Regular Representations of Lattice Ordered Semigroups

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COSy

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**Theorem (Sz. Nagy 1961)**

Let $T_1 \in \mathcal{B}(\mathcal{H})$ be a contraction. Then there exists a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$ and a unitary operator $U_1 \in \mathcal{B}(\mathcal{K})$ such that $T_1^k = P_\mathcal{H}U_1^k|_\mathcal{H}$ for all $k \geq 0$. 

Equivalently, we can consider a contractive representation $T : \mathbb{N} \to \mathcal{B}(\mathcal{H})$ where $T(k) = T^k$. Nagy's result implies that there exists a unitary representation $U : \mathbb{Z} \to \mathcal{B}(\mathcal{K})$ such that for all $k \geq 0$, $T(k) = P_\mathcal{H}U(k)|_\mathcal{H}$.
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Equivalently, we can consider a contractive representation $T : \mathbb{N} \to \mathcal{B}(\mathcal{H})$ where $T(k) = T_1^k$. Nagy’s result implies that there exists a unitary representation $U : \mathbb{Z} \to \mathcal{B}(\mathcal{K})$ such that for all $k \geq 0$, $T(k) = P_{\mathcal{H}}U(k)|_{\mathcal{H}}$

Question

What happens to contractive representations of other semigroups?
Theorem

If $G$ is a group, let $S : G \to \mathcal{B}(\mathcal{H})$ be a unital contractive map. Then the following are equivalent:

1. There exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a unitary representation $U : G \to \mathcal{B}(\mathcal{K})$ such that $S(g) = P_\mathcal{H} U(g)|_\mathcal{H}$.

2. For any $g_1, \ldots, g_n \in G$, the operator matrix $[S(g_i g_j^{-1})] \in M_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^n)$ is positive.

We call such map completely positive definite.
**Theorem**

*If G is a group, let S : G → B(ℋ) be a unital contractive map. Then the following are equivalent:*

1. *There exists a Hilbert space K ⊇ ℋ and a unitary representation U : G → B(K) such that S(g) = PℋU(g)|ℋ.*

2. *For any g₁, ⋯ , gₙ ∈ G, the operator matrix [S(gᵢg⁻¹ⱼ)] ∈ Mₙ(B(ℋ)) ≅ B(ℋⁿ) is positive.*

*We call such map completely positive definite.*

Let P be a semigroup inside a group G and T : P → B(ℋ) be a representation. If we can extend T to a completely positive definite map Ŵ : G → B(ℋ), then T has a unitary dilation. We call such representation T **completely positive definite**.
**Example**

1. \( T(k) = T_1^k \) for some contraction \( T_1 \) defines a representation on \( \mathbb{N} \subseteq \mathbb{Z} \). Sz.Nagy showed that the extension

\[
\tilde{T}(k) = \begin{cases} 
T_1^k & k \geq 0 \\
(T_1^*)^{-k} & k < 0 
\end{cases}
\]

is a completely positive map on \( \mathbb{Z} \)

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2. \( T(k_1, k_2) = T_1^{k_1} T_2^{k_2} \) for two commuting contractions \( T_1, T_2 \) defines a representation on \( \mathbb{N}^2 \subseteq \mathbb{Z}^2 \). Then Ando’s theorem shows that \( T \) is a completely positive definite. However, we do not know its completely positive definite extension \( \tilde{T} \) explicitly.
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3. \( T(k_1, k_2, k_3) = T_1^{k_1}T_2^{k_2}T_3^{k_3} \) for three commuting contractions \( T_1, T_2, T_3 \) defines a representation on \( \mathbb{N}^3 \subseteq \mathbb{Z}^3 \). By Parrott’s example, \( T \) might not be completely positive definite.
Definition

Let $G$ be a group

1. A unital semigroup $P \subseteq G$ is called a **cone**.
2. A cone $P$ is **spanning** if $PP^{-1} = G$, and is **positive** if $P \cap P^{-1} = \{e\}$. 
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1. A unital semigroup $P \subseteq G$ is called a **cone**.
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**Definition**

A positive spanning cone $P$ induces a partial order where $x \leq y$ whenever $x^{-1}y \in P$. $P$ is called a **lattice ordered semigroup** in $G$ if for every $g \in G$, $gPg^{-1} \subseteq P$ and the induced partial order is a lattice (In other words, for any two elements $g, h \in G$, they have a least upper bound $g \lor h$ and a greatest lower bound $g \land h$).
Proposition

(Properties of Lattice Ordered Group) Let $P$ be a lattice ordered semigroup in a group $G$. Then for any $a, b, c \in G$

1. $a(b \lor c) = (ab) \lor (ac)$ and $(b \lor c)a = (ba) \lor (ca)$. A similar distributive law holds for $\land$. 


