# Bounding Prime Models

Barbara F. Csima, Denis R. Hirschfeldt,
Julia F. Knight, and Robert I. Soare*

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## Contents



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**Abstract**

A set $X$ is *prime bounding* if for every complete atomic decidable (CAD) theory $T$ there is a prime model $\mathfrak{A}$ of $T$ decidable in $X$. It is easy to see that $X = 0'$ is prime bounding. Denisov claimed that every $X <_{\mathrm{T}} 0'$ is *not* prime bounding, but we discovered this to be incorrect. Here we give the correct characterization that the prime bounding sets $X \leq_{\mathrm{T}} 0'$ are exactly the sets which are not $\mathrm{low}_2$. Recall that $X$ is $\mathrm{low}_2$ if $X'' \leq_T 0''$. To prove that a $\mathrm{low}_2$ set $X$ is *not* prime bounding we use a $0'$-computable listing of the array of sets $\{Y : Y \leq_{\mathrm{T}} X\}$ to build a CAD theory $T$ which diagonalizes against all potential $X$-decidable prime models $\mathfrak{A}$ of $T$. To prove that any *non*$\mathrm{low}_2$ $X$ is indeed prime bounding, we fix a function $f \leq_{\mathrm{T}} X$ which dominates every total $0'$-computable function $g$. Given a CAD theory $T$, we use $f$ to eventually find, for every formula $\varphi(\overline{x})$ consistent with $T$, a principal type which contains it, and hence to build an $X$-decidable prime model of $T$. We prove the prime bounding property equivalent to several other combinatorial properties, including some related to the limitwise monotonic functions which have been introduced elsewhere in computable model theory.

# 1 Introduction

All of our languages are computable, and all of our structures are countable with universe $\omega$. Given a set $X \subseteq \omega$ a structure $\mathfrak{A}$ is $X$*-decidable* if its elementary diagram is computable in $X$ ($D^e(\mathfrak{A}) \leq_{\mathrm{T}} X$) and is $X$*-computable* if its atomic diagram is $X$-computable ($D(\mathfrak{A}) \leq_{\mathrm{T}} X$). We write *decidable* and *computable* for 0-decidable and 0-computable, respectively. A countable complete theory $T$ is *atomic* if, for each formula $\varphi(\overline{x})$ consistent with $T$, there is a principal type containing $\varphi(\overline{x})$, or equivalently if the isolated points of the Stone space $S_n(T)$ are dense. (See §4.1.) A model $\mathfrak{A}$ is *prime* if it can be elementarily embedded in every model of $\mathrm{Th}(\mathfrak{A})$. It is well known that a countable complete theory $T$ has a prime model iff $T$ is atomic. Our main result concerns theories $T$ which are complete, atomic, and decidable (CAD).

It is clear that any complete decidable theory $T$ has a decidable model. (The Henkin construction can be carried out effectively.) However, that model is not always prime. Goncharov and Nurtazin [1973] and independently Harrington [1974] gave an elegant criterion (Theorem 4.2 below) for a complete decidable theory $T$ to have a decidable prime model. Millar [1978] constructed a CAD theory $T$ which has no *computable* (much less decidable) prime model, and furthermore, such that the types of $T$ are all computable.

The *degree spectrum of* $\mathfrak{A}$, written $\mathrm{DgSp}(\mathfrak{A})$ (*elementary degree spectrum of* $\mathfrak{A}$, written $\mathrm{DgSp}^e(\mathfrak{A})$) is the set of degrees of atomic diagrams (elementary diagrams) of isomorphic copies $\mathfrak{B}$ of $\mathfrak{A}$, *i.e.*,

$$\mathrm{DgSp}(\mathfrak{A}) = \{\deg(D(\mathfrak{B})) : \mathfrak{B} \cong \mathfrak{A}\}, \text{ and}$$

$$\mathrm{DgSp}^e(\mathfrak{A}) = \{\deg(D^e(\mathfrak{B})) : \mathfrak{B} \cong \mathfrak{A}\}.$$

Knight [1986] showed that these two degree spectra are closed upwards for structures $\mathfrak{A}$ which are *automorphically nontrivial*, where $\mathfrak{A}$ is *automorphically trivial* if there is a finite set $F$ such that all permutations of the universe which fix $F$ pointwise are automorphisms of $\mathfrak{A}$. See Knight [1998] for more on degrees of models.

Our present results on degrees below which CAD theories always have prime models grew out of results on degree spectra of prime models such as the following.

**Theorem 1.1.** *[Drobotun (1978), Millar (1978), Denisov (1989)] Any CAD theory $T$ has a prime model $\mathfrak{A}$ with $D^e(\mathfrak{A}) \leq_{\mathrm{T}} 0'$.*

This result was improved by Csima [2002], [2003] as follows.

**Theorem 1.2.** *[Prime Low Basis Theorem, Csima (2002, 2003)] If $T$ is a CAD theory, then $T$ has a prime model $\mathfrak{A}$ with $D^e(\mathfrak{A})$ low.*

In addition, if the CAD theory $T$ has its types all computable, then the prime model degree spectrum $T$ includes all nonzero $\Delta^0_2$ degrees.

**Theorem 1.3.** *[Csima (2002, 2003)] If $T$ is a CAD theory with types all computable, then for every degree $\mathbf{d}$ with $\mathbf{0} < \mathbf{d} \leq \mathbf{0}'$, there is a prime model $\mathfrak{A}$ of $T$ with $D^e(\mathfrak{A})$ of degree $\mathbf{d}$.*

See Harizanov [2002] for further discussion of these and related results.

A natural question is whether the stronger conclusion of Theorem 1.3 can be proved with the weaker hypotheses of Theorem 1.2, *i.e.*, without assuming the types are all computable. An effort to answer this question led us to the results presented in this paper.

Researchers had previously studied not only the degree spectrum of prime models for a *single* theory but also the degrees which bound prime models for *every* CAD theory $T$.

**Definition 1.4.** A set $X$ is *prime bounding* if every CAD theory has an $X$-decidable prime model $\mathfrak{A}$. For example, Theorem 1.1 asserts that $X = 0'$ is prime bounding. As we will see, it would not matter for our results if we replaced *$X$-decidable* by *$X$-computable* in this definition.

Denisov [1989] claimed that no $X <_{\mathrm{T}} 0'$ is prime bounding. He incorrectly assumed that for every $X <_{\mathrm{T}} 0'$ there is a $0'$-computable listing of the array of sets $\{Y : Y \leq_{\mathrm{T}} X\}$. This is what led us to focus on $\text{low}_2$ and $\text{nonlow}_2$ $\Delta^0_2$ sets. Our main theorem says that the $\Delta^0_2$ sets which are prime bounding are exactly those which are $\text{nonlow}_2$. (There is a further discussion of Denisov's paper in Section 11.)

**Theorem 1.5 (Main Theorem).** *Let $X$ be a $\Delta^0_2$ set (*i.e., $X \leq_{\mathrm{T}} 0'$*). Then $X$ is $\text{nonlow}_2$ (*i.e., $X'' >_{\mathrm{T}} 0''$*) iff $X$ is prime bounding.*

One consequence of this result is that the Prime Low Basis Theorem (Theorem 1.2 above) does not follow immediately from the Low Basis Theorem of Jockusch and Soare [1972]. Indeed, suppose that one could always build a $\Pi^0_1$ class of prime models of a complete decidable theory. By a result of Jockusch and Soare [1972], given any degree of a completion of Peano arithmetic and any $\Pi^0_1$ class, there is a path in the class computable in the degree. Since there are low degrees of completions of Peano arithmetic, it would follow that each theory has a prime model computable in *the same* low degree. So there would exist a low prime bounding set, contradicting Theorem 1.5.

In §3 we derive key consequences of sets being $\text{low}_2$ (Corollary 3.6), or $\text{nonlow}_2$ (Corollary 3.3) which will be used in proving the Main Theorem 1.5.

In addition to the two properties in the main theorem we consider a number of other properties equivalent to these two. For quick reference we shall list them in §2 before proceeding with the exposition and proofs.

Some of the properties we consider are related to limitwise monotonic functions. A function $f$ is *$X$-limitwise monotonic* if there is an $X$-computable binary function $g$ such that for every $x$, $g(x, y)$ is nondecreasing in $y$ and $f(x) = \lim_y g(x, y) \geq x$. A function is *limitwise monotonic* if it is 0-limitwise monotonic.

Limitwise monotonic functions were introduced into computable model theory by N. G. Khisamiev [1981], [1986], [1998] in characterizing in a mathematical way (in terms of the Ulm sequence) the reduced Abelian $p$-groups with computable copies, at least for length $< \omega^2$, using a family of limitwise computable functions of increasing complexity. (For groups of length $\omega$, there was a single computable function, for groups of length $\omega \times 2$, there were two functions, one computable and one $\Delta^0_3$, etc.) Coles, Downey, and Khoussainov [1998] used $0'$-limitwise monotonic functions. Khoussainov, Nies, and Shore [1997] and Nies [1999] used limitwise computable functions in connection with $\aleph_1$-categorical theories, and Hirschfeldt [2001] used them in connection with linear orderings.

We use standard notation, definitions, and results in computability theory from Soare [1987] and in model theory from Chang and Keisler [1990]. We also let 0 and $0'$ denote a computable set and the complete c.e. set $K = \{e : e \in W_e\}$, respectively. A set $X$ is $\Delta^0_2$ iff $X \leq_{\mathrm{T}} 0'$, and we use the two terms interchangeably.

# 2 The properties: (P0)–(P8), and (U0)

We consider the following properties, which will be shown to be equivalent for sets $X$ which are $\Delta^0_2$ (*i.e.*, $X \leq_{\mathrm{T}} 0'$). We group them by similarity and order of presentation. Every property is really a predicate of $X$. The reader is not expected to master or remember all these properties. Rather, we list them here as a handy index.

## 2.1 Definitions and notations for the properties

**Notation**. Every tree $\mathcal{T}$ will be a subtree of $2^{<\omega}$. Fix an effective numbering $\{\sigma_e : e \in \omega\}$ of all nodes of $2^{<\omega}$. We may identify a node $\sigma_e$ with its Gödel number $e$. We write $x \subseteq y$ ($x \subset f$) to denote that $\sigma_x \subseteq \sigma_y$ ($\sigma_x \subset f$). Let $[\mathcal{T}]$ denote the set of all (infinite) paths through $\mathcal{T}$, and $[\mathcal{T}_x]$ the set of paths $f \in [\mathcal{T}]$ such that $x \subset f$.

**Definition 2.1.** (i) For $f(e, y)$ a binary function, $f_e$ denotes $\lambda y\, [\, f(e, y)\, ]$.

(ii) If $f(x, y)$ is a binary function, then $\widehat{f}(x) = \lim_y f(x, y)$, if the limit exists.

(iii) For $\mathcal{C}$ a class of (unary) functions and $X$ a set, $\mathcal{C}$ is $X$-*uniform* if there is an $X$-computable binary function $f(e, y)$ with $\mathcal{C} = \{f_e : e \in \omega\}$.

### 2.1.1 Properties in the Main Theorem: (P1), (P2), and (P3)

The equivalence of the first two properties (P1) and (P2) for $X \leq_{\mathrm{T}} 0'$ constitutes the Main Theorem (Theorem 1.5).

(P1) *The nonlow*$_2$ *property.* $X$ is not low$_2$ (*i.e.*, $X'' >_{\mathrm{T}} 0''$).

(P2) *The prime bounding property.* $X$ is prime bounding. That is, every CAD theory has an $X$-decidable prime model.

(P3) *The isolated path property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes and with isolated paths dense,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x \in \mathcal{T})\,[\, g_x \in [\mathcal{T}_x] \;\;\&\;\; g_x \text{ is isolated}\,].$$

Property (P3) is the abstract tree equivalent of the prime bounding property (P2), and is often more convenient to use. It will be shown equivalent to (P2) in §4.4. The prototype for the equivalence of (P2) and (P3) for the case of $X$ computable is the well-known Theorem 4.2 of Goncharov and Nurtazin [1973] and independently Harrington [1974] on decidable prime models, which helped to launch the area of research of which this paper is a part, and which is described in §4.2.

#### 2.1.2 Helping properties: (P0) and (U0)

For the Main Theorem 1.5, the degree theoretic property we want is (P1), but (P0) and (U0) are often the key tools which we shall use when proving an implication about (P1).

(P0) *The escape property.* $(\forall g \leq_{\mathrm{T}} 0')\,(\exists f \leq_{\mathrm{T}} X)\,(\exists^{\infty} x)\,[\; g(x) \leq f(x)\;]$,

where "$(\exists^{\infty})$" denotes "there exist infinitely many".

