MEASURING COMPLEXITIES OF CLASSES OF STRUCTURES

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Abstract. How do we compare the complexities of various classes of structures? The Turing ordinal of a class of structures, introduced by Jockusch and Soare, is defined in terms of the number of jumps required for coding to be possible. The back-and-forth ordinal, introduced by Montalbán, is defined in terms of Σ_α-types. The back-and-forth ordinal is (roughly) bounded by the Turing ordinal. In this paper, we show that, if we do not restrict the allowable classes, the reverse inequality need not hold. Indeed, for any computable ordinals $\alpha \leq \beta$ we present a class of structures with back-and-forth ordinal α and Turing ordinal β. We also present an example of a class of structures with back-and-forth ordinal 1 but no Turing ordinal.

1. Introduction

When we speak of the Turing degree of a particular presentation of a computable structure, we mean the Turing degree of the atomic diagram of that presentation, where the atomic formulas have been placed into some reasonable correspondence with $\mathbb{N}$. The degree spectrum of a structure is the set of Turing degrees of all presentations (isomorphic copies with domain ω) of the structure. There has been a significant body of work on studying what kinds of degree spectra are possible, either in general, or restricted to various classes of structures. Knight [Kni86] showed that the degree spectrum must be upward closed. When the degree spectrum has a least degree $\mathbf{d}$, we can consider $\mathbf{d}$ to be the complexity of the structure. How do we measure the complexity of a class of structures? When is one class more complicated than another? There is a sense in which the class of graphs appears to be the most complicated, since it is possible to make a rather large mess in a graph. It is easy to have any degree be the least degree in the degree spectrum of a graph. Indeed, graphs are universal for many properties in computable structure theory, as explained in [HKSS02]. In fact, we might rather view the class of graphs as "easy" because we can realize anything we want with a graph. In this sense the class of linear orderings could be viewed as more difficult than that of graphs, as Richter has shown that $\mathbf{0}$ is the only degree that can be realized as the least degree in the degree spectrum of a linear order.

We will formally define the back-and-forth ordinals of Montalbán, and the Turing ordinals of Jockusch in a moment. For now, note that graphs have uncountably many existential types, and that every degree can be realized as the least degree of the degree spectrum of a graph. This means that they have back-and-forth ordinal 1, and Turing ordinal 0. Montalbán showed [Mon12] that the class of linear orders has countably many Σ_2-types, but uncountably many Σ_3-types, so that they have

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back-and-forth ordinal 3. Knight [Kni86] improved Richter's result on linear orders to show that $\mathbf{0}'$ is the only degree that can be realized as the least degree in the collection of the *jumps* of the degrees in the degree spectrum of a linear order, and in that same paper showed that any degree $\mathbf{d} > \mathbf{0}''$ *can* be realized as the least degree of the *double jumps* of degrees in the degree spectrum of a linear order, thereby showing that the class of linear orders has Turing ordinal 2. Thus under either of these measures, the class of linear orders is seen to be more complex than the class of graphs.

We follow the notations and conventions for computable structure theory as in the book by Ash and Knight [AK00]. In this paper, all structures considered are countable.

1.1. **The Back-and-forth ordinal.** Consider a class of structures, $\mathbb{K}$. Given two structures $\mathcal{A}$ and $\mathcal{B}$ from $\mathbb{K}$, not necessarily distinct, and two fixed finite tuples $\vec{a}$ and $\vec{b}$ from the respective structures, we can ask how difficult it is to distinguish the tuple $\vec{a}$ in $\mathcal{A}$ from the tuple $\vec{b}$ in $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, with an isomorphism mapping $\vec{a}$ to $\vec{b}$, then the tuples are indistinguishable. If not then, from a complexity point of view, we can ask how difficult it is to separate the two tuples. More precisely, what is the minimal complexity of a formula φ witnessing this distinction? This idea is represented in the notion of the back-and-forth relations.

Definition 1.1 (Back-and-forth relations [AK00])**.** Let $\mathcal{A}$ be a countable structure in a finite language and let $\vec{a}$ be a finite tuple from $\mathcal{A}$. Let $\mathcal{B}$ be another structure in the same language as $\mathcal{A}$ and let $\vec{b}$ be a tuple from $\mathcal{B}$ of the same length as $\vec{a}$. For all ordinals α, define the back-and-forth relations, $\leq_\alpha$, inductively as follows:

(1) $(\mathcal{A}, \vec{a}) \leq_0 (\mathcal{B}, \vec{b})$ if and only if $\vec{a}$ and $\vec{b}$ satisfy the same atomic formulas in $\mathcal{A}$ and $\mathcal{B}$ respectively, and
(2) for $\gamma \geq 1$, $(\mathcal{A}, \vec{a}) \leq_\gamma (\mathcal{B}, \vec{b})$ if and only if for each $\vec{d} \in \mathcal{B}$ and each $0 \leq \beta < \gamma$ there exists $\vec{c} \in \mathcal{A}$ such that $(\mathcal{B}, \vec{b}, \vec{d}) \leq_\beta (\mathcal{A}, \vec{a}, \vec{c})$, where $\vec{c}$ and $\vec{d}$ are of the same length.

Note that this definition includes the case where $\vec{a}$ and $\vec{b}$ are both the empty tuple. We will denote $(\mathcal{A}, \emptyset) \leq_\alpha (\mathcal{B}, \emptyset)$ simply by $\mathcal{A} \leq_\alpha \mathcal{B}$.

There is a known relationship between the back-and-forth relations on structures in a given language and the infinitary formulas in the same language.

Theorem 1.2 (Ash and Knight [AK00])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be structures in the same language and let $\vec{a}$ and $\vec{b}$ be finite tuples, from $\mathcal{A}$ and $\mathcal{B}$ respectively, with $|\vec{a}| = |\vec{b}|$. Then, for all ordinals α, the following are equivalent.*

(i) $(\mathcal{A}, \vec{a}) \leq_\alpha (\mathcal{B}, \vec{b})$
(ii) Every Σ_α formula true of $\vec{b}$ in $\mathcal{B}$ is also true of $\vec{a}$ in $\mathcal{A}$.
(iii) Every Π_α formula true of $\vec{a}$ in $\mathcal{A}$ is also true of $\vec{b}$ in $\mathcal{B}$.

We write $(\mathcal{A}, \vec{a}) \equiv_\gamma (\mathcal{B}, \vec{b})$ if both $(\mathcal{A}, \vec{a}) \leq_\gamma (\mathcal{B}, \vec{b})$ and $(\mathcal{B}, \vec{b}) \leq_\gamma (\mathcal{A}, \vec{a})$ and get the following back-and-forth structures defined in [Mon12]:

Definition 1.3 (Montalbán)**.** Let $\mathbb{K}$ be a class of structures. Let

$$\mathbf{bf}_\gamma(\mathbb{K}) = \frac{\{(\mathcal{A}, \vec{a}) : \mathcal{A} \in \mathbb{K}, \vec{a} \in \mathcal{A}\}}{\equiv_\gamma}$$

where the equivalence classes in $\mathbf{bf}_\gamma(\mathbb{K})$ are partially ordered by $\leq_\gamma$ in the obvious way.

It is not hard to see that $(\mathcal{A}, \vec{a}) \leq_\alpha (\mathcal{B}, \vec{b})$ implies $(\mathcal{A}, \vec{a}) \leq_\beta (\mathcal{B}, \vec{b})$ for all $\beta \leq \alpha$. To measure the complexity of a class of structures, we are interested in the number of back-and-forth equivalence classes and, in particular, the first ordinal α where there are a large number of different tuples up to $\equiv_\alpha$-equivalence.

Definition 1.4 (Montalbán)**.** The *back-and-forth ordinal* of a class $\mathbb{K}$ is the least ordinal α such that $\mathbf{bf}_\alpha(\mathbb{K})$ is uncountable, if such an α exists.

By Theorem 1.2, the $\equiv_\alpha$-equivalence classes correspond to Σ_α-types. It is easy to see that there are uncountably many existential types realized by tuples from graphs, and it follows that the back-and-forth ordinal of the class of graphs is 1. Montalbán analyzed the back-and-forth classes of equivalence structures and linear orderings in [Mon12] and showed that the back-and-forth ordinal of these classes are 2 and 3 respectively.

1.2. **Turing ordinal.** We now present a computability-theoretic method of comparing classes of structures based on the ease or difficulty of coding information into structures of the given class, as introduced by Jockusch and Soare.

Definition 1.5 (Jockusch)**.** Let $\mathcal{A}$ be a structure. For any computable ordinal α, we define the following.

(1) The α^{th} *jump degree spectrum of* $\mathcal{A}$ is defined as

$$\mathrm{Spec}^{(\alpha)}(\mathcal{A}) = \{\deg(\mathcal{B})^{(\alpha)} : \mathcal{B} \cong \mathcal{A}\}.$$

(2) We say that $\mathcal{A}$ has α^{th} *jump degree* $\mathbf{d}$ if $\mathbf{d}$ is the least member of $\mathrm{Spec}^{(\alpha)}(\mathcal{A})$.

Definition 1.6 (Jockusch and Soare)**.** Let T be a first order theory which has continuum many pairwise nonisomorphic countable models. We call a computable ordinal α the *Turing ordinal* of T if

(i) every degree $\geq \mathbf{0}^{(\alpha)}$ is the α^{th} jump degree of a model of T, and
(ii) for all $\beta < \alpha$, the only possible β^{th} jump degree of a model of T is $\mathbf{0}^{(\beta)}$.

