Haagerup-Schultz projections and decompositions

Ken Dykema

Department of Mathematics
Texas A&M University
College Station, TX, USA.

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In honor of George Elliott
Invariant subspaces

**Definition**
Let $A$ be a bounded linear operator on a Banach space $X$. An $A$-invariant subspace is a closed subspace $\mathcal{V} \subseteq X$ such that $A(\mathcal{V}) \subseteq \mathcal{V}$. We say $\mathcal{V}$ is nontrivial if $\mathcal{V} \neq \{0\}$ and $\mathcal{V} \neq X$.

One of the great problems in operator theory is to understand operators $A \in \mathcal{B}(X)$ by describing their invariant subspaces. (Do they even exist?)
The Invariant Subspace Problem

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- C. Read (1985) found such an example on the Banach space $\ell^1$.

Invariant Subspace Problem (ISP)
is unsolved for operators on separable, infinite dimensional Hilbert space $\mathcal{H}$: does every $A \in \mathcal{B}(\mathcal{H})$ have a nontrivial invariant subspace?
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- Aronszajn and Smith (1954): every compact operator on Banach space (of dimension $\geq 2$) has a nontrivial invariant subspace.
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**Definition (the name was coined by Douglas and Pearcy)**

For $A \in B(H)$, an $A$-hyperinvariant subspace is a closed subspace $V \subseteq H$ such that $S(V) \subseteq V$ for every $S \in B(H)$ satisfying $AS = SA$.

Example: For any $A \in B(H)$, the operator $A \oplus A \in B(H \oplus H)$ has $H \oplus 0$ as an invariant subspace, but it is not $(A \oplus A)$-hyperinvariant.
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**Definition (the name was coined by Douglas and Pearcy)**

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**The Hyperinvariant Subspace Problem (HISP)**

is unsolved for operators on separable, infinite dimensional Hilbert space $\mathcal{H}$: does every $A \in \mathcal{B}(\mathcal{H})$ that is not a scalar multiple of the identity operator have a nontrivial hyperinvariant subspace?
Subspaces / Projections

If $\mathcal{V} \subseteq \mathcal{H}$ is a closed subspace and $p \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{V}$, then $p$ is $A$-invariant if and only if $Ap = pAp$. Such $p$ are called $A$-invariant projections, and they are nontrivial if and only if $p \notin \{0, 1\}$. Similarly, projections onto $A$-hyperinvariant subspaces are called $A$-hyperinvariant projections.

Proposition

If $p$ is an $A$-hyperinvariant projection, then $p$ belongs to the von Neumann algebra $\mathcal{vN}(A)$ generated by $A$. 
Subspaces / Projections

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Hope: von Neumann algebra theory can help with the HISP.
Let $\mathcal{M}$ be a finite von Neumann algebra with normal faithful tracial state $\tau$.

**Brown measure**

The Brown measure $\nu_A$ of $A \in \mathcal{M}$ is a Borel probability measure on $\mathbb{C}$; we think of it as a sort of spectral distribution measure for $A$.

- $\text{supp} \, \nu_A \subseteq \text{spec}(A)$; equality holds in some cases, but not always.

- $\nu_A$ is characterized by $\tau(\log |A - \lambda 1|) = \int \log |z - \lambda| \, d\nu_A(z)$ ($\lambda \in \mathbb{C}$).

- In fact, $f(\lambda) := \tau(\log |A - \lambda 1|)$ is $\log \Delta(A - \lambda 1)$, the logarithm of the Fuglede–Kadison determinant (1954) of $A - \lambda 1$, and the Brown measure is $\frac{1}{2\pi}$ times the Laplacian of this function $f$ (in the sense of distributions).
In a finite von Neumann algebra $\mathcal{M}$ with trace $\tau$,

**Examples**

(a) If $A \in M_n(\mathbb{C})$ and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues (listed according to algebraic multiplicity), then $\nu_A = \frac{1}{n}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_n})$.