(U0) *The* $0'$*-uniform property.* $(\exists g \leq_{\mathrm{T}} 0')\,[\;\{\, Y : Y \leq_{\mathrm{T}} X \,\} = \{\, g_e \,\}_{e\in\omega}\;]$,

where $g_e = \lambda y\,[\,g(e,y)\,]$, and is viewed as the $e^{th}$ row of $g(e,y)$. We informally refer to (P0) as the *escape property* because it asserts that for every $g \leq_{\mathrm{T}} 0'$ there exists $f \leq_{\mathrm{T}} X$ which "escapes" domination by $g$ at least infinitely often. The $0'$-uniform property (U0) asserts that the $X$-computable sets are $0'$-uniform.

In §3 we shall use well-known theorems of Martin [1966] and Jockusch [1972] to recall the proofs that

$$(\forall X \leq_{\mathrm{T}} 0')\,[\;\;(\mathrm{P1}) \quad\Longleftrightarrow\quad (\mathrm{P0}) \quad\Longleftrightarrow\quad \neg(\mathrm{U0})\;].$$

Then to prove an implication of the form (P1) $\Longrightarrow$ (Q) we shall prove (P0) $\Longrightarrow$ (Q). Similarly, to prove that (Q) $\Longrightarrow$ (P1), or equivalently $\neg$(P1) $\Longrightarrow$ $\neg$(Q), we shall make use of (U0), although we will usually also need a bit more. Thus, all implications to or from (P1) will use the hypothesis that $X \leq_{\mathrm{T}} 0'$. In §10 we discuss the case where $X \not\leq_{\mathrm{T}} 0'$.

### 2.1.3 Tree and omitting types properties: (P4) and (P5)

(P4) *The tree property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes, and for every uniformly $\Delta_2^0$ sequence of subsets $\{S_i\}_{i\in\omega}$ all dense in $\mathcal{T}$, there exists an $X$-computable function $g(x, y)$ such that for every $x \in \mathcal{T}$, $g_x = \lambda y\,[g(x, y)]$ is a path extending $x$ and entering each $S_i$, that is,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x \in \mathcal{T})\,(\forall i)(\exists z \in S_i)[\,x \subset g_x \quad \& \quad z \subset g_x \quad \& \quad g_x \in [\mathcal{T}]\,].$$

It would make no difference if we required that $g_x$ enter each $S_i$ above $x$, that is, that

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x \in \mathcal{T})\,(\forall i)(\exists z \in S_i)[\,x \subseteq z \subset g_x \quad \& \quad g_x \in [\mathcal{T}]\,],$$

since we could replace the sequence $\{S_i\}_{i\in\omega}$ by the sequence $\{\widehat{S}_{i,x}\}_{i\in\omega, x\in\mathcal{T}}$, where $\widehat{S}_{i,x} = \{z \in S_i : z \not\subset x\}$.

Property (P4) has a topological interpretation in the Cantor Space $2^\omega$, where the basic open sets for $x \in 2^{<\omega}$ are given by

$$U_x = \{f : f \in \ 2^\omega \quad \& \ \ x \ \subset f\}.$$

**Definition 2.2.** The basic open (indeed clopen) set defined by $S \subseteq 2^{<\omega}$ is

$$U_S = \bigcup \{\ U_x : x \in S\ \}.$$

Property (P4) asserts, for $\mathcal{T}$ as above, that if $\{S_i\}_{i\in\omega}$ is a uniformly $\Delta_2^0$ sequence of $\Delta_2^0$ sets, then

$$(\exists g \leq_{\mathrm{T}} X)(\forall x \in \mathcal{T})\,(\forall i)\,[\ g_x \in \quad U_{S_i} \cap\ [\mathcal{T}_x]\ \ ].$$

Hence, (P4) says that for every $x \in \mathcal{T}$, the path $g_x \in [\mathcal{T}]$ extends $x$ and lies in every dense open set $U_{S_i}$. This says for the $\Delta_2^0$ family $\mathcal{G} = \{S_i\}_{i\in\omega}$ that $X$ can compute a $\mathcal{G}$-generic path $g$. A special case is that $X$ computes a 1-generic set. For a more detailed discussion of the Cantor space, the clopen basis $U_\sigma$, and the Stone space $S_n(T)$ of a theory see §4.1.

**Definition 2.3.** Fix a language $L$. Let $\Gamma = \Gamma(\overline{x}) \subset F_n(L)$ be a set of $L$-formulas with free variables $\overline{x}$. Let $\mathfrak{A}$ be an $L$-structure.

(i) We say that a tuple $\overline{a}$ in $\mathfrak{A}$ *realizes* $\Gamma$ if $\mathfrak{A} \models \gamma(\overline{a})$ for all $\gamma \in \Gamma$.

(ii) We say that $\mathfrak{A}$ *omits* $\Gamma$ if $\Gamma$ is not realized by any tuple in $\mathfrak{A}$.

(iii) Let $T$ be a consistent $L$-theory, not necessarily complete. We say that $\Gamma(\overline{x})$ is *principal* with respect to $T$ if there is a *generating formula* $\varphi(\overline{x})$, consistent with $T$, such that

$$(\forall \gamma \in \Gamma)\,[\,T \;\vdash\; (\forall \overline{x})\,[\,\varphi(\overline{x}) \;\longrightarrow\; \gamma(\overline{x})\,]\,].$$

(iv) We say that $\Gamma$ is *nonprincipal* with respect to $T$ if there is no such generating formula.

(P5) *The omitting types property.* For any complete decidable theory $T$ and any uniformly $\Delta^0_2$ family of sets of formulas $\{\Gamma_j(\overline{x}_j)\}_{j\in\omega}$, all nonprincipal with respect to $T$, there is an $X$-decidable model of $T$ omitting all $\Gamma_j(\overline{x}_j)$.

#### 2.1.4 Algebraic properties: (P6), (P7), and (P8)

**Definition 2.4.** A set $S \subset \omega$ is $X$*-monotonic* if there is a function $g \leq_T X$ such that for every $x$, $g(x, y)$ is nondecreasing in $y$, with limit $\widehat{g}(x) = \lim_y g(x, y)$, $\widehat{g}(x) \geq x$, and $\widehat{g}(x) \in S$. A set $S$ is $X$*-nonmonotonic* if there is no such $g$.

The following property (P6) asserts that every infinite $\Delta^0_2$ set $S$ is $X$-monotonic.

(P6) *The monotonic property.* For any infinite $\Delta^0_2$ set $S$,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x)\,(\forall y)\,[\,x \leq g_x(y) \leq g_x(y+1) \;\;\&\;\; \lim_y g(x,y)\!\downarrow\in S\,].$$

An *equivalence structure* is a structure of the form $\mathcal{A} = (A, E)$, where $E$ is an equivalence relation on $A$.

(P7) *The equivalence structure property.* For any $\Delta^0_2$ set $S \subseteq \omega - \{0\}$, there is an $X$-computable equivalence structure with one class of size $n$ for each $n \in S$, and no other classes.

A reduced Abelian $p$-group $\mathcal{G}$ is determined, up to isomorphism, by its Ulm sequence $\{u_\alpha(\mathcal{G})\}_{\alpha<\lambda(\mathcal{G})}$, where $\lambda(\mathcal{G})$ is the length of $\mathcal{G}$. (See Ash and Knight [2000, p. 317].) Here we restrict our attention to reduced Abelian $p$-groups $\mathcal{G}$ of length $\omega$ such that for all $n \in \omega$, $u_n(\mathcal{G}) \leq 1$. Define $S(\mathcal{G}) = \{n : u_n(\mathcal{G}) \neq 0\}$.

(P8) *The Abelian p-group property.* For any infinite $\Delta^0_2$ set $S \subseteq \omega - \{0\}$, there is an $X$-computable reduced Abelian $p$-group $\mathcal{G}$, of length $\omega$, and with $u_n(\mathcal{G}) \leq 1$ for all $n$, such that $S(\mathcal{G}) = S$.

### 2.2 The implications and equivalences among the properties

In §10, we consider the case $X \not\leq_{\mathrm{T}} 0'$. We shall use throughout the paper the notation (P) $\longrightarrow$ (Q) for an implication which is proved *using* the hypothesis $X \leq_{\mathrm{T}} 0'$, and (P) $\Longrightarrow$ (Q) for an implication whose proof does *not* assume $X \leq_{\mathrm{T}} 0'$. The implications denoted by $\longrightarrow$ are exactly those to or from (P1), because these proofs all use the characterizations of $\mathrm{low}_2$ and $\mathrm{nonlow}_2$ in §3, which are proved under the assumption that $X \leq_{\mathrm{T}} 0'$. We shall prove the equivalences and implications as written in the lines numbered (1), (2), and (3) below.

(1) $\neg$(U0) $\longleftrightarrow$ (P1) $\longleftrightarrow$ (P0) $\longleftarrow$ $\Longrightarrow$ (P3) $\Longleftrightarrow$ (P2).

(2) (P4) $\Longrightarrow$ (P3), (P4) $\Longleftrightarrow$ (P5), and (P6) $\Longleftrightarrow$ (P7) $\Longleftrightarrow$ (P8).

To connect line (1) with line (2) we prove

(3) (P0) $\Longrightarrow$ (P4) and (P0) $\Longrightarrow$ (P6) $\longrightarrow$ (P1).

These three lines (1), (2), and (3) establish the equivalence of all the properties (P1)–(P8) for $\Delta^0_2$ sets.

### 2.3 The plan of the paper

In §3, we develop the computability theoretic properties (P0) and $\neg$(U0) which are equivalent to (P1) if $X \leq_{\mathrm{T}} 0'$. In §4 we review some basic model theory about the Lindenbaum algebra, the Stone Space, the tree of $n$-types of a theory, and connections with Henkin models. We also prove the equivalence of the prime bounding property (P2) with its abstract tree equivalent (P3) for isolated paths, and we use (P3) thereafter. In §5, we use all these results to prove (P1) $\longleftrightarrow$ (P3), thereby establishing the Main Theorem 1.5 that (P1) $\longleftrightarrow$ (P2). In §6 and §7, we prove the implications in line (2). In §8, we prove the first two implications in line (3), that is, (P0) $\Longrightarrow$ (P4) and (P0) $\Longrightarrow$ (P6). In §9, we finish connecting the last part of line (2) to line (1) by proving (P6) $\longrightarrow$ (P1). This establishes the equivalence of all the properties (P0)–(P8) and $\neg$(U0) for $X \leq_{\mathrm{T}} 0'$, which we state in Theorem 9.2.

### 2.4 Summary of the properties

For the convenience and later reference of the reader we now summarize the properties.

(U0) *The* $0'$*-uniform property.* $(\exists g \leq_{\mathrm{T}} 0')\,[\ \{\ Y : Y \leq_{\mathrm{T}} X\ \} = \{\ g_e\ \}_{e\in\omega}\ ]$.

(P0) *The escape property.* $(\forall g \leq_{\mathrm{T}} 0')\,(\exists f \leq_{\mathrm{T}} X)\,(\exists^\infty x)\,[\ g(x) \leq f(x)\ ]$.

(P1) *The nonlow*$_2$ *property.* The set $X$ is not low$_2$ (*i.e.*, $X'' >_{\mathrm{T}} 0''$).

(P2) *The prime bounding property.* The set $X$ is prime bounding. That is, every CAD theory has an $X$-decidable prime model.

(P3) *The isolated path property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes and with isolated paths dense,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x \in \mathcal{T})\,[\,g_x \in [\mathcal{T}_x]\ \ \&\ \ g_x \text{ is isolated}\,].$$

(P4) *The tree property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes, and for every uniformly $\Delta^0_2$ sequence of subsets $\{S_i\}_{i\in\omega}$ all dense in $\mathcal{T}$, there exists an $X$-computable function $g(x,y)$ such that for every $x \in \mathcal{T}$, $g_x = \lambda y\,[g(x,y)]$ is a path extending $x$ and entering all $S_i$, that is,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x \in \mathcal{T})\,(\forall i)(\exists z \in S_i)[\,x \subset g_x\ \ \&\ \ z \subset g_x\ \ \&\ \ g_x \in [\mathcal{T}]\ ].$$

(P5) *The omitting types property.* For any complete decidable theory $T$ and any uniformly $\Delta^0_2$ family of sets of formulas $\{\Gamma_j(\overline{x}_j)\}_{j\in\omega}$, all nonprincipal with respect to $T$, there is an $X$-decidable model of $T$ omitting all $\Gamma_j(\overline{x}_j)$.

(P6) *The monotonic property.* For any infinite $\Delta^0_2$ set $S$,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x)\,(\forall y)\,[\,x \leq g_x(y) \leq g_x(y+1)\ \ \&\ \ \lim_y g(x,y)\!\downarrow\in S\,].$$

(P7) *The equivalence structure property.* For any $\Delta^0_2$ set there is an $X$-computable equivalence structure with one class of size $n$ for each $n \in S$, and no other classes.

(P8) *The Abelian p-group property.* For any infinite $\Delta^0_2$ set $S \subseteq \omega - \{0\}$, there is an $X$-computable reduced Abelian $p$-group $\mathcal{G}$, of length $\omega$, and with $u_n(\mathcal{G}) \leq 1$ for all $n$, such that $S(\mathcal{G}) = S$.