There are many natural questions that arise from this definition. One that is of particular interest in this paper is the following: Is every computable ordinal the Turing ordinal of some class of structures? And if so, how complicated must the theory of such a class be? It has been known since 1994 that, for each ordinal α satisfying $0 \leq \alpha \leq \omega$, there is a finitely axiomatizable class having Turing ordinal α. In Section 3, following the work of Ash and Knight in [AK90], we will define classes of structures having Turing ordinal α for all computable ordinals α. These will be classes of linear orderings. We will not discuss their axiomatizations, except to note that they are axiomatizable via computable infinitary formulas, the complexity of which increase as a function of α. For $\alpha > \omega$, it is still unknown whether or not there is a finitely axiomatizable class with Turing ordinal α.

1.3. **Relating the two ordinals.** We will see how the back-and-forth ordinal can provide computability-theoretic information about the given class of structures. In particular, it will help to describe the collection of sets that can be *coded* into structures in the class. We now present the necessary background for this analysis.

Definition 1.7 (Montalbán [Mon12])**.** We say that a set $X \subseteq \omega$ is *coded by a structure* $\mathcal{A}$ if X is c.e. in $\mathcal{B}$ for all $\mathcal{B} \cong \mathcal{A}$. More generally, $X \subseteq \omega$ is *coded by the* n^{th} *jump of a structure* $\mathcal{A}$ if X is c.e. in $\mathcal{B}^{(n)}$ for all $\mathcal{B} \cong \mathcal{A}$.

Montalbán also defined a slightly weaker notion of coding requiring only that the set be left-c.e. rather than c.e. in each copy.

Definition 1.8. Let $X \subseteq \omega$.

(1) For $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \leq_L \tau$ if $\sigma \subseteq \tau$ or for the least n such that $\sigma(n)$ and $\tau(n)$ are both defined and $\sigma(n) \neq \tau(n)$, we have $\tau(n) = 1$.
Note that $\leq_L$ *is total order on* $2^{<\omega}$*.*
(2) Let $\sigma \in 2^{<\omega}$ and $X, Y \in 2^{\omega}$. We write $\sigma \leq_L X$ if $\sigma \subseteq X$ or there exists a least n such that $\sigma(n)$ is defined and $\sigma(n) \neq X(n) = 1$. If there is a least n such that $\sigma(n)$ is defined and $1 = \sigma(n) \neq X(n)$ then we write $X \leq_L \sigma$. Finally, we write $X \leq_L Y$ if for the least n such that $X(n) \neq Y(n)$ we have $Y(n) = 1$.
Note that $\leq_L$ *is total order on* $2^{\leq\omega}$*.*
(3) We will write $<_L$ if we have $\leq_L$ but not equality. Observe that for any $\sigma \in 2^{<\omega}$ and $X \in 2^{\omega}$ we have either $\sigma <_L X$ or $X <_L \sigma$. Let $X_L := \{\sigma \in 2^{<\omega} : \sigma <_L X\}$. We say that X is *left-c.e.* if the set X_L is c.e.

Definition 1.9 (Montalbán [Mon12])**.** We say that a set $X \subseteq \omega$ is *weakly coded by a structure* $\mathcal{A}$ if X is left-c.e. in $\mathcal{B}$ for all $\mathcal{B} \cong \mathcal{A}$. More generally, $X \subseteq \omega$ is *weakly coded by the* n^{th} *jump of* $\mathcal{A}$ if X is left-c.e. in $\mathcal{B}^{(n)}$ for all $\mathcal{B} \cong \mathcal{A}$.

We will also be using the notion of enumeration reducibility. Informally, we want A to be enumeration reducible to B if we can computably enumerate A from an enumeration of B, where the enumeration of A does not depend on the order in which the set B is enumerated. For a formal treatment, we need a coding of pairs n, D where n is a natural number and D is a finite set of natural numbers. Fix an effective list of all finite sets of natural numbers, say $D_0, D_1, D_2, \dots$, and let $\langle n, D_j \rangle = \langle n, j \rangle$.

Definition 1.10 ([Coo04])**.** We say that a set A is *enumeration reducible* to a set B, denoted $A \leq_e B$, if for some c.e. set W_i,

$$n \in A \iff (\exists \text{ finite } D \subseteq B)[\langle n, D \rangle \in W_i].$$

If we have $A \leq_e B$ via the set W_i then we write $A = \Psi_i^B$.

Recall the following equivalent definition of enumeration reducibility due to Selman [Sel71]:

$$A \leq_e B \Leftrightarrow (\forall X)[B \text{ is c.e. in } X \rightarrow A \text{ is c.e. in } X].$$

A result of Knight's relates the two previous definitions:

Theorem 1.11 (Knight)**.** *Let* $\mathcal{A}$ *be a structure. A set* $X \subseteq \omega$ *is coded by the* n^{th} *jump of* $\mathcal{A}$ *if and only if* X *is enumeration reducible to the* Σ^c_{n+1}*-type of some tuple* $\vec{a} \in \mathcal{A}$*.*

Note that the Σ^c_{n+1}-type of $\vec{a}$ in $\mathcal{A}$ is the set of Σ^c_{n+1} formulas true of $\vec{a}$ in $\mathcal{A}$. The proof of the $n = 0$ case can be found in [AK00] and this proof can be generalized to obtain the above result for all $n \geq 0$.

1.4. **Size of the n-back-and-forth structure.** It follows from Theorem 1.11 that if there are only countably many $\equiv_{n+1}$-classes of tuples from $\mathbb{K}$, then only countably many sets can be coded by n^{th} jumps of structures in $\mathbb{K}$. It follows from a result of Silver's in [Sil80] that, if $\mathbb{K}$ is Borel class — i.e. a class axiomatizable via countably many $\mathcal{L}_{\omega_1,\omega}$ formulas — then $\mathbf{bf}_n(\mathbb{K})$ is either countable or has size continuum. The following results from [Mon12] characterize exactly when each of these two sizes occur, relative to the difficulty of coding into structures of the given Borel class.

Theorem 1.12 (Montalbán)**.** *Let $\mathbb{K}$ be a Borel class of structures, then the following are equivalent.*

(i) $|\boldsymbol{bf}_n(\mathbb{K})| = \aleph_0$
(ii) There exists an oracle relative to which the only sets of numbers that can be coded by the $(n-1)^{st}$ jump of a structure in $\mathbb{K}$ are the sets computable in the oracle.

Corollary 1.13 (Montalbán)**.** *Let $\mathbb{K}$ be class of countable structures with $|\boldsymbol{bf}_{n+1}(\mathbb{K})| = \aleph_0$. If $\mathbb{K}$ has Turing ordinal m then $n < m$ (and hence $n+1 \leq m$).*

Proof. Suppose that $\mathbf{bf}_{n+1}(\mathbb{K})$ is countable. Then by Theorem 1.12, we can only code countably many sets into the n^{th} jumps of structures in $\mathbb{K}$. It follows that structures in $\mathbb{K}$ cannot have arbitrary n^{th} jump degree. Hence the Turing ordinal (if it exists) must be strictly bigger than n by definition. □

Corollary 1.14. *If $\mathbb{K}$ has back-and-forth ordinal $n+1$ and the Turing ordinal of $\mathbb{K}$ is m then $n \leq m$.*

Proof. If $\mathbb{K}$ has back-and-forth ordinal $n+1$ then in particular $\mathbf{bf}_n(\mathbb{K})$ is countable. □

Montalbán extended this result to infinite computable ordinals using the following result of Knight's.

Theorem 1.15 (Knight)**.** *Let α be a computable ordinal. If S is c.e. in $\Delta^0_\alpha(\mathcal{B})$ for all $\mathcal{B} \cong \mathcal{A}$ then S is enumeration reducible to the Σ^c_α-type of some tuple $\vec{a} \in \mathcal{A}$.*

Theorem 1.15, along with its converse, appears without proof in [Kni98]. For a full proof of Theorem 1.15 see Knoll's thesis [Kno13].

Corollary 1.16. *Let $\mathbb{K}$ be a class of countable structures. If the Turing ordinal, γ, of $\mathbb{K}$ exists and satisfies $\omega \leq \gamma < \omega_1^{CK}$, then the back-and-forth ordinal of $\mathbb{K}$ is at most γ.*

Montalbán hoped to obtain a complete characterization of the back-and-forth ordinal in terms of coding and, along these lines, proved the following.

Theorem 1.17 (Montalbán)**.** *Let $\mathbb{K}$ be a Borel class of structures, then the following are equivalent.*

(i) $|\boldsymbol{bf}_n(\mathbb{K})| = 2^{\aleph_0}$
(ii) Relative to some fixed oracle, every set can be weakly coded into the $(n-1)^{st}$ jump of some structure in $\mathbb{K}$.

To have a proper dichotomy in Theorems 1.12 and 1.17, we would need to replace *weak coding* in Theorem 1.17 with *coding*, but unfortunately, this cannot be done.

It is clear that the direction $(ii) \Rightarrow (i)$ remains true if we replace the statement with coding, but the direction $(i) \Rightarrow (ii)$ is false. This is not as obvious. A class of structures defined by Montalbán (Example 2.17 in [Mon10]) exhibits a class with uncountably many $\equiv_1$ classes, but where arbitrary coding is not possible, even relative to any fixed oracle. Indeed, his example gives a class of structures with back-and-forth ordinal 1 and no Turing ordinal. We verify his example in Section 2.

