INTERMEDIATE BIMODULES FOR CROSSED PRODUCTS OF VON NEUMANN ALGEBRAS

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CROSSED PRODUCTS

$M \subseteq B(H)$ is a factor. A discrete group $G$ acts on $M$ by automorphisms $\alpha_g$ with $\alpha_g$ outer for $g \neq e$ (there is no $u \in \mathcal{U}(M)$ so that $\alpha_g(x) = uxu^*$ for $x \in M$). Outerness is equivalent to:

if $t \in M$ and $xt = t\alpha_g(x)$ for $x \in M$, then $t = 0$.

$\pi : M \to B(H \otimes \ell^2(G))$ is

$$\pi(x)(\xi \otimes \delta_h) = \alpha_{h^{-1}}(x)\xi \otimes \delta_h, \quad \xi \in H, \ x \in M, \ h \in G.$$ 

A representation $g \mapsto u_g$ is

$$u_g(\xi \otimes \delta_h) = \xi \otimes \delta_{gh}, \quad \xi \in H, \ g, h \in G.$$ 

$M \rtimes_\alpha G$ is generated by

$$\{\pi(x) : x \in M\} \cup \{u_g : g \in G\}$$

$$u_g\pi(x)u_g^* = \pi(\alpha_g(x)), \quad x \in M, \ g \in G.$$
We also write this as

\[ M \rtimes_\alpha G = \{ xg : x \in M, \ g \in G \} \]

subject to the relations \( gx = \alpha_g(x)g \).

There is a normal faithful conditional expectation \( E : M \rtimes_\alpha G \to M \) defined on generators by

\[
E(xg) = \begin{cases} 
  xg, & g = e, \\
  0, & g \neq e.
\end{cases}
\]

Each \( x \in M \rtimes_\alpha G \) has a Fourier series

\[
x \sim \sum_{g \in G} x_g g
\]

where

\[ x_g = E(xg^{-1}). \]

The series does not converge in the usual von Neumann algebra topologies so we need the Bures topology or \( B \)-topology (Mercer).
THE B-TOPOLOGY

$M \subseteq N$ with a faithful normal conditional expectation $E : N \to M$ (think $E : M \rtimes_{\alpha} G \to M$). For each normal state $\phi$ on $M$ there is a seminorm

$$x \mapsto (\phi \circ E(x^*x))^{1/2}, \quad x \in N.$$  

The B-topology is defined by these seminorms and $x_\lambda \to x$ in the $B$-topology if and only if

$$\lim_{\lambda} \phi \circ E((x_\lambda - x)^*(x_\lambda - x)) = 0, \quad \phi \in M^*,$$

equivalently

$$w^* - \lim_{\lambda} E((x_\lambda - x)^*(x_\lambda - x)) = 0.$$

(Mercer) For each $x \in M \rtimes_{\alpha} G$, its Fourier series converges in the $B$-topology to $x$. 
EXAMPLE

The $B$-topology is closely connected to the $w^*$-topology. Let $x_\lambda \to x$ in the $B$-topology for elements in $M \rtimes_\alpha G$. Write the Fourier series as

$$x_\lambda = \sum_{g \in G} x_{\lambda, g} g, \quad x = \sum_{g \in G} x_g g.$$

For a particular $g \in G$ and a normal state $\phi$ on $M$,

$$|\phi(x_{\lambda, g} - x_g)| = |\phi \circ E((x_\lambda - x)g^{-1})|$$

$$\leq (\phi \circ E((x_\lambda - x)^*(x_\lambda - x)))^{1/2} \quad \text{(C-S)}$$

$$\to 0 \quad \text{(limit over } \lambda)$$

so $x_g = w^* - \lim_\lambda x_{\lambda, g}$. 
Theorem (H. Choda)

Let $M$ be a factor and let $G$ act on $M$ by outer automorphisms. If $N$ is a von Neumann algebra with $M \subseteq N \subseteq M \rtimes_{\alpha} G$ and there is a faithful normal conditional expectation

$$E : M \rtimes_{\alpha} G \to N,$$

then there is a subgroup $H \subseteq G$ so that

$$N = M \rtimes_{\alpha} H.$$

For $M$ with separable predual, Izumi-Longo-Popa showed that the existence of such a conditional expectation is always true.