# 3 Properties (P1), (P0), and (U0), for $X \leq_{\mathrm{T}} 0'$

In this section we derive some useful computability theoretic characterizations of low$_2$ and nonlow$_2$ sets $X \leq_{\mathrm{T}} 0'$. In §10 we discuss the fact that the hypothesis $X \leq_{\mathrm{T}} 0'$ is necessary because (P1) $\not\Longrightarrow$ (P0) and (P0) $\not\Longrightarrow$ (P1) without it.

### 3.1 (P1) $\longleftrightarrow$ (P0) for $X \leq_{\mathrm{T}} 0'$: Martin's theorem

We say that a function $f$ *dominates* a function $g$ if $f(x) > g(x)$ for all but finitely many $x \in \omega$. A function is $X$-*dominant* if it dominates every (total) $X$-computable function, and *dominant* if it is 0-dominant. In his investigation of the degrees of maximal (c.e.) sets Martin [1966] proved the following characterization, which is also presented in Soare [1987, p. 208].

**Theorem 3.1 (Martin).** *A set $A$ satisfies $0'' \leq_{\mathrm{T}} A'$ if and only if there is a dominant function $f \leq_{\mathrm{T}} A$.*

Martin's theorem can be relativized to any $X \leq_{\mathrm{T}} 0'$. If we do this and take $A = 0'$ we get:

**Corollary 3.2.** *If $X \leq_{\mathrm{T}} 0'$, then $X'' \leq_{\mathrm{T}} 0''$ (*i.e., *$X$ is low$_2$) if and only if there is an $X$-dominant function $f \leq_{\mathrm{T}} 0'$.*

Note that both directions of the $X$-relativization in Corollary 3.2, as proved in Soare [1987, pp. 208–209], use the assumption that $X \leq_{\mathrm{T}} 0'$. This is necessary, as we discuss in §10.

**Corollary 3.3.** *Assume $X \leq_{\mathrm{T}} 0'$. Then (P1) $\longleftrightarrow$ (P0),* i.e., *$X$ is nonlow$_2$ iff $X$ satisfies:*

(4) *The escape property.* $(\forall g \leq_{\mathrm{T}} 0')\,(\exists f \leq_{\mathrm{T}} X)\,(\exists^{\infty} x)\,[\,g(x) \leq f(x)\,]$.

*Proof.* By Corollary 3.2, $X$ is nonlow$_2$ iff there is no $0'$-computable $X$-dominant function. But this is precisely the escape property. ☐

The "escape property" (4) of Corollary 3.3 will allow us to produce a prime model $\mathfrak{A}$ with $D^e(\mathfrak{A}) \leq_{\mathrm{T}} X$. Note that Theorem 1.1 (the case $X = 0'$) is easily proved by observing that the set $A$ of atoms of the Lindenbaum algebra $B_n(T)$ (defined in §4.1), or equivalently the set of generators of the principal types, is $\Pi^0_1$, and hence is $0'$-computable. Thus, the standard effective omitting types theorem (relativized to $0'$) produces a $0'$-decidable model which omits all the nonprincipal types and therefore is prime. In Theorem 5.1, where we show the general case for $X$ nonlow$_2$, we can no longer $X$-compute the set $A$ of atoms. However, using $f$ in the escape property (4) we can approximate $A$ often enough to build an $X$-decidable atomic model.

### 3.2 (P1) $\longleftrightarrow$ ¬(U0) for $X \leq_T 0'$: Jockusch's theorem

The earlier Definition 2.1 and the following theorem are due to Jockusch [1972] and are presented in Soare [1987, pp. 254–255].

**Theorem 3.4 (Jockusch).** *For any set $A$ the following statements are equivalent:*

(i) $A' \geq_T 0''$ (i.e., *$A$ is high);*

(ii) *the computable functions are $A$-uniform;*

(iii) *the computable sets are $A$-uniform.*

**Theorem 3.5.** *¬(P1) $\longrightarrow$ (U0). If a set $X \leq_T 0'$ is low$_2$ then the $X$-computable functions (and hence also sets) are $0'$-uniform.*

*Proof.* Let $X \leq_T 0'$ be low$_2$. Then $0'' \geq_T X''$. That is, $0'$ is high over $X$. So to see that the $X$-computable functions (sets) are $0'$-uniform, it suffices to relativize Theorem 3.4 to $X$, replacing 0 by $X$ and $A$ by $0'$. By Corollary 3.2 we may choose a $0'$-computable function $g$ which dominates every total $X$-computable function $\varphi_e^X$. Since $X \leq_T 0'$ we can $0'$-computably define $f(\langle e, i\rangle, x) = \varphi^X_{e,i+g(x)}(x)$ if $\varphi^X_{e,i+g(y)}(y)\downarrow$ for all $y \leq x$ and $f(\langle e, i\rangle, x) = 0$ otherwise. Now either $f_{\langle e,i\rangle} = \varphi_e^X$ is a total function, or $f_{\langle e,i\rangle}$ is finitely nonzero. In either case $f_{\langle e,i\rangle}$ is $X$-computable. If $\varphi_e^X$ is total then $g(x)$ dominates $c(x) = (\mu s)\,[\varphi^X_{e,s}(x)\downarrow]$, so $\varphi_e^X = f_{\langle e,i\rangle}$ for some $i$. □

**Corollary 3.6.** *If $X \leq_T 0'$ is low$_2$, there is a computable function $f(e, y, s)$ such that $\widehat{f}(e, y) = \lim_s f(e, y, s)$ exists for all $e$ and $y$, and*

$$\{Y : Y \leq_T X\} = \{\widehat{f}_e : e \in \omega\}, \tag{5}$$

*where we identify a set $Y$ with its characteristic function $\chi_Y$ as in Soare [1987].*

*Proof.* Use Theorem 3.5 to see that $\widehat{f}(e, y)$ exists, and then the Limit Lemma in Soare [1987, p. 57] to derive $f(e, y, s)$ from $\widehat{f}(e, y)$. □

For a fixed low$_2$ set $X$, we can think of $\widehat{f}(e, y)$ as a $0'$-*matrix* with rows $\{f_e(y)\}_{e\in\omega}$, which is approximated at every stage $s$ in our computable construction by $\lambda\, e\, y\,[\, f(e, y, s)\,]$, and which in the limit correctly gives (5). In §5 we shall use this *dynamic matrix* approximation,

$$\{\ \lambda\, e\, y\,[\ f(e, y, s)\ ]\ \}_{s\in\omega}$$

to produce a CAD theory with no $X$-decidable prime model, thereby proving one half of Theorem 1.5. Millar [1978] constructed a CAD theory with no decidable prime model, and our construction is based on his. The next proposition says that the assumption $X \leq_{\mathrm{T}} 0'$ is necessary in order to apply Corollary 3.6.

**Proposition 3.7.** *A set $X$ satisfies (U0) iff $X \leq_{\mathrm{T}} 0'$ and $X$ is low$_2$.*

*Proof.* ($\Longleftarrow$). Apply Corollary 3.6.

($\Longrightarrow$). If $f$ is a computable function satisfying (5) then $Y = X$ *itself* is one of the rows $\widehat{f}_e$ for some $e$, but $\widehat{f} \leq_{\mathrm{T}} 0'$, so $X \leq_{\mathrm{T}} 0'$. Using $\widehat{f} \leq_{\mathrm{T}} 0'$ we can define a $0'$-function which dominates every $X$-computable function. Now by Corollary 3.2, $X'' \leq_{\mathrm{T}} 0''$. □

# 4 Prime models and trees: (P2) $\Longleftrightarrow$ (P3)

The abstract isolated path tree property (P3) was designed to capture what is needed to construct a prime model. Now we show the equivalence of properties (P3) and (P2). To accomplish this, we review in §4.1 certain basic model-theoretic definitions and facts, most of which can be found in Chang and Keisler [1990]. Most of the results proved and quoted here will use this terminology and background implicitly or explicitly. Then we examine some lemmas in §4.3 which enable us to pass from a theory to a tree, and vice-versa.

## 4.1 Review of model theory, types, and the Stone space

We review and expand on some notions of model theory found in Chang and Keisler [1990] and Sacks [1972]. Let $T$ be a (consistent) decidable theory in a decidable language $L$. We replace $L$ if necessary by $L_c$, a computable expansion obtained by adding an infinite set of new (Henkin) constants $C = \{c_j\}_{j \in \omega}$, and let $T_c$ be the theory in $L_c$ consisting of $T$ together with Henkin axioms of the form $(\exists x)\varphi(x) \longrightarrow \varphi(c)$ with $c \in C$ for every formula $(\exists x)\,\varphi(x) \in T_c$, added in the usual way so that $T_c$ is a conservative extension of $T$ and is decidable if $T$ is. From now on we shall assume that $T$ has been so Henkinized, and we shall write $T$ for $T_c$ and $L$ for $L_c$.

Let $F_n(L)$ be all the formulas $\theta(\overline{x})$ in the $n$ free variables $x_0, \ldots, x_{n-1}$ in $L$. Let $\{\, \theta_n(\overline{x}) \,\}_{n \in \omega}$ be an effective listing of $F_n(L)$. For every string $\sigma \in 2^{<\omega}$ define

$$\theta_\sigma \quad = \quad \bigwedge \{\, \theta_i^{\sigma(i)} : i < \mathrm{lh}(\sigma) \,\}, \tag{6}$$

where $\mathrm{lh}(\sigma)$ denotes the length of $\sigma$, and where we let $\theta^1 = \theta$ and $\theta^0 = \neg\theta$. Define

$$F_n(T) = \{\, \theta(\overline{x}) \;:\; \theta(\overline{x}) \in F_n(L) \quad \& \quad T \cup \{\, (\exists \overline{x})\, \theta(\overline{x}) \,\} \text{ is consistent} \,\}.$$

Now define the *equivalence class* of $\theta(\overline{x}) \in F_n(T)$ under $T$-provability, $\vdash_T$.

$$[\, \theta(\overline{x}) \,] \quad = \quad \{\, \gamma(\overline{x}) \;:\; \vdash_T \; (\forall \overline{x})\, [\, \theta(\overline{x}) \;\leftrightarrow\; \gamma(\overline{x}) \,] \,\}$$

For each $n$ define the *Lindenbaum algebra,*

$$B_n(T) \quad = \quad \{\, [\, \theta(\overline{x}) \,] \;:\; \theta(\overline{x}) \in F_n(T) \,\}.$$

If $T$ is decidable, then these equivalence classes $[\,\theta\,]$ are decidable, uniformly in $\theta$, so $B_n(T)$ is a decidable Boolean algebra, and we can identify a formula $\theta(\overline{x})$ with its equivalence class $[\,\theta(\overline{x})\,]$.

Naturally associated with $T$, under our fixed numbering $\{\theta_n(\overline{x})\}_{n\in\omega}$, is the following subtree of $2^{<\omega}$,

$$\mathcal{T}_n(T) \;=\; \{\, \sigma \;:\; \theta_\sigma \in F_n(T) \,\}. \tag{7}$$

Every string $\sigma \in \mathcal{T}_n(T)$ is associated with the formula $\theta_\sigma \in F_n(T)$ (but really with the equivalence class $[\theta_\sigma] \in B_n(T)$). We identify a formula $\theta_\sigma \in F_n(T)$ (and its equivalence class $[\,\theta_\sigma\,] \in B_n(T)$) with the corresponding string $\sigma \in \mathcal{T}_n(T)$, and generally carry out any computability theoretic analysis in the more abstract tree setting $\mathcal{T}_n(T)$. Let $[\mathcal{T}]$ denote the set of (infinite) paths through a tree $\mathcal{T} \subseteq 2^{<\omega}$.

**Definition 4.1.** (i) An *n-type* $\Gamma$ of a theory $T$ is a maximal consistent subset of $F_n(T)$. Let $S_n(T)$ denote the set of $n$-types of $T$. An $n$-type $\Gamma$ corresponds to a path $f \in [\mathcal{T}_n(T)]$ where $\theta_\sigma \in \Gamma$ iff $\sigma \subset f$. We shall make this identification, and also identify $S_n(T)$ with $[\mathcal{T}_n(T)]$. We sometimes write *complete* type to distinguish a type from a partial type defined as follows.

(ii) A *partial type* $\Gamma$ is a subset of a (complete) type, *i.e.*, a consistent subset $\Gamma(\overline{x}) \subseteq F_n(T)$.

(For $n$-types see Chang and Keisler [1990, p. 77], and for the Lindenbaum algebra see p. 47. Note that $F_n(T)$, $B_n(T)$, $S_n(T)$, and $n$-types are here as defined in Sacks [1972, pp. 71–72]. For more on types and omitting types see §6.2 below.)

The *Cantor space* is $2^\omega$ with the discrete topology on $\{0, 1\}$ and with the product topology on the whole space. Hence, the class of clopen sets is generated by these clopen sets, $U_\sigma = \{ f : \sigma \subset f \}$, for every $\sigma \in 2^{<\omega}$. The *Stone space* $S_n(T)$ is the set of $n$-types, *i.e.*, the set of maximal filters in the Boolean algebra $B_n(T)$, endowed with the Cantor set topology. The well-known theorem by M. H. Stone gives the duality between the Boolean algebra $B_n(T)$ and its Stone space $S_n(T)$.