Corollary 1.18. *There is a class of structures $\mathbb{K}$ such that $|\boldsymbol{bf}_1(\mathbb{K})| = 2^{\aleph_0}$ but such that there is no fixed oracle relative to which every set can be coded in some $\mathcal{A} \in \mathbb{K}$.*

1.5. **Lower Bound.** After [Mon12], we had the following concrete examples of classes where both the Turing ordinals and back-and-forth ordinals were known, or easy to calculate:

Class of structures	Turing ordinal	Back-and-forth ordinal
Abelian groups	0 [Ric81]	1
Graphs	0 [Ric81]	1
Algebraic fields	0 [CHS07]	1
Partial orderings	0 [Ric81]	1
Lattices	0 [Ric81]	1
Equivalence structures	1 [Ric81], [Mon12]	2 [Mon12]
Linear orderings	2 [Ric81], [Kni86]	3 [Mon12]
Boolean algebras	ω [JS94]	ω [AK00]

As we can see in the table, every case where the ordinals are finite satisfies that the back-and-forth ordinal is equal to the successor of the Turing ordinal. In the only infinite case, we have equality. It is natural to ask whether there is a reason for this pattern. By Corollary 1.14, for every finite case, the successor of the Turing ordinal is an upper bound for the back-and-forth ordinal, and by Corollary 1.16, in the infinite case, the Turing ordinal is an upper bound for the back-and-forth ordinal. This leads to the following questions:

Question 1.19. If the back-and-forth ordinal of a Borel class of structures, $\mathbb{K}$, is $n + 1$, must $\mathbb{K}$ have Turing ordinal n?

Question 1.20. If the back-and-forth ordinal of a Borel class of structures, $\mathbb{K}$, is $\alpha \geq \omega$, must $\mathbb{K}$ have Turing ordinal α?

For a negative answer to Question 1.19, we can look at a well-known class. It is known that the Turing ordinal of the class of models of Peano arithmetic (PA) is 1. (A standard model of PA has degree $\mathbf{0}$ and Proposition 3.4 from [Kni98] asserts that any nonstandard model of PA has no degree. The fact that every jump degree is realizable is explained in the Introduction of [Kni86]). A quick analysis of the existential types of models of PA shows that the back-and-forth ordinal of the class is also 1.

In Section 3, we will provide a negative answer to Question 1.20. More precisely, for computable successor ordinals $3 \leq \alpha \leq \beta$, we will give a class of linear orderings $\mathbb{K}_{\alpha,\beta}$ with Turing ordinal β (or $\beta - 1$ if finite) and back-and-forth ordinal α.

2. Class with no Turing ordinal

In this section we present a class of structures, defined by Montalbán in [Mon10], having back-and-forth ordinal 1 but no Turing ordinal. We take this opportunity to fill in some proofs about properties of Montalbán's class.

Definition 2.1 (Montalbán)**.** Let $\mathcal{L} = \{U, V, f, \{c_\sigma : \sigma \in 2^{<\omega}\}\}$ where U and V are unary relations, f is a unary function and each c_σ is a constant. Let $\mathbb{K}_W$ be the class of countable $\mathcal{L}$ structures, $\mathcal{A}$, that satisfy the following properties:

(i) U and V partition $|\mathcal{A}|$
(ii) x is named by a constant iff $x \in V$
(iii) If $\sigma \neq \tau$ then $c_\sigma \neq c_\tau$
(iv) $rng(f) \subseteq V$
(v) $f\restriction_U$ is 1-1
(vi) $f\restriction_V = id$, and
(vii) If $\sigma <_L \tau$ and $(\exists x \in U)[f(x) = c_\tau]$ then $(\exists x \in U)[f(x) = c_\sigma]$.

For each $\mathcal{A} \in \mathbb{K}_W$, consider the set $R_\mathcal{A} := \{\sigma : \mathcal{A} \models (\exists x \in U)[f(x) = c_\sigma]\}$. Recall that $R_\mathcal{A}$ is coded in $\mathcal{A}$ if and only if $\mathrm{Spec}(\mathcal{A}) \subseteq \{X : R_\mathcal{A}$ is c.e. in $X\}$.

Proposition 2.2. *For every $\mathcal{A} \in \mathbb{K}_W$, Spec$(\mathcal{A}) = \{X : R_\mathcal{A}$ is c.e. in $X\}$.*

Proof. Clearly, $R_\mathcal{A}$ is c.e. in $\mathcal{A}$. Suppose $\mathcal{B} \cong \mathcal{A}$. Then

$$R_\mathcal{A} = \{\sigma : \mathcal{A} \models (\exists x \in U)[f(x) = c_\sigma]\} = \{\sigma : \mathcal{B} \models (\exists x \in U)[f(x) = c_\sigma]\} = R_\mathcal{B}.$$

As $R_\mathcal{B}$ is c.e. in $\mathcal{B}$, so is $R_\mathcal{A}$, and hence $R_\mathcal{A}$ is coded in $\mathcal{A}$.

It remains to show that $\mathrm{Spec}(\mathcal{A}) \supseteq \{X : R_\mathcal{A}$ is c.e. in X$\}$. Suppose that $R_\mathcal{A}$ is c.e. in X. We want to build an X-computable copy $\mathcal{B}$ of $\mathcal{A}$. Let $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ be a computable listing of all strings in $2^{<\omega}$. By properties (ii) and (iii), the set V must be infinite. First, let $Y = \{b_0, b_1, b_2, \ldots\}$ be a (coinfinite) computable subset of ω, declare $b_i \in V^\mathcal{B}$ for all $i \in \omega$ and let $c^\mathcal{B}_{\sigma_i} = b_i$. Let $\{R^s_\mathcal{A}\}_{s\in\omega}$ be an X-computable enumeration of $R_\mathcal{A}$. At stage s, use X to compute $R^s_\mathcal{A}$ and let

$$R^s_\mathcal{A} - R^{s-1}_\mathcal{A} = \{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}.$$

Take the first k numbers that are not in Y and not yet in the domain of $f^\mathcal{B}$, say $a_1, a_2, \ldots, a_k$. Declare $a_j \in U^\mathcal{B}$ for all $j = 1, \ldots, k$, and let $f^\mathcal{B}(a_j) = b_{i_j}$ and $f^\mathcal{B}(b_{i_j}) = b_{i_j}$.

By construction, the structure $\mathcal{B}$ is computable from X and satisfies properties (i)-(vii). Let's define a map, π, between $\mathcal{A}$ and $\mathcal{B}$ as follows: For each $v \in V^\mathcal{A}$, we have $v = c^\mathcal{A}_\sigma$ for some σ. Let $\pi(v) = \pi(c^\mathcal{A}_\sigma) = c^\mathcal{B}_\sigma$. For each $u \in U^\mathcal{A}$, we must have $f^\mathcal{A}(u) = v = c^\mathcal{A}_\sigma$ for some $v \in V^\mathcal{A}$ and some $\sigma \in 2^{<\omega}$. There must exist exactly one $\tilde{u} \in U^\mathcal{B}$ such that $f^\mathcal{B}(\tilde{u}) = c^\mathcal{B}_\sigma$ and so we let $\pi(u) = \tilde{u}$. This map, π, is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. □

Proposition 2.3 (Montalbán)**.** *Every set $D \subseteq \omega$ is weakly coded in some $\mathcal{A} \in \mathbb{K}_W$.*

Proof. Let $D \subseteq \omega$. Consider the set $E = \{\sigma : \sigma <_L D\} = \{\sigma_0, \sigma_1, \sigma_2, \ldots\}$. We will define the structure $\mathcal{A}$ as follows: Let U consist of the even numbers, and V the odd numbers. Let $c^\mathcal{A}_{\sigma_i} = 4i + 1$ and for $\sigma \notin E$, let $c^\mathcal{A}_\sigma = 4\sigma + 3$ (under some coding of $2^{<\omega}$ into ω). Let $f^\mathcal{A}(2i + 1) = 2i + 1$ and $f^\mathcal{A}(2i) = 4i + 1$. Then $R_\mathcal{A} = E$. As $R_\mathcal{A} = E$ is coded in $\mathcal{A}$, the set D is weakly coded in $\mathcal{A}$. □

By Therorem 1.17, we must have $|\mathbf{bf}_1(\mathbb{K}_W)| = 2^{\aleph_0}$. In particular, the back-and-forth ordinal of $\mathbb{K}_W$ is 1.

Theorem 2.4. *There is a set $D \subset \omega$ such that D is not coded in any structure $\mathcal{A} \in \mathbb{K}_W$.*

First note that, for any set D and any $\mathcal{A} \in \mathbb{K}_W$, we have

$$\begin{aligned} D \text{ is coded in } \mathcal{A} &\Leftrightarrow \mathrm{Spec}(\mathcal{A}) \subseteq \{X : D \text{ is c.e. in } X\} \\ &\Leftrightarrow \{X : R_{\mathcal{A}} \text{ is c.e. in } X\} \subseteq \{X : D \text{ is c.e. in } X\} \\ &\Leftrightarrow (\forall X)[R_{\mathcal{A}} \text{ is c.e. in } X \to D \text{ is c.e. in } X] \\ &\Leftrightarrow D \leq_e R_{\mathcal{A}} \end{aligned}$$

Therefore, to prove Theorem 2.4, we need to show that

$$\bigcup_{\mathcal{A} \in \mathbb{K}} \{D : D \leq_e R_{\mathcal{A}}\} \neq 2^\omega.$$

In the proof of Proposition 2.3, we show that for every $X \subseteq \omega$ there is a structure $\mathcal{A} \in \mathbb{K}_W$ such that $R_{\mathcal{A}} = X_L$. Conversely, for every structure $\mathcal{A} \in \mathbb{K}_W$, we have $R_{\mathcal{A}} = X_L$ for some $X \subseteq \omega$. It follows from this observation that

$$\bigcup_{\mathcal{A} \in \mathbb{K}} \{D : D \leq_e R_{\mathcal{A}}\} = \bigcup_{X \subseteq \omega} \{D : D \leq_e X_L\}.$$

We will prove Theorem 2.4 by showing that $\bigcup_{X \subseteq \omega} \{D : D \leq_e X_L\} \neq 2^\omega$.