Question

What can be said about intermediate $w^*$-closed $M$-bimodules? Easier to answer for $B$-closed $M$-bimodules, a subclass of the $w^*$-closed ones.
Recall that a masa $A$ in a factor $M$ is Cartan if its unitary normalizers in $M$ generate $M$, and there is a normal conditional expectation onto $A$. Put $M$ in standard position on $H$ with a cyclic and separating vector $\xi$ and modular conjugation $J$.

$$\mathcal{A} = (A \cup JAJ)^{''},$$

masa in $B(H)$. Each $w^*$-closed $A$-bimodule $X \subseteq M$ gives an $\mathcal{A}$-invariant subspace $X\xi \parallel \cdot \parallel^2$ and the projection $P_X$ onto it is in $\mathcal{A}$.

**Theorem (Cameron-Pitts-Zarikian)**

$X \mapsto P_X$ is a bijection from $B$-closed $A$-bimodules to projections in $\mathcal{A}$.
For $x = \sum_{g \in G} x_g g \in M \rtimes_\alpha G$, let

$$\text{supp}(x) = \{ g \in G : x_g \neq 0 \} \subseteq G.$$ 

For an $M$-bimodule $X \subseteq M \rtimes_\alpha G$, let

$$S_X = \bigcup \{ \text{supp}(x) : x \in X \} \subseteq G$$

and for $S \subseteq G$, let

$$X_S = \{ x \in M \rtimes_\alpha G : \text{supp}(x) \subseteq S \} \subseteq M \rtimes_\alpha G.$$
Theorem (Cameron-S.)

\[ S \mapsto X_S = \text{span}^B \{ Mg : g \in S \} \]

is a bijection between subsets \( S \subseteq G \) and \( B \)-closed \( M \)-bimodules in \( M \rtimes_\alpha G \). The inverse map is

\[ X \mapsto S_X. \]

In particular the \( B \)-closed intermediate von Neumann algebras are

\[ M \rtimes_\alpha H \]

for subgroups \( H \subseteq G \).

If \( G \) has the approximation property (AP) of Haagerup-Kraus, then the \( w^* \)-closed and \( B \)-closed \( M \)-bimodules coincide.
The predual of $L(G)$ is the Fourier algebra $A(G)$ consisting of the functions $g \mapsto \langle \ell_g \xi, \eta \rangle$ where $\xi, \eta \in \ell^2(G)$. A completely bounded multiplier is a function $f$ on $G$ such that $f \cdot A(G) \subseteq A(G)$ and the associated multiplication operator $M_f$ is completely bounded.

The space of completely bounded multipliers is denoted by $M_0(A(G))$ and is a Banach space in the $cb$-norm. It is a dual space with predual denoted by $Q(G)$.

$G$ is said to have the approximation property (AP) if the constant function 1 is in the $w^*$-closure of the space of finitely supported functions on $G$. (Haagerup-Kraus). This is a large class of groups that contains all amenable discrete groups and also the weakly amenable groups of Cowling-Haagerup.

The AP class is closed under semidirect products so is strictly larger than the weakly amenable class.
Lemma

Let $M$ be a factor and let $X$ be a $w^*$-closed $M$-bimodule in $M \rtimes_\alpha G$. If $g_0 \in G$ is such that $x_{g_0} \neq 0$ for some

$$x = \sum_{g \in G} x_g g \in X,$$

then $g_0 \in X$. In particular

$$\bigcup \{\text{supp}(x) : x \in X\} \subseteq X.$$
SPECIAL CASE

Assume $M$ is hyperfinite and take an amenable subgroup $\Gamma \subseteq \mathcal{U}(M)$ that generates $M$. Fix an invariant mean and write it as

$$\int_{\Gamma} \cdot du, \quad u \in \Gamma.$$ 

Take $x = \sum x_g g \in X$ with $x_{g_0} \neq 0$. We want to pick out $g_0$. Using basic properties of factors we can first make $x_{g_0}$ a nonzero projection and then 1. Consider

$$y = \int_{\Gamma} uxg_0^{-1}(u^*) \, du \in X.$$ 

Since

$$uxg \alpha g_0^{-1}(u^*) = uxg \alpha gg_0^{-1}(u^*) g,$$

the $g_0$-coefficient of $y$ is 1 while for $g \neq g_0$

$$y_g = \int_{\Gamma} uxg \alpha gg_0^{-1}(u^*) \, du.$$
By invariance

\[ vyg\alpha_{gg_0}^{-1}(v^*) = yg, \quad v \in \Gamma, \]
\[ vyg = yg\alpha_{gg_0}^{-1}(v), \quad v \in \Gamma, \]
\[ xyg = yg\alpha_{gg_0}^{-1}(x), \quad x \in M. \]

Thus \( y_g = 0 \) for \( g \neq g_0 \) since \( \alpha_{gg_0}^{-1} \) is outer.
We get \( y = g_0 \in X. \)

The general case requires a different type of averaging.
**CHRISTENSEN-SINCLAIR THEOREM**

Let $L_{cb}(M, M)$ be the cb-maps of $M$ to $M$, $L_{cb}(M, M)_M$ the subspace of right $M$-module maps, which are $x \mapsto tx$ for some $t \in M$.