A path (type) $f \in S_n(T) = [\mathcal{T}_n(T)]$ is *isolated (principal)* if there is some formula $\theta_\sigma \subset f$ such that $f$ is the unique path in $S_n(T)$ containing $\theta$, and we say that $\theta_\sigma$ is an *atom* of $B_n(T)$, and that $\sigma$ is an *atom* of the tree $\mathcal{T}_n[T]$, because there are no two extensions $\theta_\rho$, $\theta_\tau$ of $\theta_\sigma$ (*i.e.*, extensions $\rho$, $\tau$ of $\sigma$ on $\mathcal{T}_n[T]$) which are incomparable with respect to provability in $T$ (*i.e.*, incomparable as strings on $\mathcal{T}_n[T]$). Hence, $f$ is an isolated point in the Cantor space topology on $S_n(T)$ since it is the unique point in the open set $U_\sigma$. A *generator* of a principal type $f$ is an atom $\sigma \subset f$. We say a theory is *atomic* if for every element $\sigma \in \mathcal{T}_n(T)$ there is an atom $\tau \supseteq \sigma$. This is the same as saying that every $\sigma$ has an isolated path $f \supset \sigma$, or that the isolated points of $S_n(T)$ are dense in $S_n(T)$ for all $n$.

For $n = 0$, the paths $f \in S_0(T)$ are just the complete extensions of $T$. If $T$ is complete, then there is just one such completion, but when we Henkinize and pass to $T_c$, there will likely be many complete extensions. Every path $f \in S_0(T_c)$ produces not only a complete extension but a *model* of $T$.

For a theory which is not decidable, but merely *computably enumerable (c.e.)*, such as Peano arithmetic, the tree $\mathcal{T}_0(T)$ of complete extensions, or more generally the trees $\mathcal{T}_n(T)$, might have *terminal* nodes, because we cannot computably decide whether a sentence is consistent with $T$. However, in this paper we study only *decidable* theories $T$. Hence, we do not put $\sigma$ into $\mathcal{T}_n(T)$ unless $(\exists \overline{x})\, \theta_\sigma(\overline{x})$ is consistent with $T$. Thus, every node $\sigma \in \mathcal{T}_n(T)$ will be effectively extendible to a (computable) path $f \in [\mathcal{T}_n(T)]$, *e.g.*, the lexicographically least extension. (The property of a tree $\mathcal{T}$ that every node is extendible has sometimes been written in the literature as $\mathcal{T} = \mathcal{T}^{ext}$, where $\mathcal{T}^{ext}$ denotes the set of extendible nodes.)

## 4.2 A prototype by Goncharov-Nurtazin and Harrington

Many theorems in the literature are stated in terms of the concepts in §4.1 above. For example, the following theorem of Goncharov and Nurtazin and independently Harrington is the prototype for some of our properties, theorems, and proofs. (See Harizanov [1998, Theorem 7.4].)

**Theorem 4.2 (Goncharov and Nurtazin (1973), Harrington (1974)).** *If $T$ is a complete, atomic, decidable theory, then the following* *are equivalent:*

(8) $\quad$ *$T$ has a decidable prime model $\mathfrak{A}$;*

(9) $\quad$ *There is a computable listing of the principal types of $T$.*

Note that an equivalent condition to (9) is the following.

(10) $$(\exists g \leq_{\mathrm{T}} 0)\,(\forall n)\,(\forall x \in \mathcal{T}_n(T))\,[\, x \subset g_x \in [\mathcal{T}_n(T)] \;\;\&\;\; g_x \text{ is isolated}\,],$$

where $g_x = \lambda y\,[g(x,y)]$, and is interpreted as a path in $[\mathcal{T}_n(T)]$, which is equivalent to $S_n(T)$.

Equation (10) asserts that there is a uniformly computable procedure $g(x,y)$ for finding, for every $\theta_x$, a principal type $g_x$ extending it (but not necessarily for finding effectively a *generator* $\gamma \supseteq \theta_x$ of that type). Note that (P2) and (P3) are the relativized versions of (8) and (10). Also note that the proof of Theorem 4.2 relativizes.

### 4.3 Lemmas on theories and trees

**Lemma 4.3.** *If $T$ is a complete atomic decidable theory then the trees $\mathcal{T}_n(T)$ defined in line (7) are uniformly computable, and each has all nodes extendible and isolated paths dense.*

*Proof.* The trees $\mathcal{T}_n(T)$ are uniformly computable because $T$ is decidable. Each $\mathcal{T}_n(T)$ has all nodes extendible because $T$ is consistent. If $T$ is atomic, then for every $n$ and every $\theta \in B_n(T)$ there is an atom $\rho \in B_n(T)$ extending $\theta$. Hence, the isolated paths of $\mathcal{T}_n(T)$ are dense in $\mathcal{T}_n(T)$. □

**Definition 4.4.** Let $L$ be the language with an infinite family of unary predicates $\{U_n\}_{n\in\omega}$. Given a tree $\mathcal{T} \subseteq 2^{<\omega}$, define the corresponding $L$-theory $T(\mathcal{T})$ given by the following axioms.

$$\{\ (\exists^{>m}x)\,[\,U_\sigma(x)\,]\ :\ \sigma \in \mathcal{T}\ \ \&\ \ m \in \omega\ \},\ \text{and}$$

$$\{\ \neg(\exists x)\,[\,U_\sigma(x)\,]\ :\ \sigma \notin \mathcal{T}\ \},$$

where $U_\sigma$ is defined as in (6) but with $U_i$ in place of $\theta_i$. (This is essentially the same conversion of tree to theory used by Millar as presented in Harizanov [1998, p. 31].)

**Lemma 4.5.** *Let $\mathcal{T} \subset 2^{<\omega}$ be a tree with $T(\mathcal{T})$ as in Definition 4.4.*

*(i) If $\mathcal{T}$ is a computable tree with no terminal nodes, then $T(\mathcal{T})$ is a complete decidable theory and admits elimination of quantifiers.*

*(ii) If the isolated paths are dense in $\mathcal{T}$, then $T(\mathcal{T})$ has a prime model.*

*(iii) If $\mathfrak{A}$ is a prime model of $T(\mathcal{T})$, then there is an enumeration $g(x, y)$ of the isolated paths $g_x \in [\mathcal{T}]$ such that $g \leq_{\mathrm{T}} D(\mathfrak{A})$.*

*Proof.* (i) This follows easily from the properties of $\mathcal{T}$ and the construction of $T(\mathcal{T})$. For a formal treatment see Harizanov [1998, p. 31].

(ii) Let $\mathfrak{A}$ be such that $|\mathfrak{A}| = \{a_{f,n} : n \in \omega,\ f \in [\mathcal{T}],\ f \text{ isolated}\}$. Say $a_{f,n} \in U_i^{\mathfrak{A}}$ iff $f(i) = 1$. Then since the isolated paths are dense, $\mathfrak{A}$ is certainly a model of $T(\mathcal{T})$. It is also clearly prime.

(iii) If $\mathfrak{A}$ is a prime model of $T$, then for every $a \in |\mathfrak{A}|$ choose the 1-type $p_a(y) \in S_1(T)$ which $a$ realizes. Let $g(a, y) = p_a(y)$. Then $g \leq_{\mathrm{T}} D(\mathfrak{A})$, and $g(a, y)$ is an enumeration of all principal types of $T$, *i.e.*, all isolated paths of $\mathcal{T}$ (where we identify formulas $\theta_\sigma$ with their strings $\sigma$). □

## 4.4 (P2) $\iff$ (P3): Prime bounding and isolated paths

**Theorem 4.6.** *(P2) $\iff$ (P3). A set $X$ is prime bounding iff $X$ has the isolated path property.*

*Proof.* (P3) $\implies$ (P2). Suppose that $X$ has the isolated path property (P3) and $T$ is a CAD theory. Then for every $n$ the tree $\mathcal{T}_n(T)$ of (7) is a computable tree with no terminal nodes and with the isolated points dense by Lemma 4.3. Let $\alpha_n = 1^n\widehat{\ }0$. Construct a computable tree $\mathcal{U}$ by putting $\alpha_n\widehat{\ }\sigma$ on $\mathcal{U}$ iff $\sigma \in \mathcal{T}_n(T)$. So $\mathcal{U}$ is a tree gluing together the trees $\mathcal{T}_n(T)$. Applying (P3) to $\mathcal{U}$, choose $g \leq_{\mathrm{T}} X$ such that for all $x \in \mathcal{U}$, $g_x \in [\mathcal{U}_x]$, and $g_x$ is isolated. Since $R = \{(x, g_x) : x \in \mathcal{U}\}$ is an $X$-computable enumeration of all principal types of $T$, $T$ has an $X$-decidable prime model $\mathfrak{A}$ by the relativization to $X$ of Theorem 4.2.

(P2) $\implies$ (P3). Assume $X$ has the prime bounding property (P2). Let $\mathcal{U}$ be a computable tree with no terminal nodes and with isolated paths dense as in (P3). Let $T(\mathcal{U})$ be the theory in Definition 4.4 above. Hence, $T(\mathcal{U})$ is a CAD theory and has an $X$-decidable prime model $\mathfrak{A}$ by (P2). For every $x \in \mathcal{U}$, let the formula $U_x$ be as in Definition 4.4 and find some $a_x \in |\mathfrak{A}|$ such that the 1-type $p_x(y)$ of $a_x$ contains $U_x$. Define $g(x, y) = p_x(y)$. Then $g$ witnesses the fact that $X$ satisfies (P3). □

Note that in the proof of (P2) $\implies$ (P3), it would suffice to have $\mathfrak{A}$ be $X$-computable. Hence, if we had $X$-computable instead of $X$-decidable in the definition of prime bounding, the prime bounding property would still be equivalent to the isolated path property.

# 5 The Main Theorem: (P1) $\longleftrightarrow$ (P2)

We now prove the Main Theorem that (P1) $\longleftrightarrow$ (P2). We shall actually prove that (P0) $\implies$ (P3) and that (P3) $\longrightarrow$ (P1). The Main Theorem then follows by applying Theorem 4.6 ((P2) $\iff$ (P3)) and Corollary 3.3 ((P1) $\longleftrightarrow$ (P0)).

## 5.1 (P0) $\implies$ (P3): Escape and isolated paths

**Theorem 5.1.** *(P0) $\implies$ (P3): If $X$ has the escape property then $X$ has the isolated path property.*

*Proof.* Assume $X$ satisfies the escape property (P0),

$$(\forall h \leq_{\mathrm{T}} 0')\,(\exists f \leq_{\mathrm{T}} X)\,(\exists^\infty x)\,[\; h(x) \leq f(x) \;]. \tag{11}$$

Let $\mathcal{T} \subseteq 2^{<\omega}$ be a computable tree with no terminal nodes and with isolated paths dense. Our goal is to produce an $X$-computable function $g(x,s)$ such that for all $x \in \mathcal{T}$, $g_x$ is an isolated path extending $x$. Let $S$ be the set of *atoms* of $\mathcal{T}$, *i.e.*, nodes $x$ with a unique extension $f \in [\mathcal{T}_x]$. Since $S$ is $\Pi^0_1$ and hence $\Delta^0_2$, there is a computable sequence $\{S_s\}_{s\in\omega}$ such that $S(x) = \lim_s S_s(x)$ for all $x$. We may assume that for every $z \in \mathcal{T}$ and every $s$, $S_s$ contains an element extending $z$.

For every $z \in \mathcal{T}$ define the *target,*

$$y_z = (\mu y)[z \subset y \;\;\&\;\; y \in S], \qquad \text{and} \qquad y^s_z = (\mu y)[z \subset y \;\;\&\;\; y \in S_s],$$

its computable approximation at stage $s$. Define the $0'$-computable function

$$h(n) = (\mu s)\,(\forall z)_{|z|\leq n}\,(\forall w \leq y^s_z)\,(\forall t \geq s)[\, S_t(w) = S_s(w) = S(w)\,]. \tag{12}$$

(Note that $h$ is total because for each $n$ we examine the finitely many $z$ with $|z| \leq n$, and for each $z$ and each $s$ the apparent target $y^s_z$ until this stabilizes using $S(x) = \lim_s S_s(x)$.) By the escape property (P0) in (11),

$$(\exists f \leq_{\mathrm{T}} X)\,(\exists \text{ an infinite set } T\,)\,(\forall t \in T\,)\,[\; h(t) \leq f(t) \;]. \tag{13}$$

We may assume that $f$ is monotonic. We refer to $T$ as the set of *true stages*.