We wish to build a set D such that $D \not\leq_e X_L$ for all $X \subseteq \omega$. We will build D satisfying the following requirements, for all $e \in \omega$:

$$R_e : \; D \neq \Psi_e^{X_L} \text{ for all } X \subseteq \omega.$$

Given a set $X \subseteq \omega$, finite subsets of X_L will be finite sets of strings $\{\sigma_1, \ldots, \sigma_k\}$ such that $\sigma_i <_L X$ for all $0 \leq i \leq k$. As such, $\vec{\sigma} := \{\sigma_1 \ldots, \sigma_k\}$ is a subset of X_L if and only if the "rightmost" string in $\vec{\sigma}$ is in X_L. Let $R(\vec{\sigma}) := \{\sigma \in \vec{\sigma} : \tau \leq_L \sigma \text{ for all } \tau \in \vec{\sigma}\}$ denote the rightmost string of $\vec{\sigma}$.
Recall that we write $D = \Psi_e^{X_L}$ if for all $n \in \omega$,

$$n \in D \iff (\exists \text{ finite } \vec{\sigma} \subseteq X_L)[\langle n, \vec{\sigma}\rangle \in W_e].$$

__**To meet Requirement R_e:**__ We will use the numbers $\langle 0, e\rangle$ and $\langle 1, e\rangle$ to meet the requirement R_e.
Let $S_0^e := \{\vec{\sigma} : \langle\langle 0, e\rangle, \vec{\sigma}\rangle \in W_e\}$ and let $S_1^e := \{\vec{\sigma} <_L S_0^e : \langle\langle 1, e\rangle, \vec{\sigma}\rangle \in W_e\}$, where we write "$\vec{\sigma} <_L S$" for some set $S \subseteq 2^{<\omega}$ if $R(\vec{\sigma}) <_L R(\vec{\tau})$ for all $\vec{\tau} \in S$.

Definition 2.5. We now define D as follows, depending on which, if any, of the sets S_0^e and S_1^e are empty:

(1) If $S_0^e = \emptyset$ then set $D(\langle 0, e\rangle) = 1$.
(2) If $S_0^e \neq \emptyset$ and $S_1^e = \emptyset$, then set $D(\langle 0, e\rangle) = 0$ and $D(\langle 1, e\rangle) = 1$.
(3) If $S_0^e \neq \emptyset$ and $S_1^e \neq \emptyset$, then set $D(\langle 0, e\rangle) = 1$ and $D(\langle 1, e\rangle) = 0$.

For any x for which $D(x)$ has not been defined by the above conditions, set $D(x) = 0$.

Lemma 2.6. *The set D defined above satisfies $D \not\leq_e X_L$ for all $X \subseteq \omega$.*

Proof. We will show that R_e is met for each $e \in \omega$ by cases:

Case 1 ($S_0^e = \emptyset$): If S_0^e is empty then, by definition of S_0^e, we have $\Psi_e^{X_L}(\langle 0, e\rangle) = 0$ for all $X \subseteq \omega$. So since $D(\langle 0, e\rangle) = 1$ we satisfy R_e.

Case 2 ($S_0^e \neq \emptyset$ and $S_1^e = \emptyset$): We have two subcases:

(1) There is some $\vec{\tau} \in S_0^e$ satisfying $R(\vec{\tau}) <_L X$:

In this case we have $\vec{\tau} \subset X_L$ with $\langle\langle 0, e\rangle, \vec{\tau}\rangle \in W_e$ and hence

$$\Psi_e^{X_L}(\langle 0, e\rangle) = 1 \neq 0 = D(\langle 0, e\rangle).$$

(2) $X <_L R(\vec{\tau})$ for all $\vec{\tau} \in S_0^e$:

In this case, for every $\vec{\rho} \in 2^{<\omega}$, we must have either $\vec{\rho} \not\subseteq X_L$ or $\langle\langle 1, e\rangle, \vec{\rho}\rangle \notin W_e$. Suppose for a contradiction that we have both $\vec{\rho} \subset X_L$ and $\langle\langle 1, e\rangle, \vec{\rho}\rangle \in W_e$. Then we have $R(\vec{\rho}) <_L X <_L R(\vec{\tau})$ for all $\vec{\tau} \in S_0^e$, or in other words, $\vec{\rho} <_L S_0^e$. As $\langle\langle 1, e\rangle, \vec{\rho}\rangle \in W_e$, it follows that $\vec{\rho} \in S_1^e = \emptyset$ which is a contradiction. Therefore we have $\langle\langle 1, e\rangle, \vec{\rho}\rangle \notin W_e$ for all $\vec{\rho} \subseteq X_L$ and hence $\Psi_e^{X_L}(\langle 1, e\rangle) = 0 \neq 1 = D(\langle 1, e\rangle)$.

Case 3 ($S_0^e \neq \emptyset$ and $S_1^e \neq \emptyset$): Let $S^e = S_0^e \cup S_1^e$. Again we have two subcases:

(1) There is some $\vec{\sigma} \in S^e$ such that $R(\vec{\sigma}) <_L X$:

If $\vec{\sigma} \in S_1^e$, then $D(\langle 1, e\rangle) = 0 \neq 1 = \Psi_e^{X_L}(\langle 1, e\rangle)$ and we are done. If $\vec{\sigma} \in S_0^e$, then as $S_1^e \neq \emptyset$, we can choose a string $\vec{\tau} \in S_1^e$ such that $\vec{\tau} <_L \vec{\sigma} <_L X$ and hence $\vec{\tau} \subset X_L$ and $\langle\langle 1, e\rangle, \vec{\tau}\rangle \in W_e$. Thus we again have $D(\langle 1, e\rangle) = 0 \neq 1 = \Psi_e^{X_L}(\langle 1, e\rangle)$.

(2) $X <_L R(\vec{\sigma})$ for all $\vec{\sigma} \in S^e$:

We will show in this case that $\langle\langle 0, e\rangle, \vec{\tau}\rangle \notin W_e$ for all $\vec{\tau} \subset X_L$. Suppose that $\vec{\tau} \subset X_L$. Then we have $R(\vec{\tau}) <_L X$ and hence $R(\vec{\tau}) <_L X <_L R(\vec{\sigma})$ for all $\vec{\sigma} \in S^e$. In particular, we have $R(\vec{\tau}) <_L R(\vec{\sigma})$ for all $\vec{\sigma} \in S_0^e$ and so $\vec{\tau} \notin S_0^e$. The only way we could have $\vec{\tau} \notin S_0^e$ is if $\langle\langle 0, e\rangle, \vec{\tau}\rangle \notin W_e$. So we have $\Psi_e^{X_L}(\langle 0, e\rangle) = 0 \neq 1 = D(\langle 0, e\rangle)$.

In all cases, R_e is met. □

Remark 2.7. It should be noted that the proof of Theorem 2.4 can be relativized to include an arbitrary fixed oracle. In other words, if we fix an oracle Y, then we can build a set D such that D is not coded in any structure in $\mathbb{K}_W$, even relative to the oracle Y. We amend the previous construction as follows: We write $A \leq_e^Y B$ if there is some e such that for all $n \in \omega$,

$$n \in A \Longleftrightarrow (\exists \text{ finite } D \subseteq B)\left[\langle n, D\rangle \in W_e^Y\right].$$

Then for any structure $\mathcal{A} \in \mathbb{K}_W$, we have

D is coded in $\mathcal{A}$ relative to $Y \Leftrightarrow (\forall X)[R_{\mathcal{A}}$ is c.e. in $X \to D$ is c.e. in $X \oplus Y] \Leftrightarrow D \leq_e^Y R_{\mathcal{A}}$.

The first equivalence follows immediately from previous work, and the second equivalence is a relativization of Selman's theorem. Now we can prove (the relativized version of) Theorem 2.4 by fixing any oracle Y, and building a set D such that $D \not\leq_e^Y X_L$ for all $X \subseteq \omega$. The construction and verification are the same, except that every occurrence of the set W_e must be replaced by the set W_e^Y.

Corollary 2.8. *There is a class of structures* $\mathbb{K}$ *such that* $|\boldsymbol{bf}_1(\mathbb{K})| = 2^{\aleph_0}$ *but such that there is no fixed oracle relative to which every set can be coded in some* $\mathcal{A} \in \mathbb{K}$*.*

Proof. Let $\mathbb{K}_W$ be the previously defined class. As every set can be weakly coded into some $\mathcal{A} \in \mathbb{K}_W$ then, by Theorem 1.17, we must have $|\mathbf{bf}_1(\mathbb{K}_{\mathbb{W}})| = 2^{\aleph_0}$.
The set D from Definition 2.5 is not coded in any $\mathcal{A} \in \mathbb{K}_W$ (even relative to a fixed oracle) by Lemma 2.6, Remark 2.7 and earlier observations. □

Corollary 2.9. *The Turing ordinal of the class* $\mathbb{K}_W$*, if it exists, is strictly greater than* 0*.*

Proof. Let D be the set we constructed in Theorem 2.4. Then there is no structure $\mathcal{A} \in \mathbb{K}_W$ of degree $\mathbf{d} = \deg(D)$. Since there is at least one degree that cannot be realized as the degree of a structure in $\mathbb{K}_W$, the Turing ordinal cannot equal 0. □

The following observation of Joe Miller shows that the class $\mathbb{K}_W$ has no Turing ordinal.