Take all strings $\beta = (m_j)$ where $\sum_j m_j m_j^* = 1$, $m_j \in M$.

For a cb-map $\phi : M \to M$, form

$$\phi^\beta(x) = \sum_j \phi(xm_j)m_j^*, \quad x \in M.$$ 

**Theorem (Chris.-Sin.)**

(i) There exists a contractive projection

$$\rho : L_{cb}(M, M) \to L_{cb}(M, M)_M.$$ 

(ii) There is a net $(\beta)$ so that

$$\rho\phi(x) = w^* - \lim_{\beta} \phi^\beta(x), \quad x \in M.$$
For an $x = \sum_{g \in G} x_g g$ with $x_{g_0} = 1$, form

$$\sum_j m_j x_{\alpha_g^{-1}(m_j^*)} = \sum_{g \in G} \left( \sum_j m_j x_g \alpha_{gg_0^{-1}}(m_j^*) \right) g,$$

and take the limit over $(\beta)$ to get rid of $x_g$ for $g \neq g_0$, and leaving $g_0$. This is achieved by applying the C-S theorem to the cb-map

$$\phi(x) = \alpha_{g_0g^{-1}}(x) \alpha_{g_0g^{-1}}(x_g), \quad x \in M.$$
GENERAL CHODA THEOREM

If $M \subseteq N \subseteq M \rtimes_{\alpha} G$, $M$ a factor, $N$ a von Neumann algebra then

$$H = \{g \in G : g \in N\}$$

is a subgroup and

$$M \rtimes_{\alpha, r} H = \overline{\text{span}}\|\cdot\|\{Mg : g \in H\}$$

$$\subseteq N \subseteq \overline{N^B} = M \rtimes_{\alpha} H.$$ 

Taking $w^*$-closures gives

$$N = M \rtimes_{\alpha} H$$

so all intermediate von Neumann algebras are $M \rtimes_{\alpha} H$ for subgroups $H$ and there is always a faithful normal conditional expectation

$$E : M \rtimes_{\alpha} G \to M \rtimes_{\alpha} H = N$$

without a separable predual assumption.
NONFACTOR CASE

Let \( \{ z_g : g \in G \} \) be a set of central projections in \( M \) and let

\[
N = \overline{\text{span}}^{w^*} \{ Mz_g g : g \in G \} \subseteq M \rtimes_\alpha G.
\]

When is \( N \) a von Neumann algebra?

(i) \textit{Unital}: so \( z_e = 1 \).

(ii) \textit{Self-adjoint}: so

\[
(Mz_g g)^* = g^{-1} Mz_g = M\alpha_{g^{-1}}(z_g)g^{-1}
\]

\[
= Mz_{g^{-1}} g^{-1} \implies \alpha_{g^{-1}}(z_g) = z_{g^{-1}}.
\]

(iii) \textit{Closed under multiplication}: so

\[
z_g g z_h h = z_g \alpha_g(z_h) gh \implies z_g \alpha_g(z_h) \leq z_{gh}.
\]
Theorem (Cameron-S.)

*Every intermediate von Neumann algebra is*

\[
N = \overline{\text{span}}^{w^*} \{ Mz_g g : g \in G \}
\]

*where \( \{ z_g : g \in G \} \) satisfy (i)-(iii). There is a faithful normal conditional expectation \( E : M \rtimes_\alpha G \to N \) given by*

\[
E(\sum x_g g) = \sum x_g z_g g.
\]
When $M$ is a factor, $z_g \in \{0, 1\}$. Put

$$H = \{ g \in G : z_g = 1 \}.$$

(ii) $\alpha_{g^{-1}}(Z_g) = Z_{g^{-1}}$

$\implies$ $H$ is closed under inverses,

(iii) $z_g \alpha_g(z_h) \leq z_{gh}$

$\implies$ $H$ is closed under multiplication

so the intermediate algebras of Choda reappear as

$M \rtimes_{\alpha} H$. 
Let $A$ be abelian. It is Cartan in $A \rtimes_\alpha G$.
Assume $\tau$ is a trace on $A$ and the $\alpha_g$ are trace preserving. Also the action of $G$ is free and ergodic so $A \rtimes_\alpha G$ is a $\text{II}_1$ factor.