Now use $f$ to "speed up" the computable sequence $\{S_s\}_{s\in\omega}$ to an $X$-computable sequence $\{\widehat{S}_s\}_{s\in\omega}$ by defining $\widehat{S}_s = S_{f(s)}$. Define $\widehat{y}_z^s = y_z^{f(s)}$, which is $X$-computable as a function of $z$ and $s$. Note that any *apparent* target $\widehat{y}_z^t$ at a true stage $t \in T$ is the *true* target $y_z$, *i.e.*,

$$(\forall t \in T)\,(\forall z)_{|z|\leq t}\,(\forall v \geq t)[\,\widehat{y}_z^t = \widehat{y}_z^v = y_z\,]. \tag{14}$$

For $s \leq |x|$ define $g(x,s) = x\restriction s$. Fix $s \geq |x|$ and assume we are given $g(x,s)$ with $|g(x,s)| = s$. Define $g(x,s+1) = \widehat{y}^s_{g(x,s)}\restriction(s+1)$. (That is, let $g(x,s+1)$ strictly extend $g(x,s)$ and also take one more step toward the apparent target $\widehat{y}^s_{g(x,s)}$.) Notice that for every $s > |x|$ we have $x \subset g(x,s) \subset g(x,s+1)$, and $|g(x,s)| = s$ for all $s$.

The key point is that we have not only (14), but also that if $t \in T$ and $y = \widehat{y}^t_{g(x,t)}$, then for every $s$ with $t < s < v = |y|$, we have $\widehat{y}^s_{g(x,s)} = y$, because $y$ will be the most attractive target for $g(x,s)$ since no elements $w \leq y$ enter or leave $S$ after stage $t$. Hence, if $t \in T$, then the sequence $\{\,g(x,s) : t < s \leq v\,\}$ marches inexorably from $g(x,t)$ toward $y$ until hitting it at stage $v$, even though the intermediate stages $s$ with $t < s < v$, need *not* be in $T$. Hence, $g_x \in U_S$, and so $g_x$ is an isolated path. □

## 5.2 The Millar theorem on prime models

Millar [1978, Theorem 4] (see also Harizanov [1998, p. 31]) proved that there is a CAD theory $T$ which has no computable prime model. He did this by proving the following theorem and then applying a close variant of Lemma 4.5. In the next section our proof will use the analogue for $X$-computable functions of his strategy for computable functions, which we now review.

**Theorem 5.2 (Millar).** *There is a computable tree $\mathcal{T}$ with no terminal nodes and with isolated points dense, such that there is no computable function $h$ such that $\{\lambda y\,[\,h(n,y)\,]\}_{n\in\omega}$ is an enumeration of the isolated paths of $[\mathcal{T}]$.*

*Proof.* Let $\varphi_0, \varphi_1, \ldots$ be an effective listing of all partial computable binary functions. We must meet for all $e$ the requirement $R_e$: $\{\lambda y\,[\,\varphi_e(n,y)\,]\}_{n\in\omega}$ is not an enumeration of the isolated paths of $[\mathcal{T}]$. Let $\varphi_e^n(y) = \varphi_e(n,y)$. Construct $\mathcal{T}$ by first putting $\alpha_e = 1^e{}^\frown 0$ on $\mathcal{T}$. To ensure $\mathcal{T}$ has no terminal nodes, we put $\sigma^\frown 0$ on $\mathcal{T}$ for every $\sigma \in \mathcal{T}$. We use the nodes and paths above $\alpha_e$ to satisfy $R_e$.

We wait until there is some $n$ such that $\varphi_e^n$ extends $\alpha_e$. If no such $n$ exists, then the extension of $\alpha_e$ will be an isolated path not listed by $\varphi_e$. To defeat $R_e$, we use $\varphi_e^n$ as follows. We introduce a splitting $\sigma, \tau$ along the path extending $\alpha_e$. Then $\varphi_e^n$ must eventually extend $\sigma$ or $\tau$, or fail to list a path on the tree. Say $\varphi_e^n$ extends $\sigma$. Then we introduce another splitting along the extension of $\sigma$. We continue to build $\mathcal{T}$ in this way. Then $\varphi_e^n$ will either list a non-isolated path, or else fail to be total. □

## 5.3 (P3) $\longrightarrow$ (P1): Isolated paths and nonlow$_2$

**Theorem 5.3.** *(P3) $\longrightarrow$ (P1). For $X \leq_{\mathrm{T}} 0'$, if $X$ has the isolated path property, then $X$ is not low$_2$.*

*Proof.* Let $X$ be low$_2$. We shall build a computable tree $\mathcal{T}$ with no terminal nodes and isolated paths dense such that

$$\neg\,(\exists g \leq_{\mathrm{T}} X)\,(\forall x \in \mathcal{T})\,[\,g_x \in \mathcal{T}_x \;\;\&\;\; g_x \text{ is isolated}\,].$$

Let $f(x,e,s)$ be a uniformly $\Delta^0_2$ approximation of the $X$-computable functions, as in Corollary 3.6. Let $f_{e,s}(x) = f(x,e,s)$ and $f_e(x) = \lim_s f_{e,s}(x)$. We think of the $f_{e,s}$ as binary functions by writing $f_{e,s}(n,y)$ for $f_{e,s}(\langle n,y\rangle)$. Let $f^n_{e,s}(y) = f_{e,s}(n,y)$ and $f^n_e(y) = f_e(n,y)$.

For each $e$, we must satisfy the requirement $R_e$ which says that $f_e$ is not an enumeration of the isolated paths of $\mathcal{T}$. As in the proof of Theorem 5.2, we first put $\alpha_k = 1^k{}^\frown 0$ on $\mathcal{T}$ for all $k$. To ensure $\mathcal{T}$ has no terminal nodes, we also ensure that $\sigma^\frown 0$ is in $\mathcal{T}$ for every $\sigma \in \mathcal{T}$. We will still be able to employ disjoint portions of $\mathcal{T}$ to satisfy the various requirements, but we will now need infinitely many places at which to attempt to satisfy $R_e$. Specifically, for each $i$ we use the nodes and paths above $\alpha_{\langle e,i\rangle}$ to satisfy $R_e$.

Since $X$ is low$_2$, there exists a $\Delta^0_2$ function $h$ which dominates all $X$-computable functions. In particular, suppose that for each $i$ there is an $n$ such that $f^n_e$ extends $\alpha_{\langle e,i\rangle}$. Then there is an $i$ (in fact, cofinitely many $i$) for which there is an $n < h(i)$ such that $f^n_e$ extends $\alpha_{\langle e,i\rangle}$. With this in mind, let $\psi^i_{e,s}$ be $f^n_{e,s}$ for the least $n < h_s(i)$ such that $f^n_{e,s}$ extends $\alpha_{\langle e,i\rangle}$, if such an $n$ exists, and undefined otherwise. Since $h$ and $f_e$ are $\Delta^0_2$, either $\psi^i_{e,s}$ is undefined for all sufficiently large $s$ or there is an $n_{e,i}$ such that $\psi^i_{e,s} = f^{n_{e,i}}_{e,s}$ for all sufficiently large $s$, so that $\lim_s \psi^i_{e,s} = f^{n_{e,i}}_e$.

We now proceed as in the proof of Theorem 5.2. For each $\alpha_{\langle e,i\rangle}$, we wait for $\psi^i_{e,s}$ to be defined, and then introduce a splitting $\sigma, \tau$ along the path extending $\alpha_{\langle e,i\rangle}$. We then wait for $\psi^i_{e,t}$ to extend $\sigma$ or $\tau$ for some $t > s$. Say $\psi^i_{e,t}$ extends $\sigma$. Then we introduce another splitting along the path extending $\sigma$. Of course, we might later find that $\psi^i_{e,u}$ extends $\tau$ for some $u > t$. In this case we introduce a splitting along the path extending $\tau$. The approximation of $\psi^i_e$ can shift between extending $\sigma$ and extending $\tau$ only finitely often. If it finally settles on extending $\sigma$, say, then we only build finitely many splittings extending $\tau$. This ensures that the isolated paths are dense in $\mathcal{T}$.

To see that $R_e$ is satisfied, suppose for a contradiction that $f_e$ is an enumeration of the isolated paths of $\mathcal{T}$. Then for each $i$ there is an $n$ such that $f^n_e$ extends $\alpha_{\langle e,i\rangle}$. Let $i$ be large enough so that there is an $n < h(i)$ such that $f^n_e$ extends $\alpha_{\langle e,i\rangle}$. Then for the least such $n$ we have $\lim_s \psi^i_{e,s} = f^n_e$, so the construction of $\mathcal{T}$ ensures that $f^n_e$ is not an isolated path of $\mathcal{T}$. $\square$

# 6 Tree and omitting types properties

In this section we prove the first two parts of the second main implication (2) of §2.2, which we now restate:

$$\text{(P4)} \implies \text{(P3)} \text{ and } \text{(P4)} \iff \text{(P5)}$$

## 6.1 (P4) $\implies$ (P3): The tree property and isolated paths

**Proposition 6.1.** *(P4) $\implies$ (P3). If $X$ has the tree property then $X$ has the isolated path property.*

*Proof.* Let $\mathcal{T}$ be a computable tree with no terminal nodes and with isolated paths dense. If $S$ is the set of nodes that belong to just one path, then $S$ is $\Pi^0_1$ and hence $\Delta^0_2$. $\square$

## 6.2 (P4) $\iff$ (P5): The tree property and omitting types

The familiar Omitting Types Theorem, stated below, gives sufficient conditions for $T$ to have a model omitting a countable family of sets of formulas $\{\, \Gamma_j(\overline{x}_j) \,\}_{j\in\omega}$.

**Theorem 6.2 (Omitting Types).** *Let $T$ be a countable consistent theory, and let $\{\Gamma_j(\overline{x}_j)\}_{j\in\omega}$ be a countable family of sets of formulas, all nonprincipal with respect to $T$. Then $T$ has a model omitting all $\Gamma_j$.*

When we say that a theory is *complete*, we suppose that it is also consistent. Consider a complete decidable theory $T$. If $\Gamma$ is a noncomputable complete type, then any decidable model of $T$ automatically omits it. If $\Gamma$ is a computable type (possibly partial) which is nonprincipal with respect to $T$, then by the Effective Omitting Types Theorem of Millar [1983, Theorem 1, p. 172] $T$ has a decidable model omitting $\Gamma$.

We now show how to use the tree property to produce an $X$-decidable model omitting a uniformly $\Delta^0_2$ family of sets of formulas. We begin by showing that the tree property remains true when the uniformly $\Delta^0_2$ sets mentioned in the definition of that property are replaced by uniformly $\Sigma^0_2$ sets.

**Proposition 6.3.** *Suppose that $X$ has the tree property (P4). For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes, and for every uniformly $\Sigma^0_2$ sequence of subsets $\{S_i\}_{i\in\omega}$ all dense in $\mathcal{T}$, there exists an $X$-computable function $g(x, y)$ such that for every $x \in \mathcal{T}$, $g_x = \lambda y\,[g(x, y)]$ is a path extending $x$ and entering all $S_i$.*

*Proof.* Fix a $0'$-computable enumeration of the $S_i$ and let $S_{i,s}$ be the set of all elements enumerated into $S_i$ by stage $s$. Let $\widehat{S}_i$ be the set of all $\tau \in \mathcal{T}$ such that $\tau \supseteq \sigma$ for some $\sigma \in S_{i,|\tau|}$. The $\widehat{S}_i$ are uniformly $\Delta^0_2$, and any path entering $\widehat{S}_i$ must also enter $S_i$. So the theorem follows from the tree property applied to the $\widehat{S}_i$. □

**Theorem 6.4.** *Suppose that $X$ has the tree property (P4). Let $T$ be a complete decidable theory and let $\{\Gamma_j(\overline{x}_j)\}_{j\in\omega}$ be a uniformly $\Delta^0_2$ family of sets of formulas, all nonprincipal with respect to $T$. Then there is an $X$-decidable model of $T$ omitting all $\Gamma_j(\overline{x}_j)$. In other words, (P4) $\Longrightarrow$ (P5).*

*Proof.* Let the set of Henkin constants $C = \{c_j\}_{j\in\omega}$ and the Henkinization $T_c$ of $T$ be as in §4.1. Let $\psi_0, \psi_1, \ldots$ be an effective listing of all the sentences in the language of $T_c$. For every string $\sigma \in 2^{<\omega}$ let

$$\psi_\sigma \quad = \quad \bigwedge\!\!\!\bigwedge \{\, \psi_i^{\sigma(i)} : i < \mathrm{lh}(\sigma) \,\},$$

where $\psi^1 = \psi$ and $\psi^0 = \neg\psi$, and define

$$\mathcal{T} \;=\; \{\; \sigma \;:\; T_c \nvdash \neg\psi_\sigma \;\}.$$

The tree $\mathcal{T}$ is a computable tree with no terminal nodes, and each path in $\mathcal{T}$ corresponds to a maximal consistent extension of $T_c$, which can be effectively converted into a model of $T$ as in the standard proof of the completeness theorem.