(b) If $N$ is normal operator (i.e., $N^*N = NN^*$), then $\nu_N = \tau \circ (\text{spectral measure of } N)$.

**Thm. [Brown] — Brown measure and invariant subspaces**

Suppose $p$ is an $A$-invariant projection and $p \in \mathcal{M}$, then we may write $A = \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}$, where $X = Ap = pAp$ and $Y = (1 - p)A = (1 - p)A(1 - p)$. Letting $\alpha = \tau(p)$, we have

$$\nu_A = \alpha \nu_X + (1 - \alpha)\nu_Y.$$
In a finite von Neumann algebra $\mathcal{M}$ with trace $\tau$, Brown's version of Lidskii's theorem:

Let $A \in \mathcal{M}$. Then

$$\tau(A) = \int_{\mathbb{C}} z \, d\nu_A(z).$$
In a finite von Neumann algebra $\mathcal{M}$ with trace $\tau$,

**Thm. [Haagerup–Schultz ’09]**

Let $A \in \mathcal{M}$. If $B$ is a Borel subset of $\mathbb{C}$ then there is a unique $A$-invariant projection $p_B \in \mathcal{M}$ such that

- $\tau(p_B) = \nu_A(B)$
- if $\nu_A(B) > 0$, then the Brown measure of $Ap_B = p_B Ap_B$ equals $\nu_A(B)^{-1} \nu_A \upharpoonright B$
- if $\nu_A(B) < 1$, then the Brown measure of $(1 - p_B)A = (1 - p_B)A(1 - p_B)$ equals $(1 - \nu_A(B))^{-1} \nu_A \upharpoonright B^c$.

Moreover, $p_B$ is $A$–hyperinvariant.

As a matrix of operators:

$$A = \begin{pmatrix} Ap_B & * \\ 0 & (1 - p_B)A \end{pmatrix}$$
Corollary

If \( A \in \mathcal{M} \), and if \( \nu_A \) is not of the form \( \delta_\lambda \) for some \( \lambda \in \mathbb{C} \), then \( A \) has a nontrivial hyperinvariant subspace.
Corollary

If $A \in \mathcal{M}$, and if $\nu_A$ is not of the form $\delta_{\lambda}$ for some $\lambda \in \mathbb{C}$, then $A$ has a nontrivial hyperinvariant subspace.

A special case of a Haagerup–Schultz projection

Suppose the Borel set $B$ is $D_r = \{|z| \leq r\}$. Then $p_{D_r}$ is the projection onto the subspace

$$\{\xi \in \mathcal{H} \mid \exists \xi_n \to \xi \text{ such that } \limsup_{n \to \infty} \|A^n \xi_n\|^{1/n} \leq r\}.$$ 

The above subspace is used in the proof of their general result, together with some free probability theory and ultrapower techniques.

Thm. [Haagerup, Schultz ’09]

The Brown measure of $A \in \mathcal{M}$ is equal to $\delta_0$ if and only if

$$\text{s.o.t.-} \lim_{n \to \infty} ((A^*)^n A^n)^{1/2n} = 0,$$

where s.o.t. stands for strong operator topology.
Definition

We say $A \in \mathcal{B}(\mathcal{H})$ is s.o.t.-quasinilpotent if

$$\text{s.o.t.-} \lim_{n \to \infty} ((A^*)^n A^n)^{1/2n} = 0,$$

holds, (so, for $A$ in a finite v.N. algebra, if and only if $\nu_A = \delta_0$).

Compare

$A \in \mathcal{B}(\mathcal{H})$ is quasinilpotent if any of the following equivalent conditions hold:

(i) $\text{spec}(A) = \{0\}$

(ii) $\lim_{n \to \infty} \|A^n\|^{1/n} = 0$

(iii) $\lim_{n \to \infty} \|(A^*)^n A^n)^{1/2n}\| = 0$.

