A bounded jump for the bounded Turing degrees

Bernard A. Anderson
Department of Pure Mathematics
University of Waterloo
b7anders@uwaterloo.ca
www.math.uwaterloo.ca/∼b7anders

Barbara F. Csima *
Department of Pure Mathematics
University of Waterloo
csima@math.uwaterloo.ca
www.math.uwaterloo.ca/∼csima

Abstract

We define the bounded jump of A by $A^b = \{x \in \omega \mid \exists i \leq x[\varphi_i(x)\downarrow \wedge \Phi_x^{A\restriction\varphi_i(x)}(x)\downarrow]\}$ and let A^{nb} denote the n-th bounded jump. We demonstrate several properties of the bounded jump, including that it is strictly increasing and order preserving on the bounded Turing (bT) degrees (also known as the weak truth-table degrees). We show that the bounded jump is related to the Ershov hierarchy. Indeed, for $n \geq 2$ we have $X \leq_{bT} \emptyset^{nb} \iff X$ is ω^n-c.e. $\iff X \leq_1 \emptyset^{nb}$, extending the classical result that $X \leq_{bT} \emptyset' \iff X$ is ω-c.e.. Finally, we prove that the analogue of Shoenfield inversion holds for the bounded jump on the bounded Turing degrees. That is, for every X such that $\emptyset^b \leq_{bT} X \leq_{bT} \emptyset^{2b}$, there is a $Y \leq_{bT} \emptyset^b$ such that $Y^b \equiv_{bT} X$.

1 Introduction

In computability theory, we are interested in comparing the relative computational complexities of infinite sets of natural numbers. There are many ways of doing this, and which method is used often depends on the purpose of the study, or how fine a comparison is desired. Two sets of the same computational complexity ($X \leq Y$ and $Y \leq X$) are said to be in the same *degree*. The computable sets form the lowest degree for all of the reducibilities we consider here.

Some of the most natural reducibilities are m-reducibility and 1-reducibility. Recall that a set A is m-reducible (1-reducible) to a set B if there is a computable (injective) function f such that for all x, $x \in A$ iff $f(x) \in B$. The major failing of these reducibilities is that a set need not be reducible to its complement.

The most commonly studied reducibility is that of Turing reducibility, where A is Turing reducible to B if there is a program that, with reference to an infinite oracle tape containing B, computes A. Though each computation of a Turing

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reduction views only finitely much of the oracle tape, there is no computable bound on how much of the tape can be viewed in a computation.

Many natural Turing reductions have the property that the use of the oracle is bounded by a computable function. We will refer to such reductions as *bounded Turing* reductions, and write $A \leq_{bT} B$. This is also commonly known in the literature as weak truth-table (wtt) reducibility.

A truth-table reduction is a pair of computable functions f and g, such that, for each x, $f(x)$ supplies a finite list $x_1, ..., x_n$ of positions of the oracle, and $g(x)$ provides a truth-table on n variables (a map $2^n \to 2$). A set A is said to be truth-table reducible to B if there is a truth-table reduction f, g such that, for every x, $x \in A$ iff the row of the truth table g obtained by viewing B on the positions $x_1, ..., x_n$ has value 1. It is easy to see that $A \leq_{tt} B$ iff A is Turing reducible to B via a functional that is total on all oracles. Note that if a functional is total on all oracles, then there is a computable bound on the use for each input. From this, it is easy to see that bounded Turing reducibility is weaker than tt-reducibility, and this is where the name "weak truth-table reducibility" originated. However, since the weakening has nothing to do with the truth-table, we follow the notation of bT, as used in Soare [16] [15].

The halting set is the first natural example of a non-computable set. The Turing jump operator works by relativizing the halting set to other oracles. Basic properties of the Turing jump include that it is strictly increasing with respect to Turing reducibility, and that it maps a single Turing degree into a single 1-degree. This later property shows that the Turing jump is a well-defined operator on all of the degree structures we have mentioned so far.

The strictly increasing property of the Turing jump implies that the Turing jump of any set must compute the halting set. There are a variety of "jump inversion" results, that show that the range of the Turing jump is maximal (with respect to a restricted domain). Friedberg jump inversion states that for every $X \geq_T \emptyset'$ there exists A with $A' \equiv_T X \equiv_T A \oplus \emptyset'$. Shoenfield [14] demonstrated that for every Σ_2 set $X \geq_T \emptyset'$ there is a set $Y \leq_T \emptyset'$ such that $Y' \equiv_T X$.

What about jump inversion for strong reducibilities? Mohrherr [9] showed that for any $X \geq_{tt} \emptyset'$, there exists A with $A' \equiv X$. Anderson [1] showed that the full analogue of Friedberg jump inversion holds: for every $X \geq_{tt} \emptyset'$, there exists A with $A' \equiv_{tt} X \equiv_{tt} A \oplus \emptyset'$. Both Mohrherr's and Anderson's proofs work with bT in place of tt. However, Csima, Downey, and Ng [5] have proved that the analogue of Shoenfield jump inversion fails to hold for the tt and bounded Turing degrees. Indeed, they showed that there is a Σ_2 set $C >_{tt} \emptyset'$ such that for every $D \leq_T \emptyset'$ we have $D' \not\equiv_{bT} C$. The proof exploits the fact that the Turing jump is defined with respect to Turing (and not bounded Turing) reducibilities.

Our goal for this paper was to develop a jump operator for the bounded Turing degrees. We wanted this jump to be bounded in its use of the oracle, and to hold all of the properties usually associated with a jump operator (in particular, strictly increasing and order preserving). In this paper, we will define such a jump, examine its properties, and show it is distinct from already used operators. We will prove Shoenfield inversion holds for the bounded Turing degrees with this jump.

The sets that are computable from the n-th Turing jump of $\emptyset$ have a very nice characterization – they are exactly the Δ^0_{n+1} sets. The n-th bounded jumps of $\emptyset$ also have a natural characterization. In this case, the connection is with the Ershov hierarchy. For $n \geq 2$, the sets that can be bT computed from (indeed are tt or 1-below) the n-th iterate of the bounded jump are exactly the ω^n-c.e. sets.

There have been other jumps for strong reducibilities introduced in the past, and we discuss some of these in Section 7. It has recently come to our attention that Coles, Downey, and Laforte [3] had studied an operator similar to our bounded jump (defined as A^{b_1} in this paper), but unfortunately no written record of their work exists, beyond a proof that their jump is strictly increasing.

2 Notation

We mainly follow the standard notation for computability theory as found in Cooper [4] and Soare [16] [15]. We let $\varphi_0, \varphi_1, \varphi_2, ...$ be an effective enumeration of the partial computable functions, and let $\Phi_0, \Phi_1, \Phi_2, ...$ be an effective enumeration of the Turing functionals.

We let $\emptyset' = \{x \mid \varphi_x(x)\downarrow\}$, and for an arbitrary set A, let $A' = \{x \mid \Phi^A_x(x)\downarrow\}$. In the case that the enumeration $\{\varphi_n\}_{n\in\omega}$ is such that $\varphi_n = \Phi^\emptyset_n$, then there is no confusion with the two definitions of $\emptyset'$. But under any enumeration, the two definitions are 1-equivalent.

For a set A, we let $A \upharpoonright x = \{n \in A \mid n \leq x\}$. We follow an expression with a stage number in brackets (i.e. $[s]$) to indicate the stage number applies to everything in the expression that is indexed by stage.

For sets A and B we write that $A \leq_{bT} B$, and say is A is bounded Turing reducible to B, if there exist i and j such that φ_j is total and for all x, $A(x) = \Phi_i^{B\upharpoonright\varphi_j(x)}(x)\downarrow$. This agrees with the informal definition of bT given in the introduction.

3 The bounded jump

Definition 1. Given a set A we define the bounded jump

$$A^b = \{x \in \omega \mid \exists i \leq x[\varphi_i(x)\downarrow \ \wedge \ \Phi_x^{A\upharpoonright\varphi_i(x)}(x)\downarrow]\}$$

We let A^{nb} denote the n-th bounded jump.

Remark 3.1. $\emptyset^b \equiv_1 \emptyset'$

This holds since bounding the use of an empty oracle has no effect. We will use $\emptyset^b$ and $\emptyset'$ interchangeably from now on.

We consider a more general definition of the bounded jump.

Definition 2. $A^{b_0} = \{\langle e, i, j\rangle \in \omega \mid \varphi_i(j)\downarrow \ \wedge \ \Phi_e^{A\upharpoonright\varphi_i(j)}(j)\downarrow\}$

We show that, up to truth table equivalence, A^b and A^{b_0} are the same. We will at times identify one with the other.

Remark 3.2. *For any set A we have $A^b \leq_{tt} A^{b_0}$.*

This is true since $x \in A^b \iff \exists i \leq x[\langle x, i, x\rangle \in A^{b_0}]$.

Theorem 3.3. *For any set A we have $A^{b_0} \leq_1 A^b$.*

Proof. We define the function k by $\varphi_{k(i,j)}(x) = \varphi_i(j)$. We then define the function g by

$$\Phi^C_{g(\langle e,i,j\rangle)}(x) = \begin{cases} \Phi_e^{C\restriction\varphi_{k(i,j)}(x)}(j) & \varphi_i(j)\downarrow \\ \uparrow & \text{else} \end{cases}$$

By the padding lemma we may assume without loss of generality that for all e, i, and j we have $g(\langle e, i, j\rangle) \geq k(i, j)$.