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(Properties of Lattice Ordered Group) Let $P$ be a lattice ordered semigroup in a group $G$. Then for any $a, b, c \in G$

1. $a (b \vee c) = (ab) \vee (ac)$ and $(b \vee c)a = (ba) \vee (ca)$. A similar distributive law holds for $\wedge$.

2. $(a \wedge b)^{-1} = a^{-1} \vee b^{-1}$ and similarly $(a \vee b)^{-1} = a^{-1} \wedge b^{-1}$.
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5. If $a, b, c \in P$, then $a \wedge (bc) \leq (a \wedge b)(a \wedge c)$. 
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5. If $a, b, c \in P$, then $a \land (bc) \leq (a \land b)(a \land c)$.

6. For any $g \in G$, there exists a unique $g_+, g_- \in P$ such that $g_+ \land g_- = e$ and $g = g_+g_-^{-1} = g_-^{-1}g_+$. 
### Example

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2. If $P_i$ is a totally ordered semigroup in $G_i$, then their product $\prod P_i$ is a lattice ordered semigroup in $\prod G_i$.
3. $C^+([0, 1])$ is a lattice ordered semigroup in $C([0, 1])$. 

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**Background**

**Lattice Ordered Groups**

**Main Result**

**Regular Representation**
Example

1. For any $k \geq 0$, $\mathbb{N}^k$ is a lattice ordered semigroup in $\mathbb{Z}^n$.
2. If $P_i$ is a totally ordered semigroup in $G_i$, then their product $\prod P_i$ is a lattice ordered semigroup in $\prod G_i$.
3. $C^+([0, 1])$ is a lattice ordered semigroup in $C([0, 1])$.
4. Let $X$ be a totally ordered set and $G$ be the set of all order-preserving automorphisms on $X$. Define $P = \{\sigma \in G : \sigma(x) \geq x, \forall x \in X\}$, then $P$ is a non-abelian lattice ordered semigroup in $G$. 
Example

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5. $\mathbb{F}^+_n$ is NOT a lattice ordered semigroup in $\mathbb{F}_n$. 
If $T : P \to \mathcal{B}(\mathcal{H})$ has a completely positive extension to $\tilde{T} : G \to \mathcal{B}(\mathcal{H})$ so that it has a unitary dilation $U : G \to \mathcal{B}(\mathcal{K})$. Then

$$\tilde{T}(g) = P_{\mathcal{H}}U(g)|_{\mathcal{H}}$$

$$= P_{\mathcal{H}}U(g^{-1}g_+)|_{\mathcal{H}}$$

$$= P_{\mathcal{H}}U(g_-)^*U(g_+)|_{\mathcal{H}}.$$

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**Definition**

A contractive representation $T : P \to \mathcal{B}(\mathcal{H})$ is called **regular** if the extension $\tilde{T}(g) = T(g-)^*T(g+)$ on $G$ is completely positive definite.
Theorem (Brehmer 1961)

If $T_1, \cdots, T_k$ is a family of commuting contractions. Define a representation $T : \mathbb{N}^k \to \mathcal{B}(\mathcal{H})$ by $T(e_i) = T_i$. Then $T$ is regular if and only if for any $J \subseteq \{1, 2, \cdots, k\}$, the operator

$$\sum_{U \subseteq J}(-1)^{|U|}T(e_U)^*T(e_U)$$

is positive.
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Example

For $k = 2$, let $T_1, T_2$ be two commuting contractions. Brehmer’s condition is equivalent to

$$I - T_1^* T_1 - T_2^* T_2 + (T_1 T_2)^* T_1 T_2 \geq 0.$$
Corollary (Brehmer 1961)

Let $T : \mathbb{N}^k \to \mathcal{B}(\mathcal{H})$ be a contractive representation where $T(e_i) = T_i$, then $T$ is regular if one of the following conditions is met:
Corollary (Brehmer 1961)

Let $T : \mathbb{N}^k \to \mathcal{B}(\mathcal{H})$ be a contractive representation where $T(e_i) = T_i$, then $T$ is regular if one of the following conditions is met:

1. $T$ is an isometric representation.
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1. $T$ is an isometric representation.
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3. $T_1, \cdots, T_k$ forms a column contraction, in the sense that $\sum_{i=1}^k T_i^* T_i \leq I$. 