In order to ensure that the model corresponding to $f \in [\mathcal{T}]$ omits a set of formulas $\Gamma(\overline{x})$, it is enough to ensure that, for each tuple $\overline{c} \in C$ of the same length as $\overline{x}$, there exist $\sigma \subset f$ and $\theta \in F_n(T_c)$ such that $\theta(\overline{x}) \in \Gamma$ and $\psi_\sigma \vdash \neg\theta(\overline{c})$.

For each pair $j, \overline{c}$ such that $\overline{c} \in C$ is a tuple of the same length as $\overline{x}_j$, let

$$S_{j,\overline{c}} \;=\; \{\, \sigma \;:\; \exists\, \theta \in F_n(T_c)\; [\theta(\overline{x}_j) \in \Gamma \quad \& \quad \psi_\sigma \vdash \neg\theta(\overline{c})] \,\}.$$

The sets $S_{j,\overline{c}}$ are uniformly $\Sigma^0_2$. Furthermore, we claim that each of these sets is dense in $\mathcal{T}$. To see that this claim is true, suppose that we have $\sigma \in \mathcal{T}$ such that $\tau \notin S_{j,\overline{c}}$ for all $\tau \supseteq \sigma$. Let $\theta(\overline{x}_j) \in \Gamma_i$. We must have $\psi_\sigma \vdash \theta(\overline{c})$, since otherwise there would be a $\tau \supseteq \sigma$ such that $\psi_\tau \vdash \neg\theta(\overline{c})$, and this $\tau$ would therefore be in $S_{j,\overline{c}}$. So $\psi_\sigma \vdash \theta(\overline{c})$ for every $\theta(\overline{x}_j) \in \Gamma_i$. Substituting $\overline{c}$ by $\overline{x}_j$ in $\psi_\sigma$ (after renaming the variables appearing in $\psi_\sigma$ if necessary to ensure that $\overline{x}_j$ is substitutable for $\overline{c}$ in $\psi_\sigma$), we obtain a generating formula for $\Gamma_i$, contradicting the fact that $\Gamma_i$ is nonprincipal.

Now apply Proposition 6.3 to $\mathcal{T}$ and the $S_{j,\overline{c}}$. Let $g(x,y)$ be the resulting $X$-computable function. Consider the path $g_\emptyset = \lambda y\,[g(\emptyset, y)]$, where $\emptyset$ denotes the empty string. As discussed above, this path determines a model of $T$, and this model omits each $\Gamma_i$, since $g_\emptyset$ hits each $S_{j,\overline{c}}$. □

We now show that the converse of the above theorem also holds.

**Theorem 6.5.** *Suppose that $X$ has the omitting types property (P5). Let $\mathcal{T} \subseteq 2^{<\omega}$ be a computable tree with no terminal nodes, and let $\{S_i\}_{i\in\omega}$ be a uniformly $\Delta^0_2$ sequence of subsets, all dense in $\mathcal{T}$. Then there exists an $X$-computable function $g(x,y)$ such that for every $x \in \mathcal{T}$, $g_x = \lambda y\,[g(x,y)]$ is a path extending $x$ and entering all $S_i$. In other words, (P5) $\Longrightarrow$ (P4).*

*Proof.* Let $T = T(\mathcal{T})$ be the complete decidable theory in Definition 4.4, and let $U_\sigma$ be as in that definition. In a model $\mathfrak{A}$ of $T$, each element $a$ corresponds to a path $f_a = \{\, \sigma \;:\; \mathfrak{A} \models U_\sigma(a) \,\}$. Define the sets

$$\Gamma_i(\overline{x}) \;=\; \{\, \neg\, U_\sigma(x) \;:\; \sigma \in S_i \,\}.$$

Since the $S_i$ are uniformly $\Delta^0_2$, so are the $\Gamma_i$. The fact that the $S_i$ are dense implies that each $\Gamma_i$ is nonprincipal with respect to $T$. Indeed, if there were a generating formula for $\Gamma_i$ then, since $T$ admits elimination of quantifiers, there would be a $\tau \in \mathcal{T}$ such that $T \vdash \forall x\,[U_\tau(x) \to \neg U_\sigma(x)]$ for all $\sigma \in S_i$, whence $\sigma \not\supseteq \tau$ for all $\sigma \in S_i$, contradicting the density of $S_i$.

Applying (P5) to $T$ and the $\Gamma_i$, we get an $X$-decidable model $\mathfrak{A}$ omitting all the $\Gamma_i$. Now, using $\mathfrak{A}$, we obtain the required $X$-computable function $g(\sigma, s)$ as follows. For $\sigma \in \mathcal{T}$, we first locate $a \in \mathfrak{A}$ such that $\mathfrak{A} \models U_\sigma(a)$.

Then we let $g(\sigma, y)$ be the element at level $y$ on the path $f_a$ corresponding to $a$.

$\square$

**Remark.** In $S_n(T)$ the complement of the clopen set $U_\sigma$ is the clopen set

$$V_{\overline{\sigma}} = \bigcup \{U_\tau : \tau \in S_n(T) \ \& \ |\tau| = |\sigma| \ \& \ \tau \neq \sigma\}.$$

For $S \subseteq 2^{<\omega}$, the complement of the open set $U_S$ is the closed set

$$C_S = \bigcap \{V_{\overline{\sigma}} : \sigma \in S\}.$$

A set of formulas $\Gamma = \{\varphi_j(\overline{x})\}_{j\in\omega}$ corresponds to a closed set $C_\Gamma$ of paths extending $\varphi_j$ for every $j \in \omega$, namely the intersection of the clopen sets corresponding to $\varphi_j$ for every $j$. To hit an open set $U_S$ is to avoid (omit) all paths in the closed set $C_S$ (*i.e.*, to omit the closed set $C_\Gamma$ corresponding to $C_S$), and conversely. A similar analysis holds for the tree $\mathcal{T}$ defined in the proof of Theorem 6.4.

In the proof of Theorem 6.4, we began with sets of formulas $\Gamma_j$, passed to the corresponding closed sets $C_{S_{j,\overline{c}}}$ (in the notation of that proof), and applied the tree property (P4) to their complements, the open sets $U_{S_{j,\overline{c}}}$.

In the proof of Theorem 6.5, we began with open sets $U_{S_i}$, passed to their complements $C_{S_i}$, and defined the corresponding sets of formulas $\Gamma_i$ so that the closed sets $C_{\Gamma_i}$ corresponded to $C_{S_i}$. This could result in a *partial* type $\Gamma_i$ in case $C_{S_i}$ has more than one element, or an inconsistent set of formulas $\Gamma_i$ in case $C_{S_i}$ is empty.

However, if $\Gamma_i$ is required to be a *complete* type as in part (i) of Definition 4.1 then the corresponding closed set $C_{S_i} \subseteq [T]$ is a singleton. If $|\,[T] - U_{S_i}\,| > \aleph_0$, then the omitting types theorem, even if we omit a countably infinite sequence of such types, will not suffice to prove that the constructed path lies in $U_{S_i}$. (The extended omitting types theorem in Chang and Keisler [1990, p. 84] allows a countable sequence of nonprincipal (complete) types to be omitted.) Therefore, we have been scrupulous with our terminology to avoid giving the impression that the conventional omitting (complete) types principle (as in Chang and Keisler [1990, p. 80] or Sacks [1972, p. 97]) implies the tree property (P4). Hence, property (P5) has been stated for omitting sets of sentences, potentially including *partial* types and inconsistent sets of formulas. (Of course, inconsistent sets of formulas are automatically omitted, but given a uniformly $\Delta^0_2$ family $\mathcal{F}$ of sets of formulas, the subfamily consisting of the consistent sets in $\mathcal{F}$ may not be uniformly $\Delta^0_2$.)

# 7 Algebraic properties

In this section we prove the last part of the second main implication (2) of §2.2, namely

$$\text{(P6)} \iff \text{(P7)} \iff \text{(P8)}.$$

## 7.1 (P6) $\iff$ (P7): Monotonicity, equivalence structures

Recall Definition 2.4 of a set $S$ being $X$-monotonic. Note that (P6) asserts that every infinite $\Delta^0_2$ set $S$ is $X$-monotonic.

**Proposition 7.1.** *For a set $X$, the following are equivalent:*

*(i) $X$ satisfies the monotonic property (P6).*

*(ii) No infinite $\Delta^0_2$ set is $X$-nonmonotonic.*

*(iii) No infinite $\Sigma^0_2$ set is $X$-nonmonotonic.*

*Proof.* Clearly, (i) $\Leftrightarrow$ (ii), and (iii) $\implies$ (ii). An infinite $\Sigma^0_2$ set has an infinite $\Delta^0_2$ subset, and if the former is $X$-nonmonotonic, so is the latter. Therefore, (ii) $\implies$ (iii). $\square$

So far as we know, the first appearance of limitwise monotonic functions was in work of N. G. Khisamiev [1981], [1986], characterizing the Abelian $p$-groups of bounded length ($< \omega^2$) having computable copies. (See also the survey paper by N. G. Khisamiev [1998].) Equivalence structures, discussed in Ash and Knight [2000], form a simpler class with the same features as groups of length $\omega$. Below, we state special cases of the results from these references, and derive from them two further properties equivalent to property (P6). We begin with equivalence structures.

**Definition 7.2.** An *equivalence structure* is a structure of the form $\mathcal{A} = (A, E)$, where $E$ is an equivalence relation on $A$.

There are obvious mathematical invariants associated with such structures. For an equivalence structure $\mathcal{A}$, let $c_n(\mathcal{A})$ be the number of equivalence classes of size $n$. Define

$$R(\mathcal{A}) = \{(n, k) : c_n(\mathcal{A}) \geq k\} \quad \text{and} \quad S(\mathcal{A}) = \{n : c_n(\mathcal{A}) \neq 0\}.$$

The result below, taken from Ash and Knight [2000], characterizes the equivalence structures with computable copies in terms of their invariants.

**Theorem 7.3.** *Let $\mathcal{A}$ be an equivalence structure.*

*(i) If $\mathcal{A}$ has infinitely many infinite classes, then $\mathcal{A}$ has a computable copy iff $R(\mathcal{A})$ is $\Sigma^0_2$.*

*(ii) If $\mathcal{A}$ has only finitely many infinite classes, then there is a computable copy iff $R(\mathcal{A})$ is $\Sigma^0_2$ and, in addition, there is a computable function $g(x,s)$ such that for each $x$, $g(x,s)$ is nondecreasing, with a limit $\widehat{g}(x)$ such that $\widehat{g}(x) \geq x$ and $\widehat{g}(x) \in S(\mathcal{A})$.*

*Sketch of Proof.* It is clear that if $\mathcal{A}$ has a computable copy, then $R(\mathcal{A})$ is $\Sigma^0_2$. For (i), suppose $\mathcal{A}$ has infinitely many infinite classes. If $R(\mathcal{A})$ is $\Sigma^0_2$, then we construct a computable copy, adding finite classes according to our approximation of $R(\mathcal{A})$, and turning our mistakes into infinite classes.

For (ii), suppose, for simplicity, that $\mathcal{A}$ has no infinite classes. If $\mathcal{A}$ has a computable copy, then we define the required function $g(x,s)$ as follows. First, we locate (in the computable copy) $x$ elements in the same class. Then, looking at the first $s$ elements at stage $s$, we let $g(x,s)$ be number of elements in this class at stage $s$.

If we have $g(x,s)$, then we can produce a computable copy of $\mathcal{A}$, adding classes based on our approximation of $R(\mathcal{A})$, and using $g$ to turn our mistakes into classes of acceptable finite sizes. □

For simplicity, we consider infinite equivalence structures $\mathcal{A}$ with no infinite classes, and with at most one class of each finite size. Relativizing part 2 of the theorem, we get the following.

**Corollary 7.4.** *Let $\mathcal{A}$ be an infinite equivalence structure with no infinite classes, and with at most one class of each finite size. Then $\mathcal{A}$ has an $X$-computable copy iff $S(\mathcal{A})$ is $\Sigma^0_2(X)$ and $X$-monotonic.*

Using the corollary, we get a statement on equivalence structures equivalent to property (P6), namely property (P7).

**Theorem 7.5.** *For a set $X$, the following are equivalent:*

*(i) $X$ has the monotonic property (P6).*

*(ii) $X$ has the equivalence structure property (P7). For any infinite $\Delta^0_2$ set $S$ with $0 \notin S$, there is an $X$-computable equivalence structure $\mathcal{A}$, with no infinite classes and with at most one class of each finite size, such that $S(\mathcal{A}) = S$.*

*Proof.* If $X$ has property (P6), then by Corollary 7.4, $X$ satisfies (P7). Suppose $X$ does not have property (P6). Let $S$ be an infinite $X$-nonmonotonic $\Delta^0_2$ set. Since $S$ is $\Delta^0_2(X)$, Corollary 7.4 yields the fact that there is no $X$-computable equivalence structure with just one class of size $n$ for each $n \in S$ and no infinite classes, so (P7) fails. □

## 7.2 (P6) $\iff$ (P8): Monotonicity and $p$-groups

Now we turn to Abelian $p$-groups. A countable Abelian $p$-group is determined by the Ulm sequence of its reduced part and the dimension of its divisible part. Below, we state a special case of N. G. Khisamiev's result, characterizing the computable reduced Abelian $p$-groups of length $\omega$. (See Ash and Knight [2000] for more on this subject.) Define

$$R(\mathcal{G}) = \{(n,k) : u_n(\mathcal{G}) \geq k\} \quad \text{and} \quad S(\mathcal{G}) = \{n : u_n(\mathcal{G}) \neq 0\}.$$

There are analogues of both parts of Theorem 7.3. The analogue for (i) involves groups with an infinite-dimensional divisible part, while the analogue for (ii) involves reduced groups. Below, we give only the analogue for (ii).