We now show that $\langle e, i, j\rangle \in A^{b_0} \iff g(\langle e, i, j\rangle) \in A^b$.

For the forward direction, we use $k(i, j)$ as the witness that $g(\langle e, i, j\rangle) \in A^b$.

$$\begin{aligned}
&\langle e, i, j\rangle \in A^{b_0} \\
\Rightarrow\ & \varphi_i(j)\downarrow \text{ and } \Phi_e^{A\restriction\varphi_i(j)}(j)\downarrow \\
\Rightarrow\ & \text{for any } x,\ \varphi_{k(i,j)}(x)\downarrow \text{ and } \Phi^{A\restriction\varphi_{k(i,j)}(x)}_{g(\langle e,i,j\rangle)}(x)\downarrow \text{ [by definitions of } g \text{ and } k] \\
\Rightarrow\ & \varphi_{k(i,j)}(g(\langle e, i, j\rangle))\downarrow \text{ and } \Phi^{A\restriction\varphi_{k(i,j)}(g(\langle e,i,j\rangle))}_{g(\langle e,i,j\rangle)}(g(\langle e, i, j\rangle))\downarrow \text{ [let } x = g(\langle e, i, j\rangle)] \\
\Rightarrow\ & \exists l \leq g(\langle e, i, j\rangle)\ \varphi_l(g(\langle e, i, j\rangle))\downarrow \text{ and } \Phi^{A\restriction\varphi_l(g(\langle e,i,j\rangle))}_{g(\langle e,i,j\rangle)}(g(\langle e, i, j\rangle))\downarrow \text{ [let } l = k(i, j)] \\
\Rightarrow\ & g(\langle e, i, j\rangle) \in A^b
\end{aligned}$$

For the backward direction, we ignore the witness l that $g(\langle e, i, j\rangle) \in A^b$, and rely on the definition of g.

$$\begin{aligned}
&g(\langle e, i, j\rangle) \in A^b \\
\Rightarrow\ & \exists l \leq g(\langle e, i, j\rangle)\ \varphi_l(g(\langle e, i, j\rangle))\downarrow \text{ and } \Phi^{A\restriction\varphi_l(g(\langle e,i,j\rangle))}_{g(\langle e,i,j\rangle)}(g(\langle e, i, j\rangle))\downarrow \\
\Rightarrow\ & \exists l \leq g(\langle e, i, j\rangle)\ \varphi_l(g(\langle e, i, j\rangle))\downarrow \text{ and } \varphi_i(j)\downarrow \text{ and } \Phi_e^{\left(A\restriction\varphi_l(g(\langle e,i,j\rangle))\right)\restriction\varphi_{k(i,j)}(g(\langle e,i,j\rangle))}(j)\downarrow \\
& \qquad \text{[by definition of } g] \\
\Rightarrow\ & \exists l \leq g(\langle e, i, j\rangle)\ \varphi_l(g(\langle e, i, j\rangle))\downarrow \text{ and } \varphi_i(j)\downarrow \text{ and } \Phi_e^{A\restriction\min\left(\varphi_l(g(\langle e,i,j\rangle)),\varphi_i(j)\right)}(j)\downarrow \\
\Rightarrow\ & \varphi_i(j)\downarrow \text{ and } \Phi_e^{A\restriction\varphi_i(j)}(j)\downarrow \\
\Rightarrow\ & \langle e, i, j\rangle \in A^{b_0}
\end{aligned}$$

□

We see later in Remark 5.7 that we cannot strengthen this to $A^b \equiv_1 A^{b_0}$.

Another possibility is a more "diagonal" definition for the bounded jump.

Definition 3. $A^{b_1} = \{x \mid \varphi_x(x)\downarrow \ \wedge \ \Phi_x^{A\restriction\varphi_x(x)}(x)\downarrow\}$

We view this definition as less desirable, since it depends heavily on the particular enumeration $\{\varphi_x\}_{x\in\omega}$ of the partial computable functions. Indeed, depending on the enumeration, one could have $A^{b_1} = \emptyset'$ for all sets A, or with a different enumeration, $A^{b_1} \equiv_1 A^{b_0}$.

Finally, we might also consider a simpler bounded jump.

Definition 4. $A^i = \{x \in \omega \mid \Phi_x^{A\restriction x}(x)\downarrow\}$

However, this definition seems unsatisfactory since it is not strictly increasing.

Remark 3.4. *Let A be a set with $A \geq_{bT} \emptyset'$. Then $A \geq_{bT} A^i$.*

Proof. We show that $A^i \leq_{tt} A \oplus \emptyset'$ for any A. Let $f(n)$ denote the maximum over all strings σ of length n, of the location of $\emptyset'$ needed to determine if $\Phi_n^\sigma(n)\downarrow$. Then $\emptyset' \restriction f(n)$ and $A \restriction n$ suffice to compute $A^i(n)$. □

4 Properties

We summarize some facts about the bounded jump. Let A be any set.

1. $\emptyset^b \equiv_1 \emptyset'$.
2. $A \leq_1 A^b$.
3. $A^b \leq_1 A'$ (since A^b is c.e. in A).
4. $\emptyset' \leq_{tt} A^b$ (as a consequence of Corollary 4.4 below).
5. $A^b \equiv_T A \oplus \emptyset'$ (by Proposition 4.1 below).
6. If $A \geq_T \emptyset'$ then $A^b \leq_T A$.
7. Let A be such that $A' \not\leq_T A \oplus \emptyset'$ (i.e. any $A \geq_T \emptyset'$). Then $A' \not\leq_T A^b$ (so $A' \not\leq_{bT} A^b$).
8. $A^b \not\leq_{bT} A$ (by Theorem 4.2 below).
9. If $A \geq_T \emptyset'$ then $A^b \not\leq_{bT} A \oplus \emptyset'$.

The effect of the bounded jump on the Turing degrees is easy to characterize.

Proposition 4.1. *Let A be any set. Then $A^b \leq_T A \oplus \emptyset'$.*

Proof. We wish to determine if a given n is such that $\exists i \leq n\, [\varphi_i(n)\downarrow \ \wedge \ \Phi_n^{A\restriction\varphi_i(n)}(n)\downarrow]$. We note the existential quantifier is bounded. Given $i \leq n$, we use $\emptyset'$ to determine if $\varphi_i(n)\downarrow$. If it does we then get $\sigma = A \restriction \varphi_i(n)$ from A and use $\emptyset'$ to determine if $\Phi_n^\sigma(n)\downarrow$. This does not require A' since the use of A is bounded. We can then determine if $n \in A^b$. □

While the bounded jump is not very interesting from the perspective of the Turing degrees, we hope to show it follows our intuition for a jump on the bounded Turing degrees.

We start by showing that the bounded jump is strictly increasing. The proof is a diagonalization argument using the Recursion Theorem.

Theorem 4.2. *Let A be any set. Then $A^b \not\leq_{bT} A$.*

Proof. Suppose not. Let Γ and g witness $A^b \leq_{bT} A$. We define a computable function f by

$$\Phi^C_{f(e)}(x) = \begin{cases} 0 & x \neq e \text{ or } (x = e \text{ and } \Gamma^C(e) = 0) \\ \Phi^C_e(e) + 1 & x = e \text{ and } \Gamma^C(e) = 1 \\ \uparrow & x = e \text{ and } \Gamma^C(e)\uparrow \end{cases}$$

By the recursion theorem, let M be an infinite computable set such that for all $m \in M$ we have $\Phi^C_m = \Phi^C_{f(m)}$. Let k be such that $g = \varphi_k$ and pick $m \in M$ such that $m > k$. We note Γ^A is total so $\Phi^A_{f(m)}$ is total and thus Φ^A_m is total.

Suppose $\Gamma^A(m) = 1$. Then $\Phi^A_m(m) = \Phi^A_{f(m)}(m) = \Phi^A_m(m) + 1$ for a contradiction.

Hence $\Gamma^A(m) = 0$. Thus $m \notin A^b$. So for all $i \leq m$ with $\varphi_i(m)\downarrow$ we have $\Phi^{A\restriction\varphi_i(m)}_m(m)\uparrow$. In particular, since $k < m$ and $\varphi_k(m) = g(m)\downarrow$ we have $\Phi^{A\restriction g(m)}_m(m)\uparrow$. Thus $\Phi^{A\restriction g(m)}_{f(m)}(m)\uparrow$ so $\Gamma^{A\restriction g(m)}(m)\uparrow$. This contradicts our choice of Γ and g.

We conclude $A^b \not\leq_{bT} A$. □

We next show the bounded jump is order-preserving on the bounded Turing degrees. The proof is a careful application of the s-m-n Theorem.

Theorem 4.3. *Let A and B be sets with $A \leq_{bT} B$. Then $A^{b_0} \leq_1 B^{b_0}$.*

Proof. Let Ψ and f witness $A \leq_{bT} B$. Let $\varphi_{h(i)}(x) = f(\varphi_i(x))$. Since f is total, $\varphi_{h(i)}(x)\downarrow \iff \varphi_i(x)\downarrow$. Without loss of generality, let h be strictly increasing.