To recapitulate: in finite von Neumann algebras, the hyperinvariant subspace problem is solved except for s.o.t.-quasinilpotent operators (and their translates).
**Definition**

We say $A \in B(\mathcal{H})$ is *s.o.t.-quasinilpotent* if

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**Compare**

$A \in B(\mathcal{H})$ is *quasinilpotent* if any of the following equivalent conditions hold:

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(ii) $\lim_{n \to \infty} \|A^n\|^{1/n} = 0$
(iii) $\lim_{n \to \infty} \|(A^*)^n A^n\|^{1/2n} = 0$.

To recapitulate: in finite von Neumann algebras, the hyperinvariant subspace problem is solved except for s.o.t.-quasinilpotent operators (and their translates). ($\exists$ nice examples of s.o.t.-quasinilpotent but not quasinilpotent . . .).
Some hyperinvariant subspaces of a particular quasinilpotent operator in a $\mathbb{II}_1$–factor

Thm[D., Haagerup ’04]

Let $T \in L(F_2) \subset B(\mathcal{H})$ be the quasinilpotent $\text{DT}$–operator. Then $T$ has a family of nontrivial hyperinvariant subspaces given for all $0 < t < 1$ by

$$H_t = \left\{ \xi \in \mathcal{H} \mid \limsup_{n \to \infty} \left( \frac{n}{e} \|T^n \xi\|^{2/n} \right) \leq t \right\}$$

$$= \left\{ \xi \in \mathcal{H} \mid \exists \xi_n \to \xi \text{ such that } \limsup_{n \to \infty} \left( \frac{n}{e} \|T^n \xi_n\|^{2/n} \right) \leq G(t) \right\}$$

for some particular function $G : [0, 1] \to [0, e]$. 
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In fact, we also have $\| T^n \| = \left( \frac{e}{n} \right)^{n/2}$.
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for some particular function $G : [0, 1] \to [0, e]$.

In fact, we also have $\|T^n\| = \left( \frac{e}{n} \right)^{n/2}$.

The proof depends on some very nice combinatorial results of P. Śniady (2003).
A candidate for an operator without nontrivial hyperinvariant subspaces (G. Tucci)

Tucci’s operators in the hyperfinite II$_1$–factor

Let $R = \bigotimes_{n=1}^{\infty} M_2(\mathbb{C}) \subseteq B(H)$ by the hyperfinite II$_1$-factor. Let

$$V_n = I \otimes \cdots \otimes I \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I \otimes I \otimes \cdots$$

for $n - 1$ factors $I$.

and for $0 < c < 1$, let $A_c = \sum_{n=1}^{\infty} c^n V_n$. 
Thm [Tucci, ’08]

$A_c$ generates the hyperfinite II$_1$–factor $R$, but for every positive sequence $(\gamma_n)$ and every $r > 0$, the hyperinvariant subspace

$$\{ \xi \in \mathcal{H} \mid \limsup_{n \to \infty} \gamma_n \| A_c^n \xi \|^{1/n} \leq r \}$$

is either $\{0\}$ or $\mathcal{H}$.
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Open Problem

Does Tucci’s operator $A_c$ have any nontrivial hyperinvariant subspaces?

(We know that no subspace whose projection belongs to a finite tensor power can be $A_c$–hyperinvariant.)
Another class of examples

Let \((X, \mu)\) be a probability space and let \(\alpha\) be an ergodic, probability measure preserving transformation of \((X, \mu)\). Let

\[ \mathcal{M} = L^\infty(X) \rtimes_\alpha \mathbb{Z} \]

be the crossed product von Neumann algebra (thus, a copy of the hyperfinite \(\text{II}_1\)–factor).

\(\mathcal{M}\) is generated by an embedded copy of \(L^\infty(X)\) and a unitary operator \(U\) such that for \(f \in L^\infty(X)\), \(U f U^{-1} = f \circ \alpha\).

