq n$ , since if something of low priority is enumerated into  $A_m$  it could cause something of high priority to want to enter  $A_n$  at a later stage. Also, we wish to only believe computations that we don't think will be injured by higher priority requirements.

Choose the least  $\langle e, n \rangle \leq s$  such that  $r_{n+1}^{\langle e, n \rangle}(\alpha_s \upharpoonright \langle e, n \rangle, s) = 0$  and there is an  $[n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k > R_n(\alpha_s \upharpoonright \langle e, n \rangle, s)$  with

$$A_n \parallel [n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k = \Phi_e^{A_{n+1}} \parallel [n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k,$$

and a sequence  $r_{n+1}, \dots, r_s$  with the following properties:

1.  $r_{n+1} = \varphi_e^{A_{n+1}}([n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k)$
2. For  $n+1 \leq m < s$ , if  $d_i^m < r_m$  then  $r_{m+1} > d$ .
3. For  $n+1 \leq m < s$ , if  $d_i^m < r_m$ ,  $\langle i, m \rangle \leq \langle e, n \rangle$ ,  $d$  has already entered  $A_{m+1}$ , and  $\alpha_s(\langle i, m \rangle) = 0$ , and  $d_i^m > R_m(\alpha_s \upharpoonright \langle i, m \rangle, s)$ , then  $d_i^m$  has already entered  $A_m$ .

If such  $\langle e, n \rangle$  exists, choose the least such  $k$  for that  $\langle e, n \rangle$ , enumerate  $[n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k$  into  $A_n$ , and set  $r_m^{\langle e, n \rangle}(\alpha_s \upharpoonright \langle e, n \rangle, s+1) = r_m$  for  $n < m \leq s$ ,  $r_n^i(\beta, s+1) = 0$  for  $\beta > \alpha_s \upharpoonright \langle e, n \rangle$ , and  $r_m^i(\beta, s+1) = r_m^i(\beta, s)$  for all other  $\beta, m, i$ .

If no such  $\langle e, n \rangle$  exists, set  $r_m^i(\beta, s+1) = r_m^i(\beta, s)$  for all  $\beta, m, i$ .

*Verification*

Let  $f \in 2^{<\omega}$  be the true path. That is,  $f(\langle e, n \rangle) = 0$  iff  $\varphi_e$  is total. We call  $t$  a *true stage* if the length of agreement between  $\alpha_t$  and  $f$  is longer than at any previous stage. We let  $T$  denote the set of true stages.

**Lemma 5.2** *For all  $\langle e, n \rangle$ ,  $\lim r_n^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s) < \infty$ . Therefore, for all  $\langle e, n \rangle$ ,  $\lim r_m^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s) < \infty$  and  $\lim_{t \in T} R_n(\alpha_t \upharpoonright \langle e, n \rangle, t) < \infty$ .*

*Proof.* By induction. Suppose holds for all  $\langle i, m \rangle < \langle e, n \rangle$ . Let  $s_0$  be a stage by which  $\alpha_s \upharpoonright \langle e, n \rangle \geq f \upharpoonright \langle e, n \rangle$  for all  $s \geq s_0$  and by which  $r_m^{\langle i, m \rangle}(f \upharpoonright \langle i, m \rangle, s)$

has reached its limit for all  $s \geq s_0$ ,  $\langle i, m \rangle < \langle e, n \rangle$ . Let  $s_1 > s_0$  be a stage such that  $r_n^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s_1) \neq r_n^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s_1 - 1) = 0$ . If no such  $s_1$  exists then  $\lim r_n^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s) = r_n^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s_0)$  or  $\lim r_n^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s) = 0$  and we are done. So assume  $s_1$  exists, i.e. that  $N_{\langle e, n \rangle}$  received attention at stage  $s_1$ . We show that the restraints  $r_m = r_m^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s_1)$  for  $n \leq m \leq s_1$  are never injured. First note that nothing of the form  $[m]_\tau^k$  will injure  $r_m$  since if  $\langle i, m \rangle < \langle e, n \rangle$  then  $[m]_\tau^k$  will not enter by induction hypothesis and if  $\langle i, m \rangle > \langle e, n \rangle$  then  $\alpha_s \upharpoonright \langle i, m \rangle \geq \alpha_s \upharpoonright \langle e, n \rangle \geq f \upharpoonright \langle e, n \rangle$  for all  $s \geq s_0$  and so if  $\beta = \alpha_s \upharpoonright \langle i, m \rangle$ ,  $R_m(\beta, s) \geq r_m^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s)$ . Hence  $r_{s_1}$  is never injured. Assume  $r_{m+1}$  is never injured. Suppose  $d_i^m < r_m$ . Then by (2),  $d < r_{m+1}$ . Since  $r_{m+1}$  is never injured,  $d_i^m$  is only a threat if  $d$  was enumerated into  $A_{m+1}$  by stage  $s_1$ . If  $\langle i, m \rangle > \langle e, n \rangle$  then  $\alpha_s \upharpoonright \langle i, m \rangle \geq \alpha_s \upharpoonright \langle e, n \rangle \geq f \upharpoonright \langle e, n \rangle$  for all  $s \geq s_0$  and so  $R_m(\alpha_s \upharpoonright \langle i, m \rangle, s) \geq r_m^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s)$ . If  $\langle i, m \rangle \leq \langle e, n \rangle$  then  $\alpha_s \upharpoonright \langle i, m \rangle \geq \alpha_{s_1} \upharpoonright \langle i, m \rangle$ , and so if  $d_i^m > R_m(\alpha_s \upharpoonright \langle i, m \rangle, s)$  then  $d_i^m > R_m(\alpha_{s_1} \upharpoonright \langle i, m \rangle, s_1)$  and so if  $\alpha_{s_1}(\langle i, m \rangle) = 0$  then  $d_i^m$  has already been enumerated into  $A_m$  by (3). If  $\alpha_{s_1}(\langle i, m \rangle) = 1$  then since  $\langle i, m \rangle \leq \langle e, n \rangle$  and  $\alpha_{s_1} \upharpoonright \langle e, m \rangle = f \upharpoonright \langle e, m \rangle$ ,  $\varphi_i$  won't converge on  $s_d$  and so  $d_i^m$  won't be enumerated into  $A_m$ . Hence  $r_m$  will never be injured.  $\blacksquare$