Proposition 2.10. *The Turing ordinal of the class* $\mathbb{K}_W$*, if it exists, is equal to* 0*.*

Proof. Let Z be the complement of $\emptyset'$ and let $\mathcal{A}$ be the $\mathbb{K}_W$ structure with $R_{\mathcal{A}} = Z_L$. Then we have

$$\mathrm{Spec}(\mathcal{A}) = \{\deg(X) : Z_L \text{ is c.e. in } X\} = \{\deg(X) : Z \text{ is left-c.e. in } X\} = \{\deg(X) : X \geq_T \emptyset'\}$$

and hence $\mathcal{A}$ has degree $\mathbf{0}'$. In fact we can realize any c.e. degree in the same manner. Since there exist structures in $\mathbb{K}_W$ having degree other than $\mathbf{0}$, the Turing ordinal of $\mathbb{K}_W$, if it exists, must be equal to 0. □

3. Arbitrary back-and-forth and Turing ordinals

In the paper [AK90], Ash and Knight showed that for each successor ordinal α such that $3 \leq \alpha < \omega_1^{\mathrm{CK}}$, there is a linear ordering which has αth jump degree but no γth jump degree for $\gamma < \alpha$. This was actually a weakening of an earlier result from [AJK90], but the orderings they used are useful to us. We will use the linear orderings from [AK90], and the results about them, to show that for each computable successor ordinal $\alpha \geq 3$ there exist classes of linear orderings $\mathbb{K}_\alpha$ with Turing ordinal α (or $\alpha - 1$ if finite) and back-and-forth ordinal α. For computable successor ordinals $3 \leq \alpha \leq \beta$ we will give a class of linear orderings $\mathbb{K}_{\alpha,\beta}$ with Turing ordinal β (or $\beta - 1$ if finite) and back and forth ordinal α.

For a fixed computable successor ordinal $\alpha \geq 3$, let λ and μ be discrete orderings such that $\mu \leq_\alpha \lambda$ but $\lambda \not\leq_\alpha \mu$. These exist, see [AK90] or [Ash86], indeed they can be taken to be of the form ω^β or $\omega^{\beta+1} + \omega^\beta$. Then for any $S \subseteq \omega$, let $\mathcal{L}_\alpha(S)$ be the shuffle sum of orderings of types $\eta + 1 + \lambda + n + 1 + \eta$ for $n \in S \oplus \overline{S}$ and $\eta + 1 + \mu + n + 1 + \eta$ for all $n \in \omega$. In [AK90] they show that, for infinite α,

$$\mathrm{Spec}(\mathcal{L}_\alpha(S)) = \{deg(D) \mid S \leq_T D^{(\alpha)}\}$$

and it follows that, for α finite, $\mathrm{Spec}(\mathcal{L}_\alpha(S)) = \{deg(D) \mid S \leq_T D^{(\alpha-1)}\}$.

For a computable successor ordinal $\alpha \geq 3$, let $\mathbb{K}_\alpha = \{\mathcal{L}_\alpha(S) \mid S \subseteq \omega\}$.

Lemma 3.1. *Let* α *be a computable ordinal. If* $\boldsymbol{d} \geq \boldsymbol{0}^{(\alpha)}$ *and* $S \in \boldsymbol{d}$ *then the set* $\mathcal{C} := \{\deg(D)^{(\alpha)} : S \leq_T D^{(\alpha)}\}$ *has least element* $\boldsymbol{d}$*.*

Proof. By jump inversion there is a set D_0 such that $S \equiv_T D_0^{(\alpha)}$ and so $D_0^{(\alpha)} \equiv_T S \in \mathcal{C}$. It is clear that S is a lower bound for the degrees in $\mathcal{C}$ by definition. □

With the following result, we can show that $\mathbb{K}_\alpha$ has Turing ordinal α (or $\alpha - 1$ if finite).

Theorem 3.2 (Ash, Jockusch and Knight [AJK90]). *Let γ be a computable ordinal and let $S \subseteq \omega$. If $B \leq_T D^{(\gamma)}$ for all D satisfying $S \leq_T D^{(\gamma+1)}$ then $B \leq_T \mathbf{0}^{(\gamma)}$. Hence if $S \not\leq_T 0^{(\gamma+1)}$, then the set $\{D^{(\gamma)} : S \leq_T D^{(\gamma+1)}\}$ has no element of least degree.*

Corollary 3.3. *If $\alpha \geq 3$ is a computable successor ordinal then the class $\mathbb{K}_\alpha$ has Turing ordinal α ($\alpha - 1$ if finite).*

Proof. The following proof assumes α is infinite; the proof for α finite is similar but off by 1 because the degree spectrum is off by 1. For part (i) of Definition 1.6, suppose $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. Let $S \in \mathbf{d}$, and consider $\mathcal{L}_\alpha(S) \in \mathbb{K}_\alpha$. Then since $\mathrm{Spec}(\mathcal{L}_\alpha(S)) = \{deg(D) \mid S \leq_T D^{(\alpha)}\}$, it follows that $\mathrm{Spec}(\mathcal{L}_\alpha(S))$ has α^{th} jump degree by Lemma 3.1. It remains to show that part (ii) of the definition is satisfied. In other words, we need to show that for any $S \subseteq \omega$ and any $\gamma < \alpha$, if $\mathcal{C} := \{D^{(\gamma)} : S \leq_T D^{(\alpha)}\}$ has an element of least degree then it is $\mathbf{0}^{(\gamma)}$. If $S \leq_T \mathbf{0}^{(\alpha)}$ then $\mathbf{0}^{(\gamma)}$ is least in $\mathcal{C}$. If $S \not\leq_T \mathbf{0}^{(\alpha)}$, then since α is a successor ordinal and $\gamma < \alpha$, $S \not\leq_T \mathbf{0}^{(\gamma+1)}$. So by Theorem 3.2, $\mathcal{C}$ has no element of least degree. □

We now show that the back-and-forth ordinal of $\mathbb{K}_\alpha$ is α.

Lemma 3.4. *Let $\mathcal{A}$ be the shuffle sum of linear orderings $\mathcal{A}_i$ for $i \in \omega$ and let $\mathcal{B}$ be the shuffle sum of linear orderings $\mathcal{B}_i$ for $i \in \omega$ such that $\mathcal{B}_i \leq_\alpha \mathcal{A}_i$ for all $i \in \omega$. For any $\beta \leq \alpha$, if $\vec{a} = \vec{a}_1 \cup \vec{a}_2 \cup \ldots \cup \vec{a}_k \in \mathcal{A}$ and $\vec{b} = \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_k \in \mathcal{B}$ satisfy*

- *(i) $\vec{a}_i < \vec{a}_{i+1}$ (that is, the greatest member of $\vec{a}_i$ is less than the least member of $\vec{a}_{i+1}$ under the ordering on $\mathcal{A}$) and $\vec{b}_i < \vec{b}_{i+1}$ for $1 \leq i \leq k-1$,*
- *(ii) $|\vec{a}_i| = |\vec{b}_i|$ for $i = 1, \ldots, k$,*
- *(iii) each of the tuples $\vec{a}_1, \ldots, \vec{a}_k$ lie in a distinct $\mathcal{A}_i$ block in $\mathcal{A}$ and each of the tuples $\vec{b}_1, \ldots, \vec{b}_k$ lie in a distinct $\mathcal{B}_i$ block in $\mathcal{B}$ (with slight abuse of notation, we assume $\vec{a}_i \in \mathcal{A}_i$ and $\vec{b}_i \in \mathcal{B}_i$), and*
- *(iv) $(\mathcal{B}_i, \vec{b}_i) \leq_\beta (\mathcal{A}_i, \vec{a}_i)$ for $1 \leq i \leq k$*

then $(\mathcal{B}, \vec{b}) \leq_\beta (\mathcal{A}, \vec{a})$.

Proof. We will prove the lemma by induction on β for all orderings and tuples at once. Let $\vec{a}$ and $\vec{b}$ be as above.

For $n = 0$: Since $(\mathcal{B}_i, \vec{b}_i) \leq_0 (\mathcal{A}_i, \vec{a}_i)$, the tuples $\vec{a}_i$ and $\vec{b}_i$ are ordered the same way in $\mathcal{A}_i$ and $\mathcal{B}_i$ respectively. As $\vec{a}_1 < \vec{a}_2 < \ldots < \vec{a}_k$ and $\vec{b}_1 < \vec{b}_2 < \ldots < \vec{b}_k$, we have that $\vec{a} = \vec{a}_1 \cup \vec{a}_2 \cup \ldots \cup \vec{a}_k$ and $\vec{b} = \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_k$ are ordered in the same way in $\mathcal{A}$ and $\mathcal{B}$ respectively and hence $(\mathcal{B}, \vec{b}) \leq_0 (\mathcal{A}, \vec{a})$.

Now assume $0 < \beta \leq \alpha$ and that the result holds for all $\gamma < \beta$ and suppose that we have tuples $\vec{a}$ and $\vec{b}$ satisfying $(i) - (iv)$ for β. We wish to show that $(\mathcal{B}, \vec{b}) \leq_\beta (\mathcal{A}, \vec{a})$. Fix $\vec{c} \in \mathcal{A}$ and $\gamma < \beta$. We need to find $\vec{d} \in \mathcal{B}$ such that $(\mathcal{A}, \vec{a}, \vec{c}) \leq_\gamma (\mathcal{B}, \vec{b}, \vec{d})$. For each i, let $\vec{c}_i$ be the portion of $\vec{c}$ that lies in the same $\mathcal{A}_i$ block as $\vec{a}_i$. Note that we could have $\vec{c}_i = \emptyset$. Since $(\mathcal{B}_i, \vec{b}_i) \leq_\beta (\mathcal{A}_i, \vec{a}_i)$ by assumption, there exists a tuple $\vec{d}_i \in \mathcal{B}_i$ such that $(\mathcal{A}_i, \vec{a}_i, \vec{c}_i) \leq_\gamma (\mathcal{B}_i, \vec{b}_i, \vec{d}_i)$.