Let $\xi$ be a tracial vector for $A$. Then $\xi \otimes \delta_e$ is a cyclic and separating vector for $A \rtimes_\alpha G$ and if

$$N = \overline{\text{span}}^{w^*} \{ Az_g \, g : g \in G \}$$

then the projection $P_N$ corresponding to $N$ has range

$$\overline{\text{span}}^\| \cdot \|^2 \{ Az_g \xi \otimes \delta_g : g \in G \}.$$
MERCER’S EXTENSION THEOREM

Recall that a masa $A$ in a factor $M$ is Cartan if its unitary normalizers in $M$ generate $M$.

Theorem (Mercer, Cameron-Pitts-Zarikian)

Let $A$ be a Cartan masa in a factor $M$. Let $X$ be a $w^*$-closed $A$-bimodule that generates $M$ and let $\theta : X \to X$ be a $w^*$-continuous isometric surjective isomorphism such that $\theta|_A$ is a $\ast$-automorphism and

\[ \theta(a_1 x a_2) = \theta(a_1) \theta(x) \theta(a_2), \quad x \in X, \quad a_i \in A. \]

Then $\theta$ extends to a $\ast$-automorphism $\overline{\theta}$ of $M$. 
Theorem (Cameron-S.)

Let $M$ be a factor. Let $X$ be a $w^*$-closed $M$-bimodule that generates $M \rtimes_\alpha G$ and let $\theta : X \to X$ be a $w^*$-continuous isometric surjective isomorphism such that $\theta|_M$ is a $*$-automorphism and

$$\theta(m_1 xm_2) = \theta(m_1)\theta(x)\theta(m_2), \quad x \in X, \ m_i \in M.$$ 

Then $\theta$ extends to a $*$-automorphism $\overline{\theta}$ of $M \rtimes_\alpha G$. 
SKETCH

\[ S = \{ g \in G : g \in X \}, \]
\[ Y = \text{span} \{ Mg : g \in S \} \subseteq M \rtimes_{\alpha, r} G, \]

and

\[ Y \subseteq X. \]

Apply \( \theta \) to

\[ gx = \alpha_g(x)g, \quad x \in M, \ g \in S, \]

\[ \theta(g)\theta(x) = \theta(\alpha_g(x))\theta(g), \quad x \in M, \ g \in S. \]

This gives

\[ \theta(g)^* \theta(g), \ \theta(g)\theta(g)^* \in M' \cap (M \rtimes_{\alpha} G) = \mathbb{C}1, \]

so \( \theta(g) \) is a unitary in \( M \rtimes_{\alpha} G \) that normalizes \( M \). Then \( \theta(g) = uh \) for some \( h \in G \) and \( u \in \mathcal{U}(M) \). We get \( h \in S \) so \( \theta \) maps \( Y \) to \( Y \).
Recall that a $\mathbb{C}^*$-algebra $A$ is said to be the $\mathbb{C}^*$-envelope of a unital operator space $X$ if there is a completely isometric unital embedding $\iota : X \rightarrow A$ so that $\iota(X)$ generates $A$, and if $B$ is another $\mathbb{C}^*$-algebra with a completely isometric unital embedding $\iota' : X \rightarrow B$ whose range generates $B$, then there is a $\ast$-homomorphism $\pi : B \rightarrow A$ so that $\pi \circ \iota' = \iota$ (which entails surjectivity of $\pi$). Every unital operator space has a unique $\mathbb{C}^*$-envelope denoted $C^*_{\text{env}}(X)$.
Lemma

The $C^*$-envelope of

$$Y = \overline{\text{span}}\|\cdot\| \{Mg : g \in S\} \subseteq M \rtimes_{\alpha,r} G$$

(where $S$ is the set of group elements in $X$) is $M \rtimes_{\alpha,r} G$ and

$\theta : Y \to Y$ extends to a $\ast$-automorphism

$$\phi : M \rtimes_{\alpha,r} G \to M \rtimes_{\alpha,r} G.$$  

Moreover, there is an automorphism $\pi$ of $G$ such that

$$\phi(Mg) = M\pi(g), \quad g \in G.$$
Lemma

Let $\phi$ be a $\ast$-automorphism of $M \rtimes_{\alpha, r} G$ such that $\phi(M) = M$. Then $\phi$ extends to a $\ast$-automorphism $\overline{\theta}$ of $M \rtimes_{\alpha} G$. On Fourier series,

$$\overline{\theta}(\sum x_g g) = \sum \phi(x_g)\phi(g).$$

To summarize: the original $\theta : X \rightarrow X$ was restricted to $Y \subseteq X$, then extended to $\phi : M \rtimes_{\alpha, r} G \rightarrow M \rtimes_{\alpha, r} G$, and extended again to $\overline{\theta} : M \rtimes_{\alpha} G \rightarrow M \rtimes_{\alpha} G$. 