**Lemma 5.3** *The requirements  $N_{\langle e, n \rangle}$  are met.*

*Proof.* By construction, certainly if  $\lim r_{n+1}^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s) \neq 0$  then, by the previous lemma, requirement  $N_{\langle e, n \rangle}$  was met. Assume for a contradiction that  $A_n = \Phi_e^{A_{n+1}}$  and hence that  $\lim r_{n+1}^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s) = 0$ . Let  $s$  be a true stage such that for all  $t \in T$ ,  $t \geq s$ ,  $\alpha_t \upharpoonright \langle e, n \rangle = f \upharpoonright \langle e, n \rangle$  and  $R_n(f \upharpoonright \langle e, n \rangle, t) = R_n(f \upharpoonright \langle e, n \rangle, s)$ . Let  $k$  be least such that  $[n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k \notin A_n$ ,  $[n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k > R_n(f \upharpoonright \langle e, n \rangle, s)$  and  $A_n \Vdash [n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k = \Phi_e^{A_{n+1}} \Vdash [n]_{\alpha_s \upharpoonright \langle e, n \rangle}^k$ . Such  $k$  must exist since  $A_n = \Phi_e^{A_{n+1}}$  and  $\lim r_{n+1}^{\langle e, n \rangle}(f \upharpoonright \langle e, n \rangle, s) = 0$ . Now let  $t > s$  be a true stage such that  $A_n \Vdash [n]_{f \upharpoonright \langle e, n \rangle}^k[t] = A_n \Vdash [n]_{f \upharpoonright \langle e, n \rangle}^k$  and

$$A_{n+1} \Vdash \varphi_e^{A_{n+1}}([n]_{f \upharpoonright \langle e, n \rangle}^k)[t] = A_{n+1} \Vdash \varphi_e^{A_{n+1}}([n]_{f \upharpoonright \langle e, n \rangle}^k).$$

Let  $r_{n+1} = \varphi_e^{A_{n+1}}([n]_{f \upharpoonright \langle e, n \rangle}^k)$ . For  $n+1 < n \leq t$  let  $r_{m+1}$  be minimal such that if  $d_i^m < r_m$  then  $r_{m+1} > d$ . Let  $t' > t$  be a true stage with  $A_m \Vdash r_m[t'] = A_m \Vdash r_m$  for  $n+1 \leq m \leq t$ . For  $t < m \leq t'$ , let  $r_m = r_t$ . Then the sequence  $r_{n+1}, \dots, r_{t'}$  clearly satisfies (1) and (2) (since by construction any future member of  $A_m$  must be greater than  $r_t$  for  $m > t$ ). The sequence also satisfies (3). Indeed, suppose  $d_i^m < r_m$ ,  $\langle i, m \rangle \leq \langle e, n \rangle$ ,  $d$  has already

entered  $A_{m+1}$ , and  $\alpha_t(\langle i, m \rangle) = 0$ , and  $d_i^m > R_m(\alpha_t \upharpoonright \langle i, m \rangle, t)$ . Note that  $n + 1 \leq m < t$ . Since  $t > s$  is a true stage and  $\alpha_t(\langle i, m \rangle) = 0$ ,  $\varphi_i$  is total. If  $d_i^m > R_m(\alpha_t \upharpoonright \langle i, m \rangle, t)$  then  $d_i^m$  will be enumerated into  $A_m$  at the next true stage after  $\varphi_i(s_d^{m+1}) \downarrow$ . But  $A_m$  has settled up to  $r_m$ , so  $\varphi_i(s_d^{m+1})$  must already have converged and  $d_i^m$  must have been enumerated into  $A_m$  by stage  $t'$ . ■

**Lemma 5.4** *The requirements  $P_{\langle e, n \rangle}$  are met.*

*Proof.* For every possible entrant  $d$  of  $A_{n+1}$  appointed after stage  $e$  there was a  $d_e^n < d$  appointed to possibly enter  $A_n$ . So for a.e.  $d$  which entered  $A_{n+1}$ , there was some  $d_e^n < d$  appointed with  $d_e^n > \lim_{t \in T} R(m_t \upharpoonright \langle e, n \rangle, t)$ . Hence,  $d_e^n$  was enumerated in  $A_n$  at the first stage after  $\varphi_e(s_d) \downarrow$  that  $R(\alpha_s \upharpoonright \langle e, n \rangle, s) < d_e^n$ . ■

If  $g$  is any computable function, the above proof can be modified to give a settling-time  $g$ -dominating sequence by ensuring that the guards  $c_0^n, \dots, c_s^n$  are such that  $g(c_0^n), \dots, g(c_s^n)$  are all less than  $c$  (this works if  $g$  is non-decreasing, which we may assume without loss of generality). ■

**Question 6** *Does there exist a uniformly computably enumerable sequence  $\{A_n\}_{n \in \omega}$  of c.e. sets that is settling-time  $g$ -dominating for all computable functions  $g$ ?*

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