Any part of $\vec{c}$ that does not lie in the same $\mathcal{A}_i$ block as one of the $\vec{a}_i$'s we deal with separately. Let $\vec{e}_j$ be the portion of $\vec{c}$ that lies together in some "new" $\mathcal{A}_j$ block in $\mathcal{A}$. Choose a copy of $\mathcal{B}_j$ in $\mathcal{B}$ such that if $\vec{a}_i < \vec{e}_j < \vec{a}_{i+1}$ then $\vec{b}_i < \mathcal{B}_j < \vec{b}_{i+1}$, or such that $\mathcal{B}_j$ is to the left (or right) of all $\vec{b}_i$ if e_j is to the left (or right) of all $\vec{a}_i$. Such a copy exists since $\mathcal{B}$ is a shuffle sum. Since $\mathcal{B}_j \leq_\alpha \mathcal{A}_j$, there is some tuple $\vec{f}_j \in \mathcal{B}_j$ such that $(\mathcal{A}_j, \vec{e}_j) \leq_\gamma (\mathcal{B}_j, \vec{f}_j)$.

Now we have chosen $\vec{d}_1 < \vec{d}_2 < \ldots < \vec{d}_k$ and $\vec{f}_1 < \vec{f}_2 < \ldots < \vec{f}_l$ corresponding to $\vec{c}_1 < \vec{c}_2 < \ldots < \vec{c}_k$ and $\vec{e}_1 < \vec{e}_2 < \ldots < \vec{e}_l$ where l is some natural number and some of the $\vec{c}_i$ (and corresponding $\vec{d}_i$) may be the empty tuple. Arrange the tuples $\{(\vec{a}_i \cup \vec{c}_i), \vec{e}_j\}_{1\leq i\leq k, 1\leq j\leq l}$ in $\mathcal{A}$ and $\{(\vec{b}_i \cup \vec{d}_i), \vec{f}_j\}_{1\leq i\leq k, 1\leq j\leq l}$ in $\mathcal{B}$ so that they satisfy property (i). (Note: We already have properties (ii) and (iii) by construction.) Recall that we have

$$(\mathcal{A}_i, \vec{a}_i, \vec{c}_i) \leq_\gamma \left(\mathcal{B}_i, \vec{b}_i, \vec{d}_i\right) \text{ for } 1 \leq i \leq k$$

and

$$(\mathcal{A}_j, \vec{e}_j) \leq_\gamma \left(\mathcal{B}_j, \vec{f}_j\right) \text{ for } 1 \leq j \leq l$$

which is property (iv). Let the tuple $\vec{d}$ include all $\vec{d}_i$'s and $\vec{f}_j$'s ordered correctly relative to the corresponding $\vec{c}_i$'s and $\vec{e}_j$'s in $\mathcal{A}$. By the induction hypothesis, we have $(\mathcal{A}, \vec{a}, \vec{c}) \leq_\gamma (\mathcal{B}, \vec{b}, \vec{d})$. This proves that $(\mathcal{B}, \vec{b}) \leq_\beta (\mathcal{A}, \vec{a})$ as desired. □

Corollary 3.5. *Let $\mathcal{A}$ be the shuffle sum of linear orderings $\mathcal{A}_i$ for $i \in \omega$ and let $\mathcal{B}$ be the shuffle sum of linear orderings $\mathcal{B}_i$ for $i \in \omega$. For any α, if $\mathcal{B}_i \equiv_\alpha \mathcal{A}_i$ for all $i \in \omega$, then $\mathcal{B} \equiv_\alpha \mathcal{A}$.*

Proof. We show $\mathcal{B} \leq_\alpha \mathcal{A}$. Fix $\vec{a} \in \mathcal{A}$ and $\beta < \alpha$. Decompose $\vec{a}$ as $\vec{a}_1 < \vec{a}_2 < \cdots < \vec{a}_k$ where each tuple $\vec{a}_i$ lies in a distinct $\mathcal{A}_i$ block in $\mathcal{A}$. For $i = 1$, pick a copy of $\mathcal{B}_1$ in $\mathcal{B}$ (again, we abuse notation). Then there exists a tuple $\vec{b}_1 \in \mathcal{B}_1$ such that $(\mathcal{A}_1, \vec{a}_1) \leq_\beta (\mathcal{B}_1, \vec{b}_1)$.

For $i = 2$, pick a copy of $\mathcal{B}_2$ in $\mathcal{B}$ such that $\vec{b}_1 < \mathcal{B}_2$. Then, similarly, there is some tuple $\vec{b}_2$ such that $(\mathcal{A}_2, \vec{a}_2) \leq_\beta (\mathcal{B}_2, \vec{b}_2)$. We continue in this way to find $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$ such that $\vec{b} = \vec{b}_1 \cup \vec{b}_2 \cup \ldots \cup \vec{b}_k$ and $\vec{a}$ satisfy properties (i)-(iv) from Lemma 3.4. Thus $(\mathcal{A}, \vec{a}) \leq_\beta (\mathcal{B}, \vec{b})$ and hence $\mathcal{B} \leq_\alpha \mathcal{A}$. □

Theorem 3.6. *The back-and-forth ordinal of $\mathbb{K}_\alpha$ is α.*

Proof. By Corollary 1.14 (if α finite) or Corollary 1.16 (if α infinite), the back-and-forth ordinal of each theory is at most α. To show that the back and forth ordinal is exactly α, we define a countable structure $\hat{\mathcal{L}}_\alpha$ and show that for any $(\mathcal{L}_\alpha(S), \vec{b})$, there exists $\vec{a} \in \hat{\mathcal{L}}_\alpha$ such that $(\mathcal{L}_\alpha(S), \vec{b}) \equiv_{\alpha-1} (\hat{\mathcal{L}}_\alpha, \vec{a})$. Let $\mathcal{A}_n = \eta+1+\mu+n+1+\eta$, and let $\mathcal{B}_n = \eta+1+\lambda+n+1+\eta$. Since $\mu \leq_\alpha \lambda$, we have $\mathcal{A}_n \leq_\alpha \mathcal{B}_n$, so in particular $\mathcal{A}_n \equiv_{\alpha-1} \mathcal{B}_n$ (by Theorem 1.2). Note that $\mathcal{L}_\alpha(S)$ is the shuffle sum of orderings $\mathcal{A}_n$ for all $n \in \omega$ and $\mathcal{B}_n$ for all $n \in S \oplus \overline{S}$. Let $\hat{\mathcal{L}}_\alpha$ be the shuffle sum of the orderings $\mathcal{A}_n$ and $\mathcal{B}_n$ for all $n \in \omega$. Fix $\vec{b} \in \mathcal{L}_\alpha(S)$. Write $\vec{b} = \vec{b_1} < \ldots < \vec{b_k}$ where each $\vec{b_i} \in \mathcal{C}_{n_i}$ for some $\mathcal{C}_{n_i} = \mathcal{A}_{n_i}$ or $\mathcal{C}_{n_i} = \mathcal{B}_{n_i}$. Choose $\vec{a} \in \hat{\mathcal{L}}_\alpha$ such that $\vec{a_1} < \ldots < \vec{a_k}$, $\vec{a_i} \in \mathcal{C}_{n_i}$ and $(\mathcal{C}_{n_i}, \vec{a_i}) \cong (\mathcal{C}_{n_i}, \vec{b_i})$. The result now follows from Lemma 3.4. □

For computable successor ordinals $3 \leq \alpha \leq \beta$, let

$$\mathbb{K}_{\alpha,\beta} = \{\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y) \mid Y \not\leq_T X^{(\beta)} \text{ or } (X \leq_T \emptyset^{(\alpha)} \text{ and } Y \leq_T \emptyset^{(\beta)})\}.$$

We now compute the Turing ordinals and back-and-forth ordinals for $\mathbb{K}_{\alpha,\beta}$.

Lemma 3.7. *$Spec(\mathcal{L}_\alpha(X)+1+\mathcal{L}_\beta(Y)) = \{\deg(D) : X \leq_T D^{(\alpha)}$ and $Y \leq_T D^{(\beta)}\}$. (For finite ordinals, subtract 1.)*

Proof. Without much loss of generality, assume α and β are infinite. Recall that $\text{Spec}(\mathcal{L}_\alpha(S)) = \{deg(D) \mid S \leq_T D^{(\alpha)}\}$. Let $\mathcal{A} = \mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y)$ and suppose we have $\mathcal{B} \cong \mathcal{A}$ with $\mathcal{B} \leq_T D$. Given the element separating the two orderings, we can use $\mathcal{B}$ to build D-computable copies of $\mathcal{L}_\alpha(X)$ and $\mathcal{L}_\beta(Y)$. From our knowledge of their degree spectra we conclude that $X \leq_T D^{(\alpha)}$ and $Y \leq_T D^{(\beta)}$.