**Prop. [D., Schultz ’09]**

Take \(f \in L^\infty(X)\) with \(f \geq 0\) and let \(A = U f \in \mathcal{M}\). If \(\log |f|\) is not integrable with respect to \(\mu\), then \(A\) is s.o.t.-quasinilpotent.

**Question**

Does \(A\) have any nontrivial hyperinvariant subspaces?
Upper triangular forms. Recall:

**Thm. [Schur]**

Every matrix $A \in M_n(\mathbb{C})$ is unitarily conjugate to an upper triangular matrix:

$$U^{-1}AU = \begin{pmatrix}
\lambda_1 & * & * & \cdots & * \\
0 & \lambda_2 & * & \cdots & * \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & * \\
& & & 0 & \lambda_{n-1} & * \\
0 & \cdots & \cdots & 0 & \lambda_n
\end{pmatrix}$$

**Corollary**

$A = N + B$ where $N = U \text{diag}(\lambda_1, \ldots, \lambda_n)U^{-1}$ is a normal operator having the same spectral data as $T$, and $B$ is nilpotent ($B^n = 0$).
Recall (von Neumann) every compact operator has a nontrivial invariant subspace.

**Theorem [Ringrose, ’62]**

If $T$ is a compact operator on separable Hilbert space, then there is a maximal increasing family of projections $(p_\lambda)_{0 \leq \lambda \leq 1}$ such that $p_0 = 0$, $p_1 = 1$ and $\forall \lambda \ T p_\lambda = p_\lambda T p_\lambda$.

Thus, letting $p_\lambda^- = \bigvee_{\mu < \lambda} p_\mu$, for each $\lambda$ either $p_\lambda = p_\lambda^-$ or $p_\lambda - p_\lambda^-$ has rank one.

Furthermore, setting $N = \sum_\lambda (p_\lambda - p_\lambda^-) T (p_\lambda - p_\lambda^-)$, we have $T = N + Q$, where $N$ is normal (in fact, diagonal) and $Q$ is quasinilpotent.

This entails that $N$ has the same spectrum (and multiplicities) as $T$ and $\text{spec}(Q) = \{0\}$. 
A Ringrose-type theorem on upper triangular forms in finite von Neumann algebras

Working in a diffuse, finite von Neumann algebra $\mathcal{M}$ (with specified normal faithful tracial state $\tau$),
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**Theorem [D., Sukochev, Zanin]**

Let $T \in \mathcal{M}$. Then there is an increasing family $(q_t)_{0 \leq t \leq 1}$ of $T$–hyperinvariant projections in $\mathcal{M}$ with $q_0 = 0$, $q_1 = 1$ and such that, letting $\mathcal{D} = v\mathcal{N}(\{q_t \mid 0 \leq t \leq 1\})$ and letting $N$ be the conditional expectation $E_{\mathcal{D}}(T)$ of $T$ onto $\mathcal{D}$, we have $T = N + Q$, where

- $N$ is normal
- $\nu_N = \nu_T$
- $Q$ is s.o.t.-quasinilpotent.

Why is this not a full analogue of Ringrose’s theorem? The family $(q_t)_{0 \leq t \leq 1}$ can have big “lumps”!
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Why is this not a full analogue of Ringrose’s theorem? The family $(q_t)_{0 \leq t \leq 1}$ can have big “lumps”!
Use a Peano curve \( \rho : [0, 1] \to (\text{a disk containing } \text{spec}(T)) \) and let \( q_t = P_{\rho([0,t])} \) be a Haagerup–Schultz projection.
Sketch of proof

Key idea of proof

Use a Peano curve $\rho : [0, 1] \to (\text{a disk containing } \text{spec}(T))$ and let $q_t = p_\rho([0,t])$ be a Haagerup–Schultz projection.

Let $D_n = \text{span} \{ q_k/2^n \mid 0 \leq k \leq 2^n \}$.

Then $D_n \subseteq D_{n+1}$ and $D = \bigcup_n D_n$.