We define a computable function g by

$$\Phi^C_{g(\langle e,k,j\rangle)}(x) = \begin{cases} \Phi_e^{(\Psi^{C\restriction\varphi_{h(i)}(j)})\restriction\varphi_i(j)}(j) & k = h(i) \text{ for some } i \text{ and } \varphi_i(j)\downarrow \\ \uparrow & \text{else} \end{cases}$$

We now note:

$$\begin{aligned} \langle e,i,j\rangle \in A^{b_0} &\iff \varphi_i(j)\downarrow \text{ and } \Phi_e^{A\restriction\varphi_i(j)}(j)\downarrow \\ &\iff \varphi_{h(i)}(j)\downarrow \text{ and } \Phi_e^{A\restriction\varphi_i(j)}(j)\downarrow \\ &\iff \varphi_{h(i)}(j)\downarrow \text{ and } \Phi_e^{(\Psi^{B\restriction\varphi_{h(i)}(j)})\restriction\varphi_i(j)}(j)\downarrow \\ &\iff \varphi_{h(i)}(j)\downarrow \text{ and } \Phi^{B\restriction\varphi_{h(i)}(j)}_{g(\langle e,h(i),j\rangle)}(j)\downarrow \\ &\iff \langle g(\langle e,h(i),j\rangle), h(i), j\rangle \in B^{b_0} \end{aligned}$$

By the s-m-n Theorem, the functions g and h are injective. Therefore $A^{b_0} \leq_1 B^{b_0}$. $\square$

Since for any set X we have $X^{b_0} \equiv_{tt} X^b$, we immediately obtain the following corollary.

Corollary 4.4. *Let A and B be sets with $A \leq_{bT} B$. Then $A^b \leq_{tt} B^b$.*

We would also like to show that, unlike with the Turing degrees, A^b is not equivalent to $A \oplus \emptyset'$ for the bounded Turing degrees. We noted earlier that this holds on the cone above $\emptyset'$. We can also demonstrate this holds elsewhere. We recall two notions of sets being "ordinary" (see Nies [10] for more information on randomness).

Definition 5. X is n-generic if for every Σ_n set $S \subseteq 2^{<\omega}$ either X meets S or there is an $l \in \omega$ such that every string σ extending $X \upharpoonright l$ is such that $\sigma \notin S$.

Definition 6. X is n-random if for every uniformly Σ_n family of sets $\langle U_i \subseteq 2^{<\omega} \mid i \in \omega \rangle$ such that $\mu(U_i) \leq 2^{-i}$ for all i, there exists an l such that X does not meet U_l.

We show that if A is 3-generic or 4-random then $A^b \not\leq_{bT} A \oplus \emptyset'$. In the proof, we will assume towards a contradiction that Ψ and f witness $A^{b_0} \leq_{bT} A \oplus \emptyset'$. We will then use the Recursion Theorem to find a computable set whose elements n are such that $f(n) + 1 \in A \iff n \in A^{b_0}$. Since $A \upharpoonright f(n)$ computes $A^{b_0}(n)$, we have $A(f(n) + 1)$ predicted by $A \upharpoonright f(n)$. This regularity property can then be used to show A is not 3-generic or 4-random, for a contradiction.

Theorem 4.5. *Let A be 3-generic. Then $A^b \not\leq_{bT} A \oplus \emptyset'$.*

Proof. Suppose not. Then $A^{b_0} \leq_{bT} A \oplus \emptyset'$. Let Ψ and f be such that $\Psi^{A \upharpoonright f(n) \oplus \emptyset' \upharpoonright f(n)}(n) = A^{b_0}(n)$ for all n.

For any e, let $\varphi_{g_e(i)}(j) = f(\langle e, i, j \rangle) + 1$. By the Recursion Theorem, let Z_e be an infinite computable set such that for all $m \in Z_e$ we have $\varphi_{g_e(m)} = \varphi_m$. Let t be a computable function such that for all e we have $t(e) \in Z_e$. Then for all $e, j \in \omega$, $\varphi_{t(e)}(j) = \varphi_{g_e(t(e))}(j) = f(\langle e, t(e), j \rangle) + 1$. In particular, $\varphi_{t(e)}$ is total.

We define a computable function h by

$$\Phi^C_{h(e)}(j) = \begin{cases} 1 & C(\varphi_{t(e)}(j)) = 1 \\ \uparrow & \text{else} \end{cases}$$

By the Recursion Theorem, let H be an infinite computable set such that for all $m \in H$ we have $\Phi^C_{h(m)} = \Phi^C_m$.

We then have for every $n \in H$ that

$$\Phi^C_n(j) = \begin{cases} 1 & C(f(\langle n, t(n), j \rangle + 1)) = 1 \\ \uparrow & \text{else} \end{cases}$$

Hence for every $n \in H$ we have

$$\begin{aligned}\langle n, t(n), j\rangle \in A^{b_0} &\iff \varphi_{t(n)}(j)\downarrow \wedge \Phi_n^{A\restriction\varphi_{t(n)}(j)}(j)\downarrow \\ &\iff \Phi_n^{A\restriction f(\langle n,t(n),j\rangle)+1}(j)\downarrow \\ &\iff A(f(\langle n,t(n),j\rangle)+1)=1\end{aligned}$$

Thus for $n \in H$ we have

$$\Psi^{A\restriction f(\langle n,t(n),j\rangle)\oplus\emptyset'\restriction f(\langle n,t(n),j\rangle)}(\langle n,t(n),j\rangle)=1 \iff A(f(\langle n,t(n),j\rangle)+1)=1$$

We define a set S by

$$S=\{\sigma\in 2^{<\omega} \mid \exists j\,\exists n\in H[\text{length}(\sigma)>f(\langle n,t(n),j\rangle) \text{ and}$$

$$\Psi^{\sigma\restriction f(\langle n,t(n),j\rangle)\oplus\emptyset'\restriction f(\langle n,t(n),j\rangle)}(\langle n,t(n),j\rangle)\downarrow\neq \sigma(f(\langle n,t(n),j\rangle)+1) \text{ or diverges}]\}$$

We note S is $\Sigma_2(\emptyset')$ so S is Σ_3. Since A is 3-generic and A does not meet S, there is an m such that for all τ extending $A\restriction m$ we have $\tau\notin S$. However, any string τ can be extended to one in S by picking a value for $\tau(f(\langle n,t(n),j\rangle)+1)$ that disagrees with the prediction of Ψ (if it converges) for some sufficiently large $n\in H$ and j. This is a contradiction so we conclude $A^b \not\leq_{bT} A\oplus\emptyset'$. □

A similar proof can be used to show that if A is 4-random then $A^b \not\leq_{bT} A\oplus\emptyset'$. Hence, for the bounded Turing degrees, the class of sets where A^b is equivalent to $A\oplus\emptyset'$ has measure zero.

Corollary 4.6. *Let A be 4-random. Then $A^b \not\leq_{bT} A\oplus\emptyset'$.*

Proof. Suppose not. Then $A^{b_0}\leq_{bT} A\oplus\emptyset'$. Let Ψ and f be such that $\Psi^{A\restriction f(n)\oplus\emptyset'\restriction f(n)}(n)=A^{b_0}(n)$ for all n.

Let t, H, and S be as in the proof Theorem 4.5. Since f, t, and H are computable, we can find a computable, strictly increasing function l such that for all $m\in\omega$ we have $l(m)=f(\langle n,t(n),j\rangle)$ for some j and some $n\in H$.

For each $i\in\omega$ let $U_i=\{\sigma\mid\sigma\notin S \text{ and } \text{length}(\sigma)=l(i)+1\}$. We note that the U_i are uniformly Π_3 since S is Σ_3 and l is computable. Since A does not meet S, we know that A meets every U_i.

We note from the definition of S that if τ is any string of length $l(m)$ for some m, then at least one of $\tau\,\hat{}\,0$ and $\tau\,\hat{}\,1$ is in S. We also note S is closed under extensions so if $\sigma\notin U_i$, $\text{length}(\sigma)\geq l(i)$, and ρ extends σ then $\rho\notin U_j$ for any $j\geq i$. Hence $\mu(U_i)\leq 2^{-i}$. We conclude that A is not 4-random, for a contradiction. Thus $A^b\not\leq_{bT} A\oplus\emptyset'$. □

5 Ershov Hierarchy

The iterates of the jump correspond to completeness in the arithmetic hierarchy; the n-th jump is Σ_n complete. We will show that the iterates of the bounded jump correspond to completeness in the Ershov hierarchy.

For $\alpha \geq \omega$, we say that a set A is α-c.e. if there is a partial computable $\psi : \omega \times \alpha \to \{0,1\}$ such that for every $n \in \omega$, there exists a $\beta < \alpha$ where $\psi(n,\beta)\downarrow$ and $A(n) = \psi(n,\gamma)$ where γ is least such that $\psi(n,\gamma)\downarrow$ [8].

It is a well known result that $X \leq_{bT} \emptyset' \iff X \leq_{tt} \emptyset' \iff X$ is ω-c.e. [10]. Using the bounded jump, this is $X \leq_{bT} \emptyset^b \iff X \leq_{tt} \emptyset^b \iff X$ is ω-c.e. We wish to extend this observation to higher powers of ω. In fact, we are able to establish a slightly stronger result.

Theorem 5.1. *For any set X and $n \geq 2$ we have $X \leq_{bT} \emptyset^{nb} \iff X$ is ω^n-c.e. $\iff X \leq_1 \emptyset^{nb}$.*

A set A is a *tt*-cylinder if for all X we have $X \leq_{tt} A \Rightarrow X \leq_1 A$ [11].