Now suppose that $X \leq_T D^{(\alpha)}$ and $Y \leq_T D^{(\beta)}$. Since $X \leq_T D^{(\alpha)}$, there a D-computable copy $\mathcal{B}_1$ of $\mathcal{L}_\alpha(X)$ and since $Y \leq_T D^{(\beta)}$, there is a D-computable copy $\mathcal{B}_2$ of $\mathcal{L}_\beta(Y)$. Then (with slight abuse of notation) $\mathcal{B}_1 + 1 + \mathcal{B}_2$ is a D-computable copy of $\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y)$. □

Given the above degree spectrum, we prove the following:

Theorem 3.8. *Fix $X, Y, B \subseteq \omega$, computable successor ordinals $3 \leq \alpha \leq \beta$, and let*

$$\mathcal{C} := \{D : X \leq_T D^{(\alpha)} \text{ and } Y \leq_T D^{(\beta)}\}.$$

If $B \leq_T D^{(\beta-1)}$ for all $D \in \mathcal{C}$ then $B \leq_T X^{(\beta-1)}$. Hence if $Y \not\leq_T X^{(\beta)}$, then $\{D^{(\beta-1)} : D \in \mathcal{C}\}$ has no element of least degree.

We will prove Theorem 3.8 using a generalization of the following claim of Ash, Jockusch and Knight from [AJK90].

Proposition 3.9. *Given $Y \subseteq \omega$ and a computable ordinal α, if $B \not\leq_T \emptyset^{(\alpha)}$ then there is a set A such that*

(i) $Y \leq_T A \oplus \emptyset^{(\alpha+1)}$, and
(ii) $B \not\leq_T A \oplus \emptyset^{(\alpha)}$.

By relativizing this result (easily) we get the following:

Corollary 3.10. *Given any sets $X, Y \subseteq \omega$ and any computable ordinal α, if $B \not\leq_T X^{(\alpha)}$ then there is a set A such that*

(i) $Y \leq_T A \oplus X^{(\alpha+1)}$, and
(ii) $B \not\leq_T A \oplus X^{(\alpha)}$.

Recall that we have strong α-jump inversion:

Lemma 3.11 (Macintyre [Mac77])**.** *For any computable ordinal α and any set X such that $X \geq_T 0^{(\alpha)}$, there exists an α-generic set S such that $S \oplus 0^{(\alpha)} \equiv_T S^{(\alpha)} \equiv_T X$.*

By relativizing Theorem 3.11 we get the following.

Corollary 3.12. *For any computable ordinal α, and any sets A, W such that $A \geq_T W^{(\alpha)}$, there exists a set $S \geq_T W$ such that $S \oplus W^{(\alpha)} \equiv_T S^{(\alpha)} \equiv_T A$.*

For our purposes, we need the following consequence of the previous corollary.

Corollary 3.13. *For any sets $A, X \subseteq \omega$ and any computable ordinal α, there is a set D such that $(D \oplus X)^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$.*

Proof. As $A \oplus X^{(\alpha)} \geq_T X^{(\alpha)}$, there is a set $D \geq_T X$ such that $D^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$, by relativized jump inversion. As $D \geq_T X$, we have $D^{(\alpha)} \equiv_T (D \oplus X)^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$. □

With these results in hand, we can prove the main lemma needed for Theorem 3.8.

Lemma 3.14. *Given $X, Y \subseteq \omega$ and any computable ordinal α, if $B \not\leq_T X^{(\alpha)}$ then there is a set D such that*
(i) $Y \leq_T (D \oplus X)^{(\alpha+1)}$, and
(ii) $B \not\leq_T (D \oplus X)^{(\alpha)}$.

Proof. Given X, Y and α, let A be as in Corollary 3.10. Given A, X and α, let D be a set such that $(D \oplus X)^{(\alpha)} \equiv_T A \oplus X^{(\alpha)}$, guaranteed by Corollary 3.13. Then we have

$$\begin{aligned} Y &\leq_T A \oplus X^{(\alpha+1)} \\ &\leq_T (A \oplus X^{(\alpha)})' \\ &\equiv_T \left((D \oplus X)^{(\alpha)}\right)' \\ &\equiv_T (D \oplus X)^{(\alpha+1)} \end{aligned}$$

and so (i) is satisfied. As $B \not\leq_T A \oplus X^{(\alpha)} \equiv_T (D \oplus X)^{(\alpha)}$ we also have (ii). □

Finally we can prove Theorem 3.8:

Proof. (of Theorem 3.8) Consider the following two sets:

$$\mathcal{C} := \{D : X \leq_T D^{(\alpha)} \text{ and } Y \leq_T D^{(\beta)}\}$$

and

$$\mathcal{C}^* := \{D : Y \leq_T (D \oplus X)^{(\beta)}\}.$$

Suppose that $B \leq_T D^{(\beta-1)}$ for all $D \in \mathcal{C}$. We claim that $B \leq_T (D \oplus X)^{(\beta-1)}$ for all $D \in \mathcal{C}^*$. For any $D \in \mathcal{C}^*$ we have $Y \leq_T (D \oplus X)^{(\beta)}$ by definition. Clearly, $X \leq_T (D \oplus X)^{(\beta)}$ and hence $D \oplus X \in \mathcal{C}$. So, by assumption, we have $B \leq_T (D \oplus X)^{(\beta-1)}$.

Now we wish to prove that $B \leq_T X^{(\beta-1)}$. Assume for a contradiction that $B \not\leq_T X^{(\beta-1)}$. Then by Lemma 3.14, there is a set D satisfying $Y \leq_T (D \oplus X)^{(\beta-1+1)} = (D \oplus X)^{(\beta)}$ and $B \not\leq_T (D \oplus X)^{(\beta-1)}$. In other words, we have $D \in \mathcal{C}^*$ with $B \not\leq_T (D \oplus X)^{(\beta-1)}$ which is a contradiction. Therefore we must have $B \leq_T X^{(\beta-1)}$ as desired.

We will prove the contrapositive of the *hence* statement. Suppose that the set $\{D^{(\beta-1)} : D \in \mathcal{C}\}$ has an element of least degree, say $D_0^{(\beta-1)}$. Then we have $X \leq_T D_0^{(\alpha)}$ and $Y \leq_T D_0^{(\beta)}$ and, for all $D \in \mathcal{C}$ we have $D_0^{(\beta-1)} \leq_T D^{(\beta-1)}$. It follows from the statement of the theorem that $D_0^{(\beta-1)} \leq_T X^{(\beta-1)}$ and hence $D_0^{(\beta)} \leq_T X^{(\beta)}$. Then, by the former statement, $Y \leq_T D_0^{(\beta)} \leq_T X^{(\beta)}$. □

With this result in hand we are ready to prove the main result:

Theorem 3.15. *For any computable successor ordinals satisfying $3 \leq \alpha \leq \beta$, the Turing ordinal of $\mathbb{K}_{\alpha,\beta}$ is β ($\beta - 1$ if finite) and the back-and-forth ordinal of $\mathbb{K}_{\alpha,\beta}$ is α.*

Proof. This result really has three parts so we will present each separately.

(1) For any $\boldsymbol{d} \geq \boldsymbol{0}^{(\beta)}$, there are sets $X, Y \subseteq$ such that $\mathcal{L}_\alpha(X)+1+\mathcal{L}_\beta(Y) \in \mathbb{K}_{\alpha,\beta}$ and $\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y)$ has β^{th} jump degree $\boldsymbol{d}$:

Fix $\boldsymbol{d} \geq \boldsymbol{0}^{(\beta)}$. We will choose our sets X and Y as follows: Let $X = \emptyset$ and by jump inversion, choose Y to be any set such that $Y \oplus \emptyset^{(\beta)} \equiv_T Y^{(\beta)} \in \boldsymbol{d}$. Then $\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y) \in \mathbb{K}_{\alpha,\beta}$, and we have $X \leq_T Y^{(\alpha)}$

and $Y \leq_T Y^{(\beta)}$ and hence $\deg(Y) \in \mathrm{Spec}(\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y))$. So we have $\boldsymbol{d} = \deg(Y)^{(\beta)}$ in the β^{th} jump spectrum of $\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y)$. Now suppose that $\deg(D) \in \mathrm{Spec}(\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y))$. By the spectrum result, we must have $Y \leq_T D^{(\beta)}$, and since $\emptyset^{(\beta)} \leq_T D^{(\beta)}$ as well, we have $Y^{(\beta)} \equiv_T Y \oplus \emptyset^{(\beta)} \leq_T D^{(\beta)}$. So $\boldsymbol{d} = \deg(Y)^{(\beta)}$ is a lower bound for the β^{th} jump spectrum of $\mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y)$.

(2) No $\mathcal{A} \in \mathbb{K}_{\alpha,\beta}$ can have a γ^{th} jump degree other than $\mathbf{0^{(\gamma)}}$ for any $\gamma < \beta$:

Fix $\mathcal{A} \in \mathbb{K}_{\alpha,\beta}$, and $\gamma < \beta$. If $\mathcal{A} \cong \mathcal{L}_\alpha(X) + 1 + \mathcal{L}_\beta(Y)$ for some X, Y satisfying $Y \not\leq_T X^{(\beta)}$, then it follows from Theorem 3.8 that the set $\{D^{(\beta-1)} : D \in \mathrm{Spec}(\mathcal{A})\}$ cannot have a least degree, and hence the structure $\mathcal{A}$ cannot have a γ^{th} jump degree. If $X \leq_T \emptyset^{(\alpha)}$ and $Y \leq_T \emptyset^{(\beta)}$, then $\mathrm{Spec}(\mathcal{L}_\alpha(X)+1+\mathcal{L}_\beta(Y))$ is all degrees, and so $\mathcal{A}$ has γ^{th} jump degree $\mathbf{0^{(\gamma)}}$.