By uniform continuity of $\rho$, taking conditional expectations we have $\|E_{D_n}(T) - E_D(T)\| \to 0$.

With respect to the minimal projections of $D_n$, $T$ is upper triangular and the Brown measures of $E_{D_n}(T)$ converge weakly to the Brown measure of $T$.

We can show that the Brown measure of $N := E_D(T)$ equals the Brown measure $\nu_T$ of $T$. 
It remains to show $Q := T - N$ is s.o.t.-quasinilpotent.

**Key Lemma**

$\Delta(T) = \Delta(E_{\mathcal{D}'}(T))$, where $\mathcal{D}'$ is the relative commutant of $\mathcal{D}$ in $\mathcal{M}$ and $E_{\mathcal{D}'}$ is the conditional expectation onto it.
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**Key Lemma**

$\Delta(T) = \Delta(E_{D'}(T))$, where $D'$ is the relative commutant of $D$ in $M$ and $E_{D'}$ is the conditional expectation onto it.

Now the Brown measure of $T - E_D(T)$ equals the Brown measure of $E_{D'}(T) - E_D(T)$ and using that $E_{D'}(T)$ and $E_D(T)$ commute, we show that that latter equals $\delta_0$. 
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Now the Brown measure of $T - E_\mathcal{D}(T)$ equals the Brown measure of $E_{\mathcal{D}'}(T) - E_\mathcal{D}(T)$ and using that $E_{\mathcal{D}'}(T)$ and $E_\mathcal{D}(T)$ commute, we show that that latter equals $\delta_0$.

Q.E.D.
Recall that the above result depended on a continuous (Peano) curve $\rho : [0, 1] \rightarrow (\text{a disk containing spec } T)$, which provided a spectral ordering for the upper triangular decomposition $T = N + Q$.

J. Noles showed that much more general spectral orderings, including the following:

**Thm. [Noles]**

One can choose any Borel isomorphism $\psi : [0, 1] \rightarrow \text{spec}(T)$ and construct a corresponding upper triangular form with respect to the increasing family $(p_{\psi([0,t])})_{0 \leq t \leq 1}$ of Haagerup-Schultz projections.
Measurable spectral ordering

Recall that the above result depended on a continuous (Peano) curve $\rho : [0, 1] \to (a \text{ disk containing } \text{spec } T)$, which provided a spectral ordering for the upper triangular decomposition $T = N + Q$. J. Noles showed that much more general spectral orderings, including the following:

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The proof uses some ideas from [DSZ] but avoids the need for continuity by directly constructing a spectral measure that generalizes the one implicitly generated by the Peano curve $\rho$. 
Affiliated to a finite von Neumann algebra

Recall, for a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, a (densely defined) closed operator $T$ is affiliated to $\mathcal{M}$ if, writing the polar decomposition $T = U|T|$, we have $U \in \mathcal{M}$ and all spectral projections of the positive operator $|T|$ lie in $\mathcal{M}$. If $\mathcal{M}$ is finite, then these form an algebra. [Murray, von Neumann].
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If $(\mathcal{M}, \tau)$ is a finite von Neumann algebra, then Brown measure is actually defined and has nice properties for unbounded operators $T$ affiliated to $\mathcal{M}$, provided that they satisfy

$$\tau(\log^+ |T|) < \infty.$$  \hspace{1cm} (We say $T \in \log(L^1)$ for this.)

This seems reasonable, recalling the definition:

$$\nu_A = \frac{1}{2\pi} \text{Lapl}_\lambda \tau(\log |A - \lambda 1|).$$ See [Haagerup, Schultz ’07].
Hyperinvariant subspaces for unbounded operators?

One possible definition

If $T$ is a densely defined, closed operator in $\mathcal{H}$ and $V \subseteq \mathcal{H}$ is a closed subspace, we say $V$ is $T$-hyperinvariant if for every bounded operator $S \in B(\mathcal{H})$ such that $ST \subseteq TS$, $V$ is invariant under $S$. 