Corollary 5.2. *For all $n \geq 2$ we have $\emptyset^{nb}$ is a tt-cylinder.*

The theorem follows from the lemmas below. We first introduce some notation. Let $+_c$ denote commutative addition of ordinals (term-wise sum of coefficients of ordinals in Cantor normal form) [2]. We will use two properties of commutative addition. First, given $\alpha_1 \ldots \alpha_n$ and $\beta_1 \ldots \beta_n$ such that $\beta_i \leq \alpha_i$ for all $i \leq n$ and $\beta_j < \alpha_j$ for some $j \leq n$ then $\beta_1 +_c \beta_2 +_c \ldots +_c \beta_n < \alpha_1 +_c \alpha_2 +_c \ldots +_c \alpha_n$. Also, if for some γ we have $\alpha_i < \omega^\gamma$ for all i, then $\alpha_1 +_c \alpha_2 +_c \ldots +_c \alpha_n < \omega^\gamma$.

We start by proving that being ω^k-c.e. is closed downward in the bounded Turing degrees. For the proof, we suppose Φ and f witness $A \leq_{bT} B$ and ψ witnesses B is ω^k-c.e. We will then build χ to witness A is ω^k-c.e. In order to estimate $A(n)$, we will estimate $B \upharpoonright f(n)$ using $\psi(i,\alpha_i)$ for $i \leq f(n)$, and record the output of Φ on this estimate at $\chi(n, \alpha_1 +_c \ldots +_c \alpha_{f(n)})$.

Lemma 5.3. *Let $k > 0$ and let A and B be sets such that $A \leq_{bT} B$ and B is ω^k-c.e. Then A is ω^k-c.e.*

Proof. Let Φ and f witness $A \leq_{bT} B$ and let ψ witness B is ω^k-c.e. We will define a function χ to witness that A is ω^k-c.e. by stages as follows. Fix n (we simultaneously follow the same procedure for each n).

At each stage s, for $i \leq f(n)$, let α_i^s be the least ordinal such that $\psi_s(i,\alpha_i^s)\downarrow$, if it exits. Define a string $\sigma_s(\alpha_0^s, \ldots, \alpha_{f(n)}^s)$ of length $f(n)$ by letting $\sigma_s(i) = \psi_s(i,\alpha_i^s)$.

Let s_0 be the least stage where $\alpha_i^{s_0}$ are defined for all $i \leq f(n)$. Set

$$\chi(n, \alpha_0^{s_0} +_c \ldots +_c \alpha_{f(n)}^{s_0})) = \Phi^{\sigma_{s_0}(\alpha_0^{s_0},\ldots,\alpha_{f(n)}^{s_0})}(n).$$

Note that $\alpha_0^{s_0} +_c \ldots +_c \alpha_{f(n)}^{s_0} < \omega^k$.

At stage $s+1 > s_0$, if $\alpha_i^{s+1} < \alpha_i^s$ for some $i \leq f(n)$ then define

$$\chi(n, \alpha_0^{s+1} +_c \ldots +_c \alpha_{f(n)}^{s+1})) = \Phi^{\sigma_{s+1}(\alpha_0^{s+1},\ldots,\alpha_{f(n)}^{s+1})}(n).$$

This is possible since $\alpha_0^{s+1} +_c \ldots +_c \alpha_{f(n)}^{s+1} < \alpha_0^s +_c \ldots +_c \alpha_{f(n)}^s$.

It is clear that χ is partial recursive. Let n be arbitrary, and for $i \leq f(n)$ let β_i be least such that $\psi(i,\beta_i)\downarrow$. Let $\gamma = \beta_0 +_c \ldots +_c \beta_{f(n)}$. Then γ is least such that $\chi(n,\gamma)\downarrow$ and $\chi(n,\gamma) = A(n)$. Thus χ witnesses A is ω^k-c.e. $\square$

We next prove that if A is ω^k-c.e. then A^b is ω^{k+1}-c.e. Combined with the previous lemma this will give us that $X \leq_{bT} \emptyset^{nb} \Rightarrow X$ is ω^n-c.e.

For the proof, we will let ψ witness that A is ω^k-c.e. and will define χ to witness that A^b is ω^{k+1}-c.e. We will start with $\chi(n, \omega^k \cdot n) = 0$ and each time we witness a new, longer $\varphi_i(n)\downarrow$ for some $i \leq n$ we will move down to a new ω^k level. At a fixed level, we will record estimates of A^b based on estimates of $A \Vert \varphi_i(n)$ in a manner similar to the previous lemma.

Lemma 5.4. *Let $k > 0$ and let A be a set such that A is ω^k-c.e. Then A^b is ω^{k+1}-c.e.*

Proof. Let ψ witness that A is ω^k-c.e. We will define a function χ to witness that A^b is ω^{k+1}-c.e. Fix n (we simultaneously follow the same procedure for each n).

For an ordinal β, let $u(\beta)$ be the coefficient of the units digit of β in Cantor normal form. We again let α_i^s be the least ordinal such that $\psi_s(i, \alpha_i^s)\downarrow$, if it exits, and define a string $\sigma_s(\alpha_0^s, \ldots, \alpha_m^s)$ of length m by letting $\sigma_s(i) = \psi_s(i, \alpha_i^s)$. Indeed, we will assume wlog that α_i^s is defined at each stage s by running the computation ψ for longer than s steps if necessary.

Let $r(l, \alpha_0^s, \ldots, \alpha_m^s) = \omega^k \cdot l +_c \alpha_0^s +_c \ldots +_c \alpha_m^s +_c u(\alpha_0^s +_c \ldots +_c \alpha_m^s)$. Note that if all $\alpha_i^s < \omega^k$, then $r(l, \alpha_0, \ldots, \alpha_m) < \omega^{k+1}$. Note also that if $\beta_i \leq \alpha_i$ for all $i \leq m$ and $l' \leq l$, and if one of the inequalities is strict, then $r(l', \beta_0, \ldots, \beta_m) + 2 < r(l, \alpha_0, \ldots, \alpha_m) + 1$.

We let $\chi(n, \omega^k \cdot n) = 0$ and set bookkeeping variables $l_0 = n$ and $m_0 = 0$. Every time we see $\varphi_i(n)\downarrow > m$ for some $i \leq n$ we will decrease l by one and let $m = \varphi_i(n)$. We note this can happen at most n many times.

At stage $s+1$: If $\varphi_{i,s+1}(n)\downarrow > m_s$ for some $i \leq n$, define $l_{s+1} = l_s - 1$ and let $m_{s+1} = \varphi_i(n)$. Otherwise, let $l_{s+1} = l_s$ and $m_{s+1} = m_s$.

If $l_{s+1} < l_s$ or if $\alpha_i^{s+1} < \alpha_i^s$ for some $i \leq m_{s+1}$, then let $\chi(n, r(l, \alpha_0, \ldots, \alpha_m) + 2)[s+1] = 0$.

If $\Phi_n^{\sigma(l, \alpha_0, \ldots, \alpha_m)}(n)\downarrow [s+1]$, then set $\chi(n, r(l, \alpha_0 \ldots \alpha_m) + 1)[s+1] = 1$.

This completes the construction.

We note χ is partial recursive. Let n be arbitrary and let m be the largest value of $\varphi_i(n)$ for $i \leq n$ such that $\varphi_i(n)\downarrow$. For $j \leq m$, let β_j be least such that $\psi(j, \beta_j)\downarrow$. Let l be least such that for some $\delta < \omega^k$ we have $\chi(n, \omega^k \cdot l + \delta)\downarrow$. Then $\chi(n, r(l, \beta_0 \ldots \beta_m) + 2) = 0$ and $\chi(n, r(l, \beta_0 \ldots \beta_m) + 1)\downarrow = 1$ iff $n \in A^b$. For all $\gamma \leq r(l, \beta_0 \ldots \beta_m)$ we have $\chi(n, \gamma)\uparrow$. Therefore χ witnesses A^b is ω^{k+1}-c.e. $\square$

We note that the proofs for the above lemmas hold for any ordinal ω^γ such that $0 < \omega^\gamma < \omega_1^{CK}$.

To complete the proof of the theorem, we wish to show that A is ω^k-c.e. $\Rightarrow A \leq_1 \emptyset^{kb}$. We start by proving the statement for $k = 2$.

For the proof, suppose ψ witnesses that A is ω^2-c.e. Let $n \in \omega$ and let m be first such that we see $\psi(n, \omega \cdot m + j)\downarrow$ for some j. To determine if $n \in A$, we need to know enough of $\emptyset^b$ to answer the Σ_1 questions $\exists j\, [\psi(n, \omega \cdot i + j)\downarrow]$ for each $i < m$. In each case, if the answer is yes, first witnessed by $\omega \cdot i + k$, we then need to know if $\psi(n, \omega \cdot i + j)\downarrow$ for all $j < k$. There is no computable

bound which can be determined in advance stating how much of $\emptyset^b$ is needed to answer all of these questions. However, we can in advance bound the indices of the computable functions needed to determine how much of $\emptyset^b$ will be used. Hence we can bound the amount of $\emptyset^{2b}$ required to have enough access to $\emptyset^b$ to answer these questions.