**Theorem 7.6 (N. G. Khisamiev).** *Suppose $\mathcal{G}$ is a reduced Abelian $p$-group of length $\omega$. Then $\mathcal{G}$ has a computable copy iff $R(\mathcal{G})$ is $\Sigma^0_2$ and there is a computable function $g(x,s)$ such that for each $x$, $g(x,s)$ is nondecreasing, with a limit $\widehat{g}(x)$ such that $\widehat{g}(x) \geq x$ and $\widehat{g}(x) \in S(\mathcal{G})$.*

For simplicity, we consider groups $\mathcal{G}$ such that for all $n$, $u_n(\mathcal{G}) \leq 1$. Relativizing Theorem 7.6, we obtain the following.

**Corollary 7.7.** *Let $\mathcal{G}$ be an infinite reduced Abelian $p$-group of length $\omega$, with the feature that $u_n(\mathcal{G}) \leq 1$ for all $n$. Then $\mathcal{G}$ has an $X$-computable copy iff $S(\mathcal{G})$ is $\Sigma^0_2(X)$ and $X$-monotonic.*

The next result gives the equivalence of Properties (P6) and (P8).

**Theorem 7.8.** *(P6) $\iff$ (P8). For a set $X$, the following are equivalent:*

*(i) $X$ has the monotonic property (P6).*

*(ii) $X$ has Abelian $p$-group property (P8). For any infinite $\Delta^0_2$ set $S$ with $0 \notin S$, there is an $X$-computable reduced Abelian $p$-group $\mathcal{G}$, of length $\omega$, and with $u_n(\mathcal{G}) \leq 1$ for all $n$, such that $S(\mathcal{G}) = S$.*

*Proof.* If $X$ has property (P6), then by Corollary 7.7, $X$ satisfies (P8). Suppose $X$ does not have property (P6). Let $S$ be an infinite $X$-nonmonotonic $\Delta^0_2$ set. Since $S$ is $\Delta^0_2(X)$, Corollary 7.7 yields the fact that there is no $X$-computable reduced Abelian $p$-group of length $\omega$ such that $u_n(\mathcal{G}) = S(n)$, so (P8) fails. □

# 8 (P0) $\implies$ (P4), (P6)

To connect lines (1) and (2) in §2.2 we now prove that (P0) $\implies$ (P4) and (P0) $\implies$ (P6) by essentially the same proof as for (P0) $\implies$ (P3) in §5.1 above.

## 8.1 (P0) $\implies$ (P4): Escape and the tree property

**Theorem 8.1.** *(P0) $\implies$ (P4): If $X$ has the escape property, then $X$ has the tree property.*

*Proof.* Let $\mathcal{T} \subseteq 2^{<\omega}$ be a computable tree with no terminal nodes and let $\{S_i\}_{i\in\omega}$ be uniformly $\Delta^0_2$ sets, all dense in $\mathcal{T}$. Let $\{S_{i,s}\}_{i,s\in\omega}$ be a computable array such that $S_i = \lim_s S_{i,s}$. We may assume that for every $z \in \mathcal{T}$ and every $i$ and $s$, $S_{i,s}$ contains an element extending $z$. We wish to show that there exists an $X$-computable function $g(x, y)$ such that

$$(\forall x \in \mathcal{T})\,(\forall i)(\exists z \in S_i)[\, x \subseteq z \subset g_x \quad \& \quad g_x \in [\mathcal{T}]\,].$$

For each $z \in \mathcal{T}$ we now define target functions $y_z$ and $y^s_z$. First we define auxiliary functions $y(z, i)$ and $y(z, i, s)$ for $-1 \le i \le |z|$ by induction on $i$ as follows. Define $y(z, -1) = y(z, -1, s) = z$. Now fix $i \ge 0$, and assume that $y(z, i-1)$ and $y(z, i-1, s)$ have been defined. Define

$$y(z, i) = (\mu w)[\, w \supseteq y(z, i-1) \quad \& \quad w \in S_i\,]$$

and

$$y(z, i, s) = (\mu w)[\, w \supseteq y(z, i-1, s) \quad \& \quad w \in S_{i,s}\,].$$

That is, $y(z, t, s)$ is the least extension of $z$ meeting all $S_{i,s}$, in order, for $i \le t$. Define

$$y_z = y(z, |z|)$$

and

$$y^s_z = y(z, |z|, s).$$

Define the $0'$-computable function

$$h(n) = (\mu s)(\forall z)_{|z|\le n}(\forall i)_{i\le n}(\forall w \le y^s_z)(\forall v \ge s)[S_{i,v}(w) = S_{i,s}(w) = S_i(w)].$$

By (P0),

$$(\exists f \le_{\mathrm{T}} X)\,(\exists \text{ an infinite set } T\,)\,(\forall t \in T\,)\,[\; h(t) \le f(t)\; ].$$

We may assume that $f$ is monotonic. Let $\widehat{y}_z^s = y_z^{f(s)}$. As in Theorem 5.1 define $g(x,s)$ as follows. For $s \leq |x|$ define $g(x,s) = x{\restriction}s$. Fix $s \geq |x|$ and assume we are given $g(x,s)$ with $|g(x,s)| = s$. Define $g(x,s+1) = \widehat{y}^s_{g(x,s)}{\restriction}(s+1)$.

We first note that if $t$ is a true stage (*i.e.*, $t \in T$), then $\widehat{y}^t_{g(x,t)} = y_{g(x,t)}$. We claim that if $t$ is a true stage then $g_x \supseteq \widehat{y}^t_{g(x,t)}$. Since there are infinitely many true stages, this will show that $g_x$ meets all the $S_i$.

Suppose $t$ is a true stage. To show that $g_x \supseteq \widehat{y}^t_{g(x,t)}$, we show that $\widehat{y}^s_{g(x,s)} \supseteq \widehat{y}^t_{g(x,t)}$ for all $s \geq t$. Suppose this holds for some $s \geq t$. Then

$$y(g(x,s+1),-1,f(s+1)) = g(x,s+1) = \widehat{y}^s_{g(x,s)}{\restriction}(s+1).$$

But $\widehat{y}^s_{g(x,s)}{\restriction}(s+1)$ either already extends $\widehat{y}^t_{g(x,t)}$ or is extended by it, by our assumption on $s$. In the former case we are done, so assume the later.

Let $z = \widehat{y}^s_{g(x,s)}{\restriction}(s+1)$. As $\widehat{y}^t_{g(x,t)} \supseteq z$, it follows that $\widehat{y}^t_{g(x,t)}$ is the least extension of $z$ meeting each $S_{i,f(t)}$, in order, for $i \leq t$. By definition, $y(g(x,s+1),t,f(s+1))$ is the least extension of $z$ meeting each $S_{i,f(s+1)}$, in order, for $i \leq t$. As $t$ is a true stage, $S_{i,v}{\restriction}\widehat{y}^t_{g(x,t)}+1 = S_i{\restriction}\widehat{y}^t_{g(x,t)}+1$ for all $i \leq t$ and $v \geq h(t)$. Since $f(s+1) \geq f(t) \geq h(t)$ we must have $y(g(x,s+1),t,f(s+1)) = \widehat{y}^t_{g(x,t)}$. So

$$\widehat{y}^{s+1}_{g(x,s+1)} = y(g(x,s+1),s+1,f(s+1)) \supseteq y(g(x,s+1),t,f(s+1)) = \widehat{y}^t_{g(x,t)},$$

as desired.

□

## 8.2 (P0) $\Longrightarrow$ (P6): Escape and the monotonic property

**Theorem 8.2.** *(P0) $\Longrightarrow$ (P6). If $X$ has the escape property, then $X$ has the monotonic property.*

*Proof.* Let the $\Delta^0_2$ set $S$ be as in (P6). Then $S = \lim_s S_s$ for a computable sequence $\{S_s\}_{s\in\omega}$ of infinite sets. Apply the same method of proof as in Theorems 5.1 and 8.1. We use an $X$-computable construction to build $g(x,s)$ such that $g \leq_{\mathrm{T}} X$ and

$$(\forall x)\,(\forall s > |x|)\,[\, x \leq g(x,s) \leq g(x,s+1) \;\;\&\;\; \lim_s g(x,s)\!\downarrow \in S\,],$$

thereby demonstrating that $S$ is $X$-monotonic. Define the targets,

$$y_z = (\mu w)\,[\, z \leq w \quad \& \quad w \in S\,], \quad \text{and} \quad y^s_z = (\mu w)\,[\, z \leq w \quad \& \quad w \in S_s\,].$$

Define $g(x,s) = x$ for $s \leq |x|$. Fix $s \geq |x|$ and assume we are given $g(x,s) \leq s$. Define $h(n)$, $f(t)$, and the set $T$ of true stages, exactly as in (12), (13), and (14), respectively. Let $\widehat{y}^s_z = y^{f(s)}_z$. Define $g(x,s+1) = g(x,s)+1$ if $g(x,s) < \widehat{y}^s_{g(x,s)}$, and $f(x,s+1) = g(x,s)$ if $g(x,s) = \widehat{y}^s_{g(x,s)}$. At a true stage $t \in T$, the apparent target $\widehat{y}^t_{g(x,t)} \geq g(x,t)$ will be the *true* target $y_{g(x,t)} \in S$ and $g(x,v)$ eventually reaches this target at some stage $v \geq t$. □

By applying the same proof technique as above and in Theorem 5.1 one can prove each instance of the following theorem.

**Theorem 8.3.** *For all $k$, $2 \leq k \leq 8$, (P0) $\Longrightarrow$ (Pk) can be proved directly without passing through any other properties.*

# 9 (P6) $\longrightarrow$ (P1)

We now finish linking the last part of (2) in §2.2 to line (1) in the same section by Theorem 9.1, which we prove in this section. In the proof we need a technique for $\text{low}_1$ sets which we review before proceeding with the theorem.

## 9.1 The Robinson method for $\text{low}_1$ sets

It is common in computability theory that we are given a coinfinite $\text{low}_1$ c.e. set $W$ and a computable enumeration of it $\{ W_s \}_{s\in\omega}$. We wish to define a computable function $m(e,s)$ such that $\widehat{m}(e) = \lim_s m(e,s) \downarrow\in \overline{W}$ and also $\widehat{m}(e) < \widehat{m}(i)$ for $e < i$. We can visualize $m(e,s)$ as the position of a "movable marker" $m^e$ which moves finitely often and settles on an element in $\overline{W}$. This technique used by Robinson is explained in more detail in Soare [1987, pp. 224–228], but we briefly review it here.

In this method we define a uniformly c.e. sequence of c.e. sets $\{ V_e \}_{e\in\omega}$, where $V_e = \{ m(e,s) \}_{s\in\omega}$, the set of all positions of marker $m^e$ over the whole construction. By the Recursion Theorem, we may assume we know a $\Sigma^0_1$ index of $V_e$ in advance. Since $W$ is $\text{low}_1$, we have $W' \leq_{\text{T}} 0'$, and hence the question, "Is $V_e \cap \overline{W} \neq \emptyset$?" is $\Delta^0_2$. Thus there is a computable function $G$ such that $\lim_s G(e,s) = 1$ if $V_e \cap \overline{W} \neq \emptyset$, and $\lim_s G(e,s) = 0$ otherwise. We never put a new marker position $m(e,s)$ into $V_e$ at stage $s$ until all the former marker positions $\{m(e,t)\}_{t<s}$ have appeared in $W_s$. Therefore,

$$(\forall s)\,[\, |V_{e,s} - W_s| \leq 1 \,],$$

and hence we are interrogating the $G$-oracle about the *current* marker position $m(e,s)$ at stage $s$, and the oracle can "lie" (*i.e.*, give the wrong answer) at most finitely often.

It is easy to combine this $G$-oracle testing with other strategies, such as moving the markers to maximize their $e$-states in the maximal set construction. In this way Robinson proved that any such set $W$ has a maximal superset as described in Soare [1987, Exer. XI.3.5].