(3) The back-and-forth ordinal of $\mathbb{K}_{\alpha,\beta}$ is α:

Let $\mathcal{A} = \mathcal{L}_\alpha(X)+1+\mathcal{L}_\beta(Y) \in \mathbb{K}_{\alpha,\beta}$, and let $\mathcal{B} = \mathcal{L}_\alpha(\tilde{X})+1+\mathcal{L}_\beta(\tilde{Y}) \in \mathbb{K}_{\alpha,\beta}$. For a tuple $\vec{a} \in \mathcal{A}$, let $\vec{a}_X$ denote the portion of $\vec{a}$ in $\mathcal{L}_\alpha(X)$, $\vec{a}_Y$ the portion of $\vec{a}$ in $\mathcal{L}_\beta(Y)$, and let $\vec{a}_1 = 1$ if the 1 not belonging to $\mathcal{L}_\alpha(X)$ or $\mathcal{L}_\beta(Y)$ belongs to $\vec{a}$, with $\vec{a}_1 = 0$ otherwise. Make the corresponding definitions for $\vec{b} \in \mathcal{B}$. It is then easy to see that for any $\vec{a} \in \mathcal{A}$, $\vec{b} \in \mathcal{B}$ and any computable ordinal γ, we have $(\mathcal{A}, \vec{a}) \equiv_\gamma (\mathcal{B}, \vec{b})$ iff $(\mathcal{L}_\alpha(X), \vec{a}_X) \equiv_\gamma (\mathcal{L}_\alpha(\tilde{X}), \vec{b}_{\tilde{X}}), (\mathcal{L}_\beta(Y), \vec{a}_Y) \equiv_\gamma (\mathcal{L}_\beta(\tilde{Y}), \vec{b}_{\tilde{Y}})$, and $\vec{a}_1 = \vec{b}_1$.

Since there are no restrictions on the set X allowable for an order $\mathcal{L}_\alpha(X)+1+\mathcal{L}_\beta(Y)$ to belong to $\mathbb{K}_{\alpha,\beta}$, it follows that the back-and-forth ordinal of $\mathbb{K}_{\alpha,\beta}$ is at most α from the fact that $\mathbb{K}_\alpha$ has back-and-forth ordinal α. Since $\beta \geq \alpha$, it follows from the fact that $\mathbb{K}_\alpha$ and $\mathbb{K}_\beta$ have back-and-forth ordinals α and β respectively that the back-and-forth ordinal of $\mathbb{K}_{\alpha,\beta}$ is at least α.

□

3.1. **Limit ordinals.** For limit ordinals α, we can define a class of linear orderings $\mathbb{K}_\alpha$ with Turing ordinal α. Unfortunately, the back-and-forth ordinal will be low.

Definition 3.16. Let α be a countable limit ordinal. A *fundamental sequence of* α is an ω-sequence which converges to α.

Let α be a computable limit ordinal and $(\alpha_n)_{n\in\omega}$ a fundamental sequence for α consisting of only successor ordinals. Then, for any $S \subseteq \omega$, let

$$\mathcal{L}_\alpha(S) := \sum_{n\in\omega} \Big(1 + \zeta + 1 + \mathcal{L}_{\alpha_n}(S_n)\Big)$$

where $S_n = \{k : \langle n, k\rangle \in S\}$ and α_n is the n^{th} member of the fundamental sequence.

Using the degree spectra results from the previous sections, we have the following.

Lemma 3.17. *Let α be a computable limit ordinal with computable fundamental sequence $(\alpha_n)_{n\in\omega}$ consisting of successor ordinals greater than 5. Then for any set $S \subseteq \omega$, we have:*

$$\mathit{Spec}\big(\mathcal{L}_\alpha(S)\big) = \big\{ \deg(D) : S_n \leq_T D^{(\alpha_n)} \textit{ uniformly in } n\big\}.$$

Proof. Suppose D is such that $S_n \leq_T D^{(\alpha_n)}$ uniformly in n. Then, uniformly in n, there is a D-computable copy of $\mathcal{L}_{\alpha_n}(S_n)$, so there is a D-computable copy of $\mathcal{L}_\alpha(S)$. Conversely, suppose there exists a D-computable copy of $\mathcal{L}_\alpha(S)$. We note that the only copies of ζ in $\mathcal{L}_\alpha(S)$ are those that separate the $\mathcal{L}_{\alpha_n}(S_n)$. (Note: the discrete orderings used from [AK90] are built using copies of ω^β for various β, and no copies of ζ.) So for each n, $D^{(5)}$ can locate the n^{th} and $(n+1)^{\text{st}}$ copy of $1+\zeta+1$ in $\mathcal{L}_\alpha(S)$, and hence, endpoints between which there is a D-computable copy of $\mathcal{L}_{\alpha_n}(S_n)$. The result then follows from the uniformity in the proof of the result on the degree spectra of the individual $\mathcal{L}_{\alpha_n}(S_n)$. □

The following is a translation into our setting of a result (Theorem 4.6) from [AJK90]. It shows that the class consisting of the $\mathcal{L}_\alpha(S)$ orderings has Turing ordinal at most α, if it exists.

Theorem 3.18 (Ash, Jockusch and Knight)**.** *Let α be a computable limit ordinal. Then for every degree $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there exists a set S such that $\mathcal{L}_\alpha(S)$ has α^{th} jump degree $\mathbf{d}$.*

Another result from [AJK90] will show that a particular subcollection of structures of the form $\mathcal{L}_\alpha(S)$ forms a class with Turing ordinal α.

Theorem 3.19 (Lemma 1.4 from [AJK90])**.** *Let α be a computable limit ordinal and let $(\alpha_n)_{n\in\omega}$ be a fundamental sequence with limit α that is picked out by a notation for α. Let $S \subseteq \omega$. Define*

$$\mathcal{C} := \{D : S_n \leq_T D^{(\alpha_n)} \text{ uniformly in } n\}$$

and suppose that, for some $\beta < \alpha$, $B \leq_T D^{(\beta)}$ for all $D \in \mathcal{C}$. Then

$$\beta < \alpha_n \implies B \leq_T (S_0 \oplus \dots S_{n-1})^{(\beta)}.$$

Hence if $\beta < \alpha_n$ and $S_n \not\leq_T (S_0 \oplus \dots S_{n-1})^{(\alpha_n)}$ then the set $\{D^{(\beta)} : D \in \mathcal{C}\}$ has no element of least degree.

Theorem 3.20. *Let α be a computable limit ordinal. Then for any computable fundamental sequence $(\alpha_n)_{n\in\omega}$ of α consisting of successor ordinals greater than 5, the class*

$$\mathbb{K}_\alpha = \{\mathcal{L}_\alpha(S) : S_k \not\leq_T (S_0 \oplus S_1 \oplus \dots \oplus S_{k-1})^{(\alpha_k)} \text{ for all } k\}$$

has Turing ordinal α.

Proof. Fix $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. By the proof of Theorem 3.18, there is an α-generic set S such that $\mathcal{L}_\alpha(S)$ has α^{th} jump degree $\mathbf{d}$. As S is α-generic, we have $S_k \not\leq_T (S_0 \oplus S_1 \oplus \dots \oplus S_{k-1})^{(\alpha_k)}$ for all k and hence $\mathcal{L}_\alpha(S)$ is in the given class.

Now fix any $\mathcal{L}_\alpha(S)$ in the class, and fix $\beta < \alpha$. Then we must have $\beta < \alpha_n$ for some $n \in \omega$ and $S_n \not\leq_T (S_0 \oplus S_1 \oplus \dots \oplus S_{n-1})^{(\alpha_n)}$. By Lemma 3.17 and Theorem 3.19, $\mathcal{L}_\alpha(S)$ has no β^{th} jump degree. Therefore the class has Turing ordinal α as desired. □

4. Conclusion

The back-and-forth ordinal of the class $\mathbb{K}_\alpha$ for limit ordinals depends on the fundamental sequence chosen. It is easy to see (similar to the calculation of the back-and-forth ordinal of the $\mathbb{K}_{\alpha,\beta}$), that it will be α_0, the first ordinal in the fundamental sequence.

We do not have examples of structures with limit back-and-forth ordinal beyond ω, where the example is the class of boolean algebras.

Note finally that if we are willing to use graphs and disjoint unions of structures, rather than sums of linear orderings, then we can include the lower numbers of $0, 1, 2$ in our examples of classes $\mathbb{K}_{\alpha,\beta}$.

In this paper, we have not analyzed the complexities of the theories of $\mathbb{K}_\alpha$ or $\mathbb{K}_{\alpha,\beta}$. We hope the reader will agree that they are "clearly" computably axiomatizable, with the complexity increasing as a function of α for $\mathbb{K}_\alpha$, and increasing as function of β for $\mathbb{K}_{\alpha,\beta}$. That is, we are pushing up the Turing ordinals by making the theories of the classes correspondingly more complex. For axiomatizations of classes similar to the ones in this paper, see Knoll's thesis [Kno13]. We close with the following questions.

Question 4.1. Is there a finitely axiomatizable class of structures with Turing ordinal equal to $\alpha > \omega$?

Question 4.2. Is there a finitely axiomatizable class of structures with the Turing ordinal strictly larger than the back-and-forth ordinal?

Question 4.3. What is the least $n \in \omega$ such that there is a Π^c_n-axiomatizable class of structures with the Turing ordinal strictly larger than the back-and-forth ordinal? We currently have a Π^c_9-axiomatizable class with this property. See [Kno13].

Question 4.4. What conditions (if any) can one put on the complexity of the axiomatization of a class of structures in order to ensure that the Turing ordinal and the back-and-forth ordinal are close?

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