Lemma 5.5. *Let A be a set such that A is ω^2-c.e. Then $A \leq_1 \emptyset^{2b}$.*

Proof. Let ψ witness that A is ω^2-c.e. We will define several functions, ending in a computable f such that $n \in A \iff f(n) \in \emptyset^{2b}$.

Let g be a computable function such that $g(n) = i$ where the first time we observe $\psi(n, \alpha)\downarrow$ is $\alpha = \omega \cdot i + j$ for some j. Let $q(i, n)$ be the first m observed such that $\psi(n, \omega \cdot i + m)\downarrow$. The function q is partial computable since it may be there is no such m for the given i.

Let $\tilde{h}(i, x, n)$ denote the spot of $\emptyset^b$ which answers the question $\exists m \leq x$ $[\psi(n, \omega \cdot i + m)\downarrow]$. Let $\tilde{r}(n, i)$ denote the spot of $\emptyset^b$ which answers the question $\exists m[\psi(n, \omega \cdot i + m)\downarrow]$. We then let $h(i, n) = \max\{\tilde{h}(i, x, n) \mid x \leq q(i, n)\}$ and $r(n) = \max\{\tilde{r}(n, x) \mid x \leq g(n)\}$. The functions $\tilde{h}, \tilde{r}$, and r are computable and h is partial computable, converging wherever q does.

Let $p(n)$ be the least i such that for some m we have $\psi(n, \omega \cdot i + m)\downarrow$. We can compute $p(n)$ from $\emptyset^b \upharpoonright r(n)$. We note that $h(p(n), n)$ exists and we can determine if $n \in A$ from $\emptyset^b \upharpoonright \max\{r(n), h(p(n), n)\}$.

Let v be a computable function defined by $\varphi_{v(i,n)}(y) = h(i, n) + r(n)$ (y is a dummy variable). Let $u(n) = \max\{v(i, n) \mid i \leq g(n)\}$. The function u is computable and if we let $j = v(p(n), n)$ then $j \leq u(n)$, the function $\varphi_j(y)\downarrow$, and $\emptyset^b \upharpoonright \varphi_j(y)$ suffices to determine if $n \in A$ (for any y).

We now define $f(n) > u(n)$ to be such that (for any y), $\Phi^{\emptyset^b}_{f(n)}(y)$ runs the calculation to determine if $n \in A$, and converges iff $n \in A$. Explicitly, we define $f(n) > u(n)$ such that $\Phi^C_{f(n)}(y)$ is the partial computable function determined by the following steps. First, we let $x \leq g(n)$ be least such that $C(\tilde{r}(n, x)) = 1$. Next, we let t be first such that we observe $\psi(n, \omega \cdot x + t)\downarrow$. We then let $z \leq t$ be least such that $C(\tilde{h}(x, z, n)) = 1$. Finally, we say $\Phi^C_{f(n)}(y)\downarrow$ if $\psi(n, \omega \cdot x + z) = 1$ and $\Phi^C_{f(n)}(y) \uparrow$ if $\psi(n, \omega \cdot x + z) = 0$ (or if any of the above steps can't be completed).

We note that f is computable, and if C is a sufficiently long initial segment of $\emptyset^b$ then $n \in A$ iff $\Phi^C_{f(n)}(y)\downarrow$. Recall that for any n, there exists $j \leq u(n) < f(n)$ such that $\varphi_j(y)\downarrow$ and $\emptyset^b \upharpoonright \varphi_j(y)$ suffices to run the calculations to determine if $n \in A$.

We observe $f(n) \in \emptyset^{2b} \iff \exists i \leq f(n)[\varphi_i(f(n))\downarrow \wedge \Phi^{\emptyset^b \upharpoonright \varphi_i(f(n))}_{f(n)}(f(n))\downarrow]$ $\iff n \in A$. Hence f witnesses $A \leq_1 \emptyset^{2b}$. $\square$

We use a similar method to prove the statement for all k.

Lemma 5.6. *Let $k > 1$ and let A be a set such that A is ω^k-c.e. Then $A \leq_1 \emptyset^{kb}$.*

Proof. We prove the statement by induction on k. The base case ($k = 2$) is given by Lemma 5.5. For the inductive case, we assume the statement holds

for k and wish to show it holds for $k+1$. We note for the procedure given in Lemma 5.5 that an index for f can be computed uniformly from an index for ψ.

The proof for the inductive case proceeds along the same lines as the proof for the base case. Let ψ witness that A is ω^{k+1}-c.e. Let g be a computable function such that $g(n) = i$ where the first time we observe $\psi(n, \alpha)\downarrow$ is $\omega^k \cdot i + \alpha$ for some $\alpha < \omega^k$. Let $p(n)$ be the least i such that for some α we have $\psi(n, \omega^k \cdot i + \alpha)\downarrow$.

Let $\chi_i(n, \alpha) = \psi(n, \omega^k \cdot i + \alpha)$ for all $\alpha < \omega^k$. We define a partial computable sequence of functions $e_i(n)$ as follows. To compute $e_i(n)$ we first search for any α such that $\chi_i(n, \alpha)\downarrow$ If there is none then we must have $e_i(n)\uparrow$. If the search halts then let

$$\tilde{\chi}(m, \alpha) = \begin{cases} \chi_i(m, \alpha) & m = n \\ 0 & \text{else} \end{cases}$$

. Let B be such that $\tilde{\chi}$ witnesses B is ω^k-c.e. and let $\tilde{f}$ be given by applying the induction hypothesis to B. We then let $e_i(n) = \tilde{f}(n)$.

Let v be a computable function defined by $\varphi_{v(i,n)}(y) = e_i(n)$ (y is a dummy variable). Let $u(n) = \max\{v(i, n) \mid i \leq g(n)\}$. The function u is computable and if we let $j = v(p(n), n)$ then $j \leq u(n)$, the function $\varphi_j(y)\downarrow$, and $\emptyset^{kb} \upharpoonright \varphi_j(y)$ suffices to determine $e_{p(n)}(n)$ and hence if $n \in A$ (for any y).

We define $f(n) > u(n)$ to be such that (for any y), $\Phi^{\emptyset^{kb}}_{f(n)}(y)$ calculates if $n \in A$, and converges iff $n \in A$. Explicitly, we define $f(n) > u(n)$ such that $\Phi^{C}_{f(n)}(y)$ is the partial computable function determined by the following steps. Let l be such that $\Phi^{\emptyset^{kb}}_{l}(m) = p(m)$. We then have $\Phi^{C}_{f(n)}(y)$ converge iff $\Phi^{C}_{l}(n)$ converges and $e_{\Phi^{C}_{l}(n)}(n) \in C$. As in the proof of Lemma 5.5, f witnesses $A \leq_1 \emptyset^{(k+1)b}$, completing the induction.

Therefore for all $n \geq 2$ we have A is ω^n-c.e. $\Rightarrow A \leq_1 \emptyset^{nb}$. □

We proved earlier that for any set A we had $A^{b_0} \leq_1 A^b$ and $A^b \leq_{tt} A^{b_0}$. However, we can use the results above to show that A^b and A^{b_0} are not always 1-equivalent.

Remark 5.7. $\emptyset^{2b} \not\leq_1 (\emptyset^b)^{b_0}$.

Proof. Suppose $\emptyset^{2b} \leq_1 (\emptyset^b)^{b_0}$. Let A be a properly ω^2-c.e. set. By Theorem 5.1, we have $A \leq_1 \emptyset^{2b}$ so $A \leq_1 (\emptyset^b)^{b_0}$.

Using an argument similar to the proof of Lemma 5.4, we can show that $(\emptyset^b)^{b_0}$ is $(\omega+1)$-c.e. Indeed, while $\varphi_i(j) \uparrow$ we believe $\langle e, i, j\rangle \notin (\emptyset^b)^{b_0}$, and once $\varphi_i(j)\downarrow$, since $\emptyset^b$ is c.e. , we can approximate $(\emptyset^b)^{b_0}(\langle e, i, j\rangle)$ with at most $2(\varphi_i(j))$ many changes. It is not hard to see that for any sets B and C and any ordinal α, if $B \leq_1 C$ and C is α-c.e., then B is α-c.e. Hence A is $(\omega+1)$-c.e., contradicting A being properly ω^2-c.e. We conclude $\emptyset^{2b} \not\leq_1 (\emptyset^b)^{b_0}$. □

6 Inversions

We examine what type of inverses exist for the bounded jump. Anderson [1] proved that strong jump inversion holds for the truth-table degrees. For any set $X \geq_{tt} \emptyset'$ there is a set Y such that $X \equiv_{tt} Y' \equiv_{tt} Y \oplus \emptyset'$. It follows as a corollary that strong bounded jump inversion holds for the truth-table degrees.

Corollary 6.1. *Let $X \geq_{tt} \emptyset^b$. Then there exists Y such that $Y^b \equiv_{tt} X \equiv_{tt} Y \oplus \emptyset^b$.*

Proof. Let X be given and let Y be given by strong jump inversion for the truth-table degrees. Then $Y' \equiv_{tt} X \equiv_{tt} Y \oplus \emptyset^b$ and from section 4 we have $Y \oplus \emptyset^b \leq_{tt} Y^b$ and $Y^b \leq_{tt} Y'$. We conclude $Y^b \equiv_{tt} X \equiv_{tt} Y \oplus \emptyset^b$. □

A close examination of the proof in [1] reveals that an equivalent statement also holds for the bounded Turing degrees. For any set $X \geq_{bT} \emptyset'$ there is a set Y such that $X \equiv_{bT} Y' \equiv_{bT} Y \oplus \emptyset'$. If we apply the proof of Corollary 6.1, we get that for any set $X \geq_{bT} \emptyset^b$ there is a set Y such that $X \equiv_{bT} Y^b \equiv_{bT} Y \oplus \emptyset^b$.