Question

Can we extend this to representations on lattice ordered semigroups?
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Can we extend this to representations on lattice ordered semigroups?
**Definition** (Davidson & Fuller & Kakariadis 2014)

A contractive representation $T$ of a lattice ordered semigroup $P$ is called **Nica-covariant** if for any $p, q \in P$ with $p \wedge q = e$, $T_p$ commutes with $T_q^*$. 

**Question** (Davidson & Fuller & Kakariadis, Question 2.5.11)

Is contractive Nica-covariant representation of abelian lattice ordered semigroups automatically regular? It is known to be regular in some cases. For example, $N_k$ [Brehmer], totally ordered semigroups [Mlak, 1966], and products of totally ordered semigroups [Fuller, 2013].
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**Definition**

Let \( T : P \to \mathcal{B}(\mathcal{H}) \) be a contractive representation of a lattice ordered semigroup \( P \). \( T \) is called **column contractive** if for any \( p_1, \ldots, p_n \in P \) where \( p_i \neq e \) and \( p_i \land p_j = e \) for all \( i \neq j \),

\[
\sum_{i=1}^{n} T^* p_i T p_i \leq I.
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**Definition**

Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation of a lattice ordered semigroup $P$. $T$ is called **column contractive** if for any $p_1, \ldots, p_n \in P$ where $p_i \neq e$ and $p_i \land p_j = e$ for all $i \neq j$,

$$\sum_{i=1}^{n} T_{p_i}^* T_{p_i} \leq I.$$  

Dually, $T$ is called **row contractive** if for such $p_i$,

$$\sum_{i=1}^{n} T_{p_i} T_{p_i}^* \leq I.$$
For a contractive representation $T : P \rightarrow \mathcal{B}(\mathcal{H})$ on a lattice ordered semigroup $P$ in $G$, define $\tilde{T}(g) = T(g^-)T(g^+)$. Recall, $T$ is regular when $\tilde{T}$ is a completely positive definite map on $G$. 
For a contractive representation $T : P \to \mathcal{B}(\mathcal{H})$ on a lattice ordered semigroup $P$ in $G$, define $\tilde{T}(g) = T(g_-)T(g_+)$. Recall, $T$ is regular when $\tilde{T}$ is a completely positive definite map on $G$.

**Theorem (L.)**

Let $P$ be a lattice ordered semigroup and $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then $T$ is regular if and only if for any $p_1, \ldots, p_n \in P$ and $g \in P$ where $g \wedge p_i = e$ for all $i = 1, 2, \ldots, n$, we have

$$\left[ T(g)\tilde{T}(p_ip_j^{-1})T(g) \right] \leq \left[ \tilde{T}(p_ip_j^{-1}) \right].$$

(*)
Proof Sketch: Assuming $T$ is a contractive representation that satisfies equation $(\ast)$.

**Lemma**

If $p_1, \ldots, p_n \in P$ satisfies $p_i \land p_j = e$ for any $i \neq j$, then $[\tilde{T}(p_ip_j^{-1})] \geq 0$. 
**Proof Sketch:** Assuming $T$ is a contractive representation that satisfies equation (\ast).

**Lemma**

If $p_1, \ldots, p_n \in P$ satisfies $p_i \land p_j = e$ for any $i \neq j$, then $[\tilde{T}(p_ip_j^{-1})] \geq 0$.

**Lemma**

Let $p_1, \ldots, p_n \in P$ where for any $J \subseteq \{1, 2, \ldots, n\}$ with $|J| > k$, $\land_{j \in J} p_j = e$. Let $g = \land_{j=1}^k p_j$ and define $q_1 = p_1g^{-1}, \ldots, q_k = p_kg^{-1}, q_{k+1} = p_{k+1}, \ldots, q_n = p_n$. Then $[\tilde{T}(p_ip_j^{-1})] \geq 0$ if $[\tilde{T}(q_iq_j^{-1})] \geq 0$. 
Corollary (L.)

Let $T: P \to \mathcal{B}(\mathcal{H})$ be a representation of a lattice ordered semigroup. Then $T$ is regular if one of the following conditions is met:
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1. \( T \) is an isometric representation.
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Let $T : P \rightarrow \mathcal{B}(\mathcal{H})$ be a representation of a lattice ordered semigroup. Then $T$ is regular if one of the following conditions is met:

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3. $T$ is column contractive.
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Corollary

For a \( C^* \)-dynamical system \((A, \alpha, P)\), the semi-crossed product algebra given by Nica-covariant pairs agrees with that given by isometric Nica-covariant pairs. In other words,

\[
A \times_{\alpha}^{nc} P \cong A \times_{\alpha}^{nc,iso} P.
\]
Thank you