Often we are concerned not only with *individual* markers $m^e$ at each stage but with a finite *set* of them. In this case we enumerate into $V_e$ not $m(e,s)$ but a canonical index $h(j)$ for a finite set $F_j = D_{h(j)} \subset \overline{W}_s$. By lowness of $W$ we can fix a computable function $G(e,s)$ such that

$$\widehat{G}(e) = \lim_s G(e,s) = \begin{cases} 1 & \text{if } (\exists j)[F_j \subset \overline{W}], \\ 0 & \text{otherwise.} \end{cases}$$

We never enumerate a new index $h(j)$ into $V_e$ until $F_i \cap W_s \neq \emptyset$ for all previously chosen finite sets $F_i$ with $i < j$. Hence, the $G$-oracle is being interrogated about the question, for the *present* $F_j \subset \overline{W}_s$, "do we have $F_j \subset \overline{W}$?" Furthermore, $G$ can give a false answer at most finitely often. (We could also simply define $F_j = \overline{W}_s \restriction s$ at stage $s$ and catch all the elements of $\overline{W}$ in which we are interested without specifying them individually in $F_j$ as above.) This finite set approach to low sets is also described in Soare [1987, Exer. IV.4.9].

## 9.2 $X$-nonmonotonicity

**Theorem 9.1.** *(P6) $\longrightarrow$ (P1). If $X \leq_{\mathrm{T}} 0'$ has the monotonic property, then $X$ is nonlow$_2$.*

*Proof.* We prove $\neg$(P1) $\longrightarrow$ $\neg$(P6). Assume $X$ is low$_2$ and $X \leq_{\mathrm{T}} 0'$. We must construct a $\Delta^0_2$ set $A$ which is $X$-nonmonotonic, *i.e.*, is infinite and

(15)
$$\neg(\exists g \leq_{\mathrm{T}} X)(\forall x)(\forall y)[\, x \leq g(x,y) \leq g(x,y+1) \quad \& \quad \lim_y g(x,y) \downarrow \in A\,].$$

## 9.3 Constructing a 0-nonmonotonic set $A$

In order to better reveal the intuition, we first consider the case $X = 0$. We must diagonalize against all partial computable functions $\varphi_e(x,y)$, for $e \in \omega$, as possible candidates for $g$ in (15). Define

(16) $$m(e,x,s) = \max\{\{x\} \cup \{\varphi_{e,s}(x,y) : y \leq s \;\;\&\;\; (\forall z)_{\leq y}[\varphi_{e,s}(x,z)\downarrow\,]\}\};$$

(17) $$W = \{ \langle e,x,z\rangle \ : \ (\exists s)[z \leq m(e,x,s)] \ \}.$$

Clearly, $m(e,x,s)$ is a computable function, and $W$ is a c.e. set. (Think of $m(e,x,s)$ as the position at the end of stage $s$ of a "movable marker" $m^e_x$ which moves monotonically in $s$, and which either comes to rest on $\widehat{m}(e,x) = \lim_s m(e,x,s)$ or goes to infinity.) We must meet for every $e$ the requirement

$$R_e: \quad (\exists x_e)\,[\ \widehat{m}(e,x_e)\!\uparrow \ \ \vee \ \widehat{m}(e,x_e)\!\downarrow\notin A \ ].$$

Let $\omega^{[e]} = \{\langle e,y\rangle : y \in \omega\}$. Choose finite sets $I_e \subset \omega^{[e]}$, $e \in \omega$, sufficiently large as determined later, and let $d_e = \max(I_e)$. We shall put at least one element $a_e$ from $I_e$ into $A$ for every $e$, thereby ensuring that $A$ is infinite. We construct the set $A \leq_{\mathrm{T}} 0'$ during the following $0'$-construction.

### 9.3.1 The basic module for $R_e$ when $X = 0$

Choose a witness $x = x_e$ for $R_e$ when $s = 0$. If $m(e,x,s) \in I_e$, ask the $0'$-oracle whether $\langle e,x,z\rangle \in W$, where $z = m(e,x,s)+1$. (That is, ask whether $m(e,x,t) > m(e,x,s)$ at some stage $t > s$.) If not, then $z$ is a good location for $a_e$. Otherwise, find such $t$ and repeat the process with $z' = m(e,x,t)$ until either:

(18) $d_e < m(e,x,t)$; (so marker $m^e_x$ moves beyond $I_e$), or

(19) $m(e,x,t) \leq d_e$ & the $0'$-oracle certifies $\langle e,x,m(e,x,t)+1\rangle \in \overline{W}$,

in which case $\widehat{m}(e,x)$ is defined, and we can select $a_e \in I_e$ such that $a_e \neq \widehat{m}(e,x)$. Define $V_e$ to be the set of elements $\langle e,x,z\rangle$ for which we query the $0'$-oracle on whether $\langle e,x,z\rangle \in W$. (While this set plays no part in this construction, the analogous set will be important in the case $X \geq_{\mathrm{T}} 0$, which we discuss below.)

### 9.3.2 The modified general strategy for $R_e$ when $X = 0$

In case (18) we must ensure by future action that $\widehat{m}(e,x) \neq a_i$ for lower priority requirements $R_i$, $i > e$. This means by the Golden Rule Principle[1] that $R_e$ must choose $a_e$ so that $a_e \neq \widehat{m}(j,x_j)$ for all $j < e$. To achieve this,

[1] The Harrington-Lachlan Golden Rule Principle says that $R_e$ must take the same action to respect higher priority requirements, $R_j$, $j < e$, which $R_e$ needs the lower priority requirements, $R_i$, $i > e$, to take toward it.

$R_e$ will repeat the same process as above until either (18) or (19) holds for every $j \leq e$. There are at most $e+1$ many values $\widehat{m}(j, x_j)$ with $j \leq e$ which $a_e$ must avoid, and we can choose the interval $I_e$ of cardinality $\geq e+2$, so $a_e$ can avoid them all.

## 9.4 Constructing an $X$-nonmonotonic set $A$ when $X >_{\mathrm{T}} 0$

Fix $X \leq_{\mathrm{T}} 0'$ and low$_2$. Define $m(e,x,s)$ and $W$ exactly as above in (16) and (17) but with $\varphi^X_{e,s}(x,y)$ in place of $\varphi_{e,s}(x,y)$. Now, $m(e,x,s)$ is $X$-computable, and hence $0'$-computable. Also $W \in \Sigma^X_1$, and hence $W \in \Sigma^{0'}_1$, because $X \leq_{\mathrm{T}} 0'$. Furthermore, the fact that $X$ is low$_2$ implies that $X'$, and hence $W$, are low over $0'$. Let $\{W_s : s \in \omega\}$ be a $0'$-enumeration of $W$.

### 9.4.1 The basic module for $R_e$ when $X >_{\mathrm{T}} 0$

The main difference over the previous construction in §9.3 for the case $X = 0$ is that previously we had $W \leq_{\mathrm{T}} 0'$ so we could $0'$-computably determine for $\langle e,x,z\rangle \in V_e$ whether $\langle e,x,z\rangle \in \overline{W}$. Here we will have $V_e \in \Sigma^X_1$, and $V_e$ low over $0'$, uniformly in $e$. We can relativize to $0'$ the Robinson technique of §9.1. Hence, there is a $0'$-computable function $G(e,s)$ such that $\widehat{G}(e) = \lim_s G(e,s) = 1$ if $V_e \cap \overline{W} \neq \emptyset$ and $\widehat{G}(e) = 0$ otherwise. Choose a witness $x = x_e$ for $R_e$ and a finite $I_e \subset \omega^{[e]}$ with largest element $d_e$. If $m(e,x,s) \in I_e$ then enumerate $\langle e,x,z\rangle$ into $V_e$ for $z = m(e,x,s)+1$. Find $t \geq s$ such that either:

$$\langle e,x,z\rangle \in W_t; \text{ or} \tag{20}$$

$$G(e,t) = 1. \tag{21}$$

In case (20) we iterate this process until either $d_e < m(e,x,t)$ or $G(e,t) = 1$, in which case we say that the $G$-oracle *certifies* that $\langle e,x,z\rangle \in \overline{W}$ for $z = m(e,x,t)+1$, and hence that $m(e,x,t) = \widehat{m}(e,x)$.

In case (21) we put some $a^s_e \neq m(e,x,t)$, a current candidate for $a_e$, into $A$. However, the $G$-oracle may have lied and we may later discover at some stage $u > t$ that $m(e,x,u) = a^s_e$, which threatens to make $\widehat{m}(e,x) = a_e$. In this case we choose new large values for $x_e$ and $I_e$ and restart the module. Because $G$ can lie only finitely often, if $\widehat{m}(e,x)$ is defined, then we eventually settle on some final value for $a_e \neq \widehat{m}(e,x)$.

### 9.4.2 The modified general strategy for $R_e$ when $X >_{\mathrm{T}} 0$

In the general strategy we must have the $G$-oracle certify not only the position of $m(e, x_e, s)$ but $m(j, x_j, s)$ for all $j \leq e$. Hence, we put our queries into finite sets $F_{\langle e,k\rangle} = D_{h(e,k)}$, where $h$ is $0'$-computable, and define $V_e = \{h(e,k) : k \in \omega\}$. Using the Recursion Theorem, we have an index $e$ for $V_e$ ahead of time. By lowness of $W$ over $0'$ we can fix a $0'$-computable function $G(e, s)$ as in §9.1 such that $\widehat{G}(e) = \lim_s G(e, s) = 1$ if $(\exists j)[F_j \subset \overline{W}]$ and $\widehat{G}(e) = 0$ otherwise. Now we do the same construction as in §9.3.2, modified as in §9.4.1. □

The implications in sections §3–§9 yield the following result.

**Theorem 9.2.** *If $X \leq_{\mathrm{T}} 0'$, then the properties (P0)–(P8) are all equivalent and they are equivalent to $\neg$(U0).*

## 10 The case where $X \not\leq_{\mathrm{T}} 0'$

Theorem 9.2 establishes the equivalence of all the properties for the case $X \leq_{\mathrm{T}} 0'$. However, if $X \not\leq_{\mathrm{T}} 0'$, then many of these implications disappear. Let us reexamine the implications in lines (1)–(3) but omitting all those implications which used $X \leq_{\mathrm{T}} 0'$. Without assuming $X \leq_{\mathrm{T}} 0'$, we have only

$$\text{(P0)} \implies \text{(P4)} \iff \text{(P5)} \implies \text{(P3)} \iff \text{(P2)}.$$

$$\text{(P0)} \implies \text{(P6)} \iff \text{(P7)} \iff \text{(P8)}.$$

This raises the question of which other implications follow without the hypothesis that $X \leq_{\mathrm{T}} 0'$. In a later paper we will examine these questions. For example, we will show that (P3) does not imply (P1) without the assumption that $X$ is $\Delta^0_2$, and that (P0) and (P1) do not imply one another, as we noted in §3.

## 11 Bounding homogeneous models

It is natural to investigate bounding degrees for other classes of models, such as homogeneous or saturated models in place of prime.

**Definition 11.1.** A set $X$ (degree $\mathbf{d}$) is *homogeneous bounding* if, for every complete decidable theory $T$, there is a homogeneous model decidable in $X$ (decidable in $\mathbf{d}$). (Notice that unlike the prime bounding case we do not require that the theory $T$ be atomic.)

As was the case for prime bounding, the definition of homogeneous bounding could be about models *computable* in $X$ without affecting the result mentioned below.

Denisov [1989] proved that $X = 0'$ is homogeneous bounding, and claimed to prove that no set $X <_{\mathrm{T}} 0'$ is. He incorrectly assumed that every such $X <_{\mathrm{T}} 0'$ has the $0'$-matrix representation given in Corollary 3.6. The reader might assume that the rest of Denisov's proof is correct, *i.e.*, that he actually proved that every set $X$ with the $0'$-matrix property (and hence, every $\mathrm{low}_2$ set) is homogeneous bounding. However, this statement is false by the following recent theorem.

**Theorem 11.2 (Csima, Harizanov, Hirschfeldt, and Soare, (ta)).** *A degree is homogeneous bounding if and only if it is the degree of a complete extension of Peano arithmetic.*

It is well-known by results of Scott that a degree $\mathbf{d}$ is a degree of a complete extension of Peano arithmetic iff every infinite computable tree $\mathcal{T} \subseteq 2^{<\omega}$ has an infinite path $f \in [\mathcal{T}]$ computable in $\mathbf{d}$. Hence, unlike the prime bounding characterization in the Main Theorem 1.5, the characterization of homogeneous bounding degrees is not in terms of equalities or inequalities involving the jump, like $\mathrm{nonlow}_n$ and $\mathrm{high}_n$. Rather, it is in terms of the ability of $\mathbf{d}$ to compute a path in an arbitrary infinite computable tree $\mathcal{T} \subseteq 2^{<\omega}$. In this case $\mathcal{T}$ may have nonextendible nodes, unlike the trees we have studied above.

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Cornell University, Ithaca, New York 14853-4201;
*email:* `csima@math.cornell.edu`
University of Chicago, Chicago, Illinois 60637-1546;
*email:* `drh@math.uchicago.edu`
Notre Dame University, South Bend, Indiana 46556;
*email:* `knight.1@nd.edu`
University of Chicago, Chicago, Illinois 60637-1546;
*email:* `soare@math.uchicago.edu`