As noted earlier, Shoenfield jump inversion [14] holds for the Turing degrees with the Turing jump, for every Σ_2 set $X \geq_T \emptyset'$ there is a $Y \leq_T \emptyset'$ such that $Y' \equiv_T X$. Csima, Downey, and Ng [5] showed that it does not hold for the bounded Turing degrees with the Turing jump.

We prove that Shoenfield jump inversion holds for the bounded Turing degrees with the bounded jump. In this example, the behavior of the bounded jump on the bounded Turing degrees more closely resembles the behavior of the Turing jump on the Turing degrees.

Theorem 6.2. *Let B be such that $\emptyset^b \leq_{bT} B \leq_{bT} \emptyset^{2b}$. Then there is a set $A \leq_{bT} \emptyset^b$ such that $A^b \equiv_{bT} B$.*

Proof. Suppose $\emptyset^b \leq_{bT} B \leq_{bT} \emptyset^{2b}$. Let ψ witness that B is ω^2-c.e. We build an ω-c.e. set A (so $A \leq_{bT} \emptyset^b$) such that $A^b \equiv_{bT} B$.

We will define A using a stage by stage construction. We will ensure that A is ω-c.e. via the function $f(x) = x + 1$. Before we start, we define a computable function g. We will have g witness that $B \leq_1 A^b$.

For each $n \in \omega$, let i_n be the first that we find such that $\psi(n, \omega \cdot i_n + j)\downarrow$.

We define a computable function h to help define g. Let $g(-1) = -1$. Let $h(n) = \Sigma_{k=0}^{n-1} h(k) + \Sigma_{k=1}^{g(n-1)} (\frac{k^2-k}{2}) + i_n$. Let $g(n)$ be such that between $g(n-1)$ and $g(n)$ there are $h(n)$-many partial computable functions $\varphi_{k(n,0)}, \ldots,$ $\varphi_{k(n,h(n)-1)}$ that we control by the recursion theorem, and such that we control $\Phi_{g(n)}$ by the recursion theorem.

We will make use of markers labeled x_n^i with $i \leq i_n$, called *n-markers*, which will move stage by stage, but reach a limit. At some stage s, we might say that a marker x_n^i becomes *defined*. The marker then maintains its value, unless it becomes *undefined* at a later stage. If it at an even later stage becomes redefined, then it will have a new, larger, value. At any moment, there will be at most one n-marker defined for each n. There will be a computable bound on the total number of times all n-markers will be defined/redefined, namely $h(n)$.

In each stage of the construction we will make numerous changes to the approximation of the set A. To ease notation, when we write "A" in the construction, we actually mean the most current approximation of A at that moment of the construction. By "A_s" we mean the approximation A at the *end* of stage s. Without loss of generality, we assume that if $\psi_s(n, \omega \cdot i + j)\downarrow$ then the stage $s > j + 1$.

Stage s:

Step 1*:* If some $\varphi_{e,s}(x)\downarrow$ for the first time at stage s, with $e \leq x \leq g(k)$, then for all $m > k$, extract all x^l_m from A, and declare them undefined.

Step 2*:* Let $n \leq s$ be least such that x^i_n is defined, but $A(x^i_n) \neq B_s(n)$, or such that $\psi(n, \alpha)\downarrow$ for some α but there is no marker defined for n. Let $\omega \cdot i + j$ be least such that $\psi_s(n, \omega \cdot i + j)\downarrow$.

(a) If x^i_n is undefined, then we perform the following steps. Define $x^i_n = s$. Extract all x^l_m with $m > n$ and all x^k_n with $k > i$ from A, and declare them undefined. Define $\varphi_{k(n,r)}(g(n))\downarrow= x^i_n$ for some r, and declare $\Phi^{\sigma\hat{}1}_{g(n)}(g(n))\downarrow$ for every string σ of length $x^i_n - 1$. Note that by our assumption, $j + 1 < s = x^i_n$. There will always be some r with $\varphi_{k(n,r)}(g(n))[s-1]\uparrow$ by our careful counting of $h(n)$.

(b) If needed, change $A_s(x^i_n)$ to ensure $x^i_n \in A_s$ iff $n \in B_s$ (so that $g(n) \in A^b_s$ iff $n \in B_s$).

This completes the construction.

Lemma 6.3. *A is ω-c.e.*

Proof. If at stage s we did not set $s = x^i_n$ for any n, then s was never enumerated into A. If at stage s we set $s = x^i_n$, then x^i_n is enumerated into A, and can be removed/enumerated into A at most j-many more times by Step 1 of the construction (where j is least such that $\psi_s(n, \omega \cdot i + j)\downarrow$). By convention $j \leq s$, so certainly s is enumerated/removed from A at most $s + 1$-many times. □

Lemma 6.4. *For each n and each $i \leq i_n$, $x^i_n = \lim x^i_n[s]$ exists, where we allow "undefined" as a possibility. Moreover, for each n, if $\tilde{\imath}_n = \mu i(\exists j)[\psi(n, \omega \cdot i + j)\downarrow]$ then x^i_n is defined iff $i = \tilde{\imath}_n$, and $x^{\tilde{\imath}_n}_n \in A \iff n \in B$. Finally, for each n, the total number of times any n-marker is defined or redefined, summing over all $i \leq i_n$, is at most $h(n)$.*

Proof. An n-marker x^i_n can only become defined (re-defined) via step 2a of the construction. Thus at the stage when x^i_n is defined (re-defined), i is least such that $\psi_s(n, \omega \cdot i + j)\downarrow$. At the moment that x^i_n is defined (re-defined), any x^k_n with $k > i$ that may have been defined is undefined, and since $k > i$, will never be re-defined at a later stage. That is, at any stage of the construction, there is at most one i with x^i_n defined, and, as a function of the stages, the index i of the n-markers that are defined is non-increasing. Since there is only one defined n-marker at any given stage, the total number of times that an n-marker is undefined by Step 1 of the construction is bounded by $\Sigma^{g(n-1)}_{k=1}(\frac{k^2-k}{2})$. Let $\hat{h}(m)$ be the total number of stages where an m-marker is defined (re-defined). A 0-marker cannot be undefined by step 1. In step 2, a 0 marker can only be

undefined if a new 0-marker, with lower index, is defined. Thus $\hat{h}(0) = i_0 = h(0)$. Similarly, $\hat{h}(n) = \Sigma_{k=1}^{g(n-1)}(\frac{k^2-k}{2}) + i_n + \Sigma_{k=0}^{n} h(k) = h(n)$.

Finally, consider $x_n^{\tilde{\imath}_n}$. Let s be a stage by which all m-markers with $m \leq n$ have reached their limits, and such that $A_t \upharpoonright x_n^{\tilde{\imath}_n} = A_s \upharpoonright x_n^{\tilde{\imath}_n}$ for all $t \geq s$. Note that by definition of $\tilde{\imath}_n$, we have that $x_n^{\tilde{\imath}_n}$ is defined at stage s. Then by step 2b of the construction we have that $x_n^{\tilde{\imath}_n} \in A$ iff $n \in B$. □

Lemma 6.5. $B \leq_1 A^b$

Proof. We claim that $n \in B$ iff $g(n) \in A^b$. Consider the stage s when $x_n^{\tilde{\imath}_n}$ was defined for the last time. At this stage, we set $\varphi_{k(n,r)}(g(n))\downarrow = x_n^{\tilde{\imath}_n}$ for some r, and declare $\Phi_{g(n)}^{\sigma\hat{\ }1}(g(n))\downarrow$ for all σ of length $x_n^{\tilde{\imath}} - 1$. Since $k(n,r) < x_n^{\tilde{\imath}_n}$, we have that if $x_n^{\tilde{\imath}_n} \in A$ then $g(n) \in A^b$. Conversely, we only ever define $\Phi_{g(n)}(g(n))$ to halt in Step 2a of the construction, and with an oracle that includes an n-marker. Since all n-markers besides $x_n^{\tilde{\imath}_n}$ were extracted from A at stage s, we have that if $x_n^{\tilde{\imath}_n} \notin A$ then $g(n) \notin A^b$. Now by the previous lemma we have $x_n^{\tilde{\imath}_n} \in A$ iff $n \in B$, so that $n \in B$ iff $g(n) \in A^b$ as desired. □

Lemma 6.6. $A^b \leq_{bT} B$

Proof. A bounded Turing reduction is like a Turing reduction, except that we must specify in advance, effectively in x, the list of possible questions of the oracle that we will need to use to determine membership of x.

Recall $x \in A^b \iff \exists e \leq x[\Phi_x^{A\upharpoonright\varphi_e(x)}(x)\downarrow]$. Recall also that $\emptyset^b \leq_{bT} B$. Let n be least such that $x < g(n)$.