It is easy to see that the projection onto the $T$-hyperinvariant subspace $V$ belongs to the von Neumann algebra generated by $T$ (namely, by all the spectral projections of $|T|$ together with the polar part).

However, it is not clear that that $V$ must be invariant under $T$, in the sense ([Albrecht, Vasilescu, 2003]) that $V \cap \text{dom}(T)$ is dense in $V$ and is mapped by $T$ into $V$.

Not even if $T$ is affiliated to a finite von Neumann algebra.

Dykema (TAMU)
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However, it is not clear that that $\mathcal{V}$ must be invariant under $T$, in the sense ([Albrecht, Vasilescu, 2003]) that $\mathcal{V} \cap \text{dom}(T)$ is dense in $\mathcal{V}$ and is mapped by $T$ into $\mathcal{V}$. Not even if $T$ is affiliated to a finite von Neumann algebra.
In fact, supposing $T$ is affiliated to a finite von Neumann algebra, it is not hard to see that the analogue of the Haagerup-Schultz projection for the disk of radius $r$,

$$\{\xi \in \mathcal{H} \mid \exists \xi_n \in \text{dom}(T^n), \xi_n \to \xi \text{ such that } \limsup_{n \to \infty} \|T^n \xi_n\|^{1/n} \leq r\}.$$ 

is $T$-hyperinvariant, but it is not clear that it is $T$-invariant.
In fact, supposing $T$ is affiliated to a finite von Neumann algebra, it is not hard to see that the analogue of the Haagerup-Schultz projection for the disk of radius $r$,

$$\{\xi \in \mathcal{H} \mid \exists \xi_n \in \text{dom}(T^m), \xi_n \to \xi \text{ such that } \limsup_{n \to \infty} \|T^n \xi_n\|^{1/n} \leq r\}.$$ 

is $T$-hyperinvariant, but it is not clear that it is $T$-invariant.

So, we have some trouble adapting the Haagerup–Schultz arguments to the case of unbounded operators $T$, even assuming $\tau(\log^+ |T|) < \infty$. 
Theorem [D., Noles, Sukochev, Zanin]

Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and suppose \(T\) is affiliated to \(\mathcal{M}\) and \(\tau(\log^+ |T|) < \infty\). Then there exists a finite von Neumann algebra \((\mathcal{M}_1, \tau_1)\) with a trace-preserving embedding \(\mathcal{M} \hookrightarrow \mathcal{M}_1\) and there exist \(N\) and \(Q\), affiliated to \(\mathcal{M}_1\), such that

\[
T = N + Q
\]

\[
\tau(\log^+ |N|) < \infty, \quad \tau(\log^+ |Q|) < \infty,
\]

\(N\) is normal, \(\nu_N = \nu_T\), \(\nu_Q = \delta_0\).
**Theorem** [D., Noles, Sukochev, Zanin]

Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra and suppose \(T\) is affiliated to \(\mathcal{M}\) and \(\tau(\log^+ |T|) < \infty\). Then there exists a finite von Neumann algebra \((\mathcal{M}_1, \tau_1)\) with a trace-preserving embedding \(\mathcal{M} \hookrightarrow \mathcal{M}_1\) and there exist \(N\) and \(Q\), affiliated to \(\mathcal{M}_1\), such that

\[
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\tau(\log^+ |N|) < \infty, \quad \tau(\log^+ |Q|) < \infty,
\]

\(N\) is normal, \(\nu_N = \nu_T\), \(\nu_Q = \delta_0\).

**Corollary**

On certain modules \(\mathcal{B}\) of operators affiliated to \(\text{II}_1\)-factor \(\mathcal{M}\) and all traces \(\phi\) on these modules, (i.e., so that \(\phi(xT) = \phi(Tx)\) for all \(T \in \mathcal{B}\) and \(x \in \mathcal{M}\)), \(\phi(T)\) depends only on \(\nu_T\).