Let k be the total number of different oracles that appear to witness $x \in A^b$ during the approximation of A. That is, k is maximal such that

$$\exists s_1 ... \exists s_k \exists \sigma_1 ... \exists \sigma_k (\sigma_i \neq \sigma_j \wedge \exists e \leq x[\varphi_{e,s_i}(x)\downarrow \wedge \sigma_i = A_{s_i} \upharpoonright \varphi_{e,s_i}(x)] \wedge \Phi_{x,s_i}^{\sigma_i}(x)\downarrow). \tag{1}$$

According to step 1 of the construction, whenever some $\varphi_e(x)\downarrow$ with $e \leq x$, all m-markers with $m > g(n)$ are extracted from A. So, if $x \in A^b$, then the only non-zero entries in the part of the oracle A that is used in the computation are those that arise from m-markers with $m \leq n$. Since the total number of times m-markers can be redefined is bounded by $h(m)$, and since each marker can either be in or out of A, the number k of possible oracles is computably bounded (it is certainly bounded by $2^{\Sigma_{l=0}^{n} h(l)}$). That is, we can bT compute k from $\emptyset^b$ and hence B using questions of the form (1).

For each $m \leq n$, using at most i_m-many questions of the form $(\exists x_1)...(\exists x_l)[x_{p+1} < x_p \wedge (\exists j)\psi(m, \omega \cdot x_p + j)\downarrow]$, we can bT compute $\tilde{\imath}_m$ from $\emptyset^b$ and hence B.

Similarly, we can bT compute from $\emptyset^b$, and hence from B, the number of pairs $e \leq y \leq g(n)$ such that $\varphi_e(y)\downarrow$. Thus we can bT compute from B the stage s by which point if $e \leq y \leq g(n)$ and $\varphi_e(y)\downarrow$ then $\varphi_{e,s}(y)\downarrow$.

We can certainly bT-compute from B the initial segment $B \upharpoonright n$.

We now put the above facts together to compute whether $x \in A^b$. If $k = 0$, then there is never any stage where it appears that $x \in A^b$, so $x \notin A^b$. So

suppose $k \neq 0$. Run the approximation of A to find the k-many different possible oracles which might witness $x \in A^b$. We know that the only possible non-zero entries in the correct oracle come from $x_m^{\tilde{\imath}_m}$ with $m \leq n$, and that $x_m^{\tilde{\imath}_m} \in A$ iff $m \in B$. Now since we have bT-computed from B all the $\tilde{\imath}_m$ for $m \leq n$, we can run the approximation of A until the least stage t greater than s where markers of the form $x_m^{\tilde{\imath}_m}$ are defined for all $m \leq n$. The location of $x_m^{\tilde{\imath}_m}$ at stage t is its final location. Now, using $B \upharpoonright n$, we have computed the true initial segment of A that is relevant for deciding whether $x \in A^b$. If this oracle extends any of the k-many halting oracles that we found, then $x \in A^b$. Otherwise, $x \notin A^b$.

□

□

We note the proof above cannot be modified to find an A such that $A^b \leq_{tt} B$.

7 Other jump operators

In 1979, Gerla [7] proposed jump operators for the truth-table and bounded truth-table degrees. We wish to compare his observations on these operators with some of the results shown so far for the bounded jump. Since the original article is available only in Italian, we briefly summarize the definitions and highlight a few of the results from the paper.

We start with some basic definitions used in studying the truth-table degrees (see Rogers [12]).

Definition 7. A tt-condition is a finite sequence $x_1 \dots x_k \in \omega$ and a function $\alpha : 2^k \to 2$. We say it is satisfied by A if $\alpha(A(x_1) \dots A(x_k)) = 1$. We define $A^{tt} = \{x \mid x \text{ is a } tt\text{-condition satisfied by } A\}$.

We note that $A^{tt} \leq_{tt} A$ and $A \leq_1 A^{tt}$. Gerla [7] uses A^{tt} to define jumps A_{tt} and A_{bk} for the truth-table degrees and bounded truth-table degrees of norm k, respectively.

Definition 8. $A_{tt} = \{x \mid \varphi_x(x)\downarrow \in A^{tt}\}$. $A_{bk} = \{x \mid \varphi_x(x)\downarrow \in A^{tt} \wedge \varphi_x(x) \leq k\}$.

The behavior of A_{tt} and A_{bk} on the truth-table and bounded truth-table degrees shares several similarities with that of A' on the Turing degrees. We state a few of the many results below.

Theorem 7.1 (Gerla [7])**.** *Let k be a number and let A and B be sets.*

1. $A_{tt} \not\leq_{tt} A$. $A_{bk} \not\leq_{bk} A$.
2. $A \leq_{tt} B \Rightarrow A_{tt} \leq_1 B_{tt}$.
3. $A <_1 A_{bk} \leq_1 A_{b(k+1)} \leq_1 A_{tt} \leq_1 A'$.
4. $\emptyset_{bk} \equiv_1 \emptyset_{tt} \equiv_1 \emptyset'$.

We demonstrated earlier the connection between the bounded jump and the Ershov hierarchy. We see that the finite levels of the Ershov hierarchy share a similar (but weaker) connection with A_{bk}.

Theorem 7.2 (Gerla [7])**.** *Let A be n-c.e. and let $B \leq_1 A_{bk}$. Then B is $(nk+1)$-c.e.*

Let $\emptyset_{n(bk)}$ denote the n-th iteration of the bk jump of the empty set. It follows from the theorem that if $A \leq_1 \emptyset_{n(bk)}$ then A is $(1+k+k^2+\ldots k^{n-1})$-c.e. [7].

Since Gerla's truth-table jump is designed for a stronger reducibility, we expect it to be weaker than the bounded jump. We prove that for every set A we have $A_{tt} \leq_1 A^b$, but there are many sets X such that $X^b \not\leq_{bT} X_{tt}$.

Proposition 7.3. $A_{tt} \leq_1 A^{b_0}$

Proof. Let f and Φ_k witness that $A^{tt} \leq_{tt} A$. Define h by $\varphi_{h(e)}(z) = f(\varphi_e(e))$.

Define j by $\Phi^C_{j(e)}(z)\downarrow$ iff $\varphi_e(e)\downarrow$ and $\varphi_e(e) \in \Phi^C_k$.

Let z represent an arbitrary dummy variable. We note the following.

$$\begin{aligned}
x \in A_{tt} &\iff \varphi_x(x)\downarrow \in A^{tt} \\
&\iff \varphi_x(x)\downarrow \in \Phi^A_k \\
&\iff \varphi_x(x)\downarrow \in \Phi^{A\restriction f(\varphi_x(x))}_k \\
&\iff \varphi_x(x)\downarrow \in \Phi^{A\restriction \varphi_{h(x)}(z)}_k \\
&\iff \Phi^{A\restriction \varphi_{h(x)}(z)}_{j(x)}(z)\downarrow \\
&\iff \langle j(x), h(x), z\rangle \in A^{b_0}
\end{aligned}$$

Thus $A_{tt} \leq_1 A^{b_0}$.

□

Corollary 7.4. $A_{tt} \leq_1 A^b$

Theorem 7.5. *There is a c.e. set A such that $A^b \not\leq_{bT} A_{tt}$.*

Proof. First note that if we have a computable approximation to a set A, then this induces an obvious approximation for A_{tt}. Namely, if $\varphi_{x,s}(x)\uparrow$ then $x \notin A_{tt}[s]$, and if $\varphi_{x,s}(x)\downarrow$ then $x \in A_{tt}[s] \iff \varphi_x(x) \in A^{tt}[s]$. We also have an approximation for A^b by $x \in A^b[s] \iff \exists i \leq x[\varphi_x(x)\downarrow \ \wedge \Phi^{A\restriction\varphi_x(x)}_x(x)\downarrow [s]]$. We note that if A is c.e. then these are both Δ^0_2 approximations.

For $k \in \omega$, let $l(k,s) = \max\{\varphi_{x,s}(x) \mid x \leq k\}$. Note that for $s < t$, if $A \restriction l(k,s)[s] = A \restriction l(k,s)[t]$, then $A_{tt} \restriction k[s] = A_{tt} \restriction k[t]$ *unless* $\varphi_{x,t}(x)\downarrow$ for some $x \leq k$ such that $\varphi_{x,s}(x)\uparrow$.

We now proceed with the construction of A. We must meet for all $n \in \omega$ the requirement

$$R_n : (\neg\forall x)[\Phi^{A_{tt}\restriction\varphi_{\pi_2(n)}(x)}_{\pi_1(n)}(x) = A^b(x)]$$

where π_1 and π_2 are projection functions for some canonical pairing function. To ease notation, we will write Φ_n and φ_n for $\Phi_{\pi_1(n)}$ and $\varphi_{\pi_2(n)}$. We note that this notation only applies to the variables n and e_i, so that Φ_{x_n} and φ_{x_n} still have their usual meaning.

Let $e_0 < e_1 < e_2 < \ldots$ be a computable list such that we control φ_{e_i} and Φ_{e_i} by the Recursion Theorem. We will use a set of movable markers x_i for $i \in \omega$ such that for all i we have $x_i = e_j$ for some j. We will also make use of a restraint function r. We start with $r(n)[0] = 0$ for all n.

Stage 0: Let $x_0 = e_0$.

Stage s+1: For each m let $r(m)[s+1] = \max\{\varphi_{x,s+1}(x) \mid x \leq \varphi_l(x_l)[s]$ for some $l < m\}$ (we say $r(m)[s+1] = 0$ if this set is empty). Let k be least such that $r(k)[s+1] > r(k)[s]$ (if no such k exists, use $k = s$). Undefine all x_m with $m \geq k$.

Case 1: There is no $n < k$ such that x_n is defined and $\Phi_n^{A_{tt}\restriction\varphi_n(x_n)}(x_n)[s] \downarrow= A^b(x_n)[s]$.

We then let m be least such that x_m is not defined, and define x_m to be the least e_i that has not been used in the construction (proceed to the next stage).

Case 2: Else.

We then let $n < k$ be least such that x_n is defined and $\Phi_n^{A_{tt}\restriction\varphi_n(x_n)}(x_n)[s] \downarrow= A^b(x_n)[s]$. Undefine all x_m with $m > n$. If it has not yet been defined (with the current value of x_n), let $\varphi_{x_n}(x_n) = r(n)[s+1] + \max\{A_s\} + \varphi_n(x_n)$.

Subcase 2A: $A^b(x_n)[s] = 0$.

Set $\Phi_{x_n}^{A_s\restriction\varphi_{x_n}(x_n)}(x_n) \downarrow$, so that $A^b(x_n)[s+1] = 1$. Do not enumerate into A.

Subcase 2B: $A^b(x_n)[s] = 1$.

Choose the least $x > r(n)$ such that $x \notin A[s]$, and enumerate $x \in A[s+1]$. We demonstrate later that we have $x \leq \varphi_{x_n}(x_n)$, so that this will cause $A^b(x_n)[s+1] = 0$.

This completes the construction of A.

It is easy to see that the set constructed is c.e. We claim that for each n, $x_n = \lim_s x_n[s]$ exists, and provides a witness for R_n. We say that a requirement R_n receives attention if we perform case 2 of the construction for some x_n.

Lemma 7.6. *For each n, $x_n = \lim_s x_n[s]$ exists, and provides a witness for R_n. Moreover, the requirement R_n receives attention at most finitely often.*

Proof. Since x_0 is never undefined, it reaches its limit at stage 0. Assume that x_l with $l < m$ have reached their limit, and if $\varphi_l(x_l)\downarrow$ then it has already done so. Then the value $r(m)$ can increase at most $\max\{\varphi_l(x_l) \mid l < m\}$-many more times, and so there must be a stage after which x_m is never undefined.

Assume for a contradiction that x_n is least such that $\Phi_n^{A_{tt}\restriction\varphi_n(x_n)}(x_n) \downarrow= A^b(x_n)$. Let s be the least stage after which no x_m with $m < n$ requires attention, and $r(n)$ has reached its limit. So at stage $s+1$ of the construction, x_n is defined, and is never again undefined. Since $\Phi_n^{A_{tt}\restriction\varphi_n(x_n)}(x_n) \downarrow= A^b(x_n)$, there is a least stage $s_0 > s+1$ such that $\Phi_n^{A_{tt}\restriction\varphi_n(x_n)}(x_n)[s_0] \downarrow= A^b(x_n)[s_0]$. Since s_0 is the first stage where R_n requires attention with this value of x_n, we define $\varphi_{x_n}(x_n) = r(n) + \max\{A_{s_0-1}\} + \varphi_n(x_n)$ at stage s_0, and we have

$A^b(x_n)[s_0-1] = 0$. So at stage s_0 of the construction, we set $\Phi_{x_n}^{A_{s_0}\upharpoonright\varphi_{x_n}(x_n)}(x_n)\downarrow$, so that $A^b(x_n)[s_0] = 1$. Note that at stage s_0 there are at least $\varphi_n(x_n)$-many numbers greater than $r(n)$ and less than $\varphi_{x_n}(x_n)$ available to enumerate into A.

Let $s_0 < s_1 < s_2 < \ldots$ be all the further stages of the construction where R_n receives attention. We will show that for all even k an element is enumerated into $\emptyset' \upharpoonright \varphi_n(x_n)$ at some stage t with $s_k < t \leq s_{k+1}$. It follows that the sequence $s_0, s_1, \ldots$ must be finite, contradicting the assumption that $\Phi_n^{A_{tt}\upharpoonright\varphi_n(x_n)}(x_n)\downarrow = A^b(x_n)$. We will also show inductively that there is sufficient room to enumerate elements into A between $r(n)$ and $\varphi_{x_n}(x_n)$, as claimed earlier.

Let k be even, and assume for our induction that there are at least $(\varphi_n(x_n) - \frac{k}{2})$-many numbers greater than $r(n)$ and less than $\varphi_{x_n}(x_n)$ available to enumerate into A. Without loss of generality, suppose that at stage s_k we ensured $A^b(x_n)[s_k] = 1$. Since all requirements R_m with $m < n$ have stopped acting, no requirement R_m with $m \leq n$ enumerated into A at any stage $s_k \leq t \leq s_{k+1}$. Furthermore, since $r(m) \geq \varphi_{x_n}(x_n)$ for all $m > n$, no requirement R_m enumerates into $A \upharpoonright \varphi_{x_n}(x_n)$ at any stage $s_k \leq t < s_{k+1}$. Hence $A^b(x_n)[s_{k+1} - 1] = 1$ and $\Phi_n^{A_{tt}\upharpoonright\varphi_n(x_n)}(x_n)[s_{k+1}-1] = 1$. So $A_{tt} \upharpoonright \varphi_n(x_n)[s_k-1] \neq A_{tt} \upharpoonright \varphi_n(x_n)[s_{k+1}-1]$.

Using our observation from the start of the proof of the theorem, to demonstrate that there is a $y \leq \varphi_n(x_n)$ such that $\varphi_{y,s_k}(y)\uparrow$ but $\varphi_{y,s_{k+1}}(y)\downarrow$, it suffices to show $A \upharpoonright l(\varphi_n(x_n), s_k - 1)[s_k - 1] = A \upharpoonright l(\varphi_n(x_n, s_{k+1} - 1)[s_{k+1} - 1]$. Between stages $s_k - 1$ and $s_{k+1} - 1$, the construction only runs subcase 2 for a requirement R_m with $m > n$. Hence no element is enumerated into $A \upharpoonright r(n+1)$. Since $r(n+1) \geq l(\varphi_n(x_n), s_k - 1)$ we have $A \upharpoonright l(\varphi_n(x_n), s_k - 1)[s_k - 1] = A \upharpoonright l(\varphi_n(x_n, s_{k+1} - 1)[s_{k+1} - 1]$, as desired. Thus some $y \leq \varphi_n(x_n)$ was added to $\emptyset'$ between stages s_k and s_{k+1}.

At stage s_{k+1} the least $x > r(n)$ such that $x \notin A[s_{k+1} - 1]$, was enumerated into $A[s_{k+1}]$. By induction hypothesis, we had $x \leq \varphi_{x_n}(x_n)$, so that $A^b(x_n)[s_{k+1}] = 0$. Note that at stage $s_{k+1}+1$ there are at least $(\varphi_n(x_n) - \frac{k}{2} - 1)$-many numbers greater than $r(n)$ and less than $\varphi_{x_n}(x_n)$ available to enumerate into A.

At stage s_{k+2}, we acted because $\Phi_n^{A_{tt}\upharpoonright\varphi_n(x_n)}(x_n)[s_{k+2}-1]\downarrow = A^b(x_n)[s_{k+2}-1] = 0$. We set $\Phi_{x_n}^{A_{s_{k+2}}\upharpoonright\varphi_{x_n}(x_n)}(x_n)\downarrow$, so that $A^b(x_n)[s_{k+2}] = 1$. There was no enumeration into A below $r(n+1)$ at any stage $s_{k+1} < t \leq s_{k+2}$, so that at stage $s_{k+2} + 1$ there are at least $(\varphi_n(x_n) - \frac{k}{2} - 1)$-many numbers greater than $r(n)$ and less than $\varphi_{x_n}(x_n)$ available to enumerate into A. This completes our induction.

Since we can only reach a stage s_k with k even if a number less than $\varphi_n(x_n)$ enters $\emptyset'$, and since we have left room to enumerate into A in the desired interval at each such stage, it follows that there can be only finitely many stages s_k, as desired. □

□

A similar proof can be used to show that every 2-generic A is such that $A^b \not\leq_{bT} A_{tt}$.

Finally, we note that the minijump operator developed by Ershov [6] works on the *pm* degrees in a manner similar to A_{tt} on the truth-table degrees (See Odifreddi [11], Volume II page 732).

8 Further Study

There is considerable room left to explore in the study of the bounded jump. We can examine to what degree do important results for the Turing jump on the Turing degrees also hold for the bounded jump on the bounded Turing degrees, particularly in cases where these results do not hold for the Turing jump on the bounded Turing degrees.

For example, Sacks [13] proved that for every Σ_2 set $X \geq_T \emptyset'$ there is a c.e. set Y such that $Y' \equiv_T X$. Csima, Downey, and Ng [5] proved that Sacks jump inversion does not hold for the Turing jump on the bounded Turing degrees. We do not know if Theorem 6.2 holds if we add the requirement that Y is c.e.

We can also look at concepts related to the Turing jump. We say that a set X is bounded-high if $X^b \geq_{bT} \emptyset^{2b}$ and bounded-low if $X^b \leq_{bT} \emptyset^b$. We can then attempt to characterize which sets are bounded-high and bounded-low. Finally, the jumps for the truth-table and bounded truth-table degrees developed by Gerla [7] could be considered in more detail.

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