

# Improvements on the star discrepancy of $(t, s)$ -sequences

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## Abstract

The term low-discrepancy sequences is widely used to refer to  $s$ -dimensional sequences  $X$  satisfying the bound  $D^*(N, X) \leq c_s(\log N)^s + O((\log N)^{s-1})$ , where  $D^*$  denotes the usual star discrepancy. In this article, we are concerned with  $(t, s)$ -sequences in base  $b$ , one of the most famous families of low-discrepancy sequences along with Halton sequences. The constants  $c_s$  for  $(t, s)$ -sequences were first computed by Sobol' and Faure in special cases and then achieved in general form by Niederreiter in the eighties. Then, quite recently, Kritzer improved these constants for  $s \geq 2$  by a factor  $\frac{1}{2}$  for an odd base and  $\frac{b}{2(b+1)}$  for an even base  $b \geq 4$ . Our aim is to further improve the result in the case of an even base by a factor  $\frac{2(b+1)}{b} \left(\frac{b-1}{b}\right)^{s-1}$  ( $s \geq 2$ ), hence obtaining the ratio  $\frac{3}{2^{s-1}}$  for  $b = 2$ . Combining this estimate with best known  $t$ -values from the database MinT, we obtain new values for Niederreiter-Xing sequences where base 2 recovers supremacy on base 3 (in Kritzer's paper). Our proof relies on a method of Atanassov to bound the discrepancy of Halton sequences. We also investigate  $(t, 1)$ -sequences for which the approach of Kritzer does not work.

## 1 Introduction

In the number theoretic setting, low-discrepancy sequences are usually compared against each other by looking at the asymptotic behavior of bounds on their discrepancy. More precisely, the term low-discrepancy sequences refers to sequences  $X$  satisfying the bound  $D^*(N, X) \leq c_s(\log N)^s + O((\log N)^{s-1})$  (see Section 1.1). A better understanding of the properties of such bounds provides useful insights on the corresponding sequences in applications to numerical analysis and to (quasi-) Monte Carlo methods. Among many publications on both these areas, the reader can consult the recent monographs [3, 12, 18] and [14] for a complete overview.

Two well-known families of multi-dimensional low-discrepancy sequences are the Halton sequences [9] and the so-called  $(t, s)$ -sequences introduced by Sobol' [21], Faure [5] and generalized by Niederreiter [13]. Recently, there has been a renewed interest in Halton sequences due to an important improvement discovered by Atanassov [1]. His results

provided a drastic change in our understanding of how bounds on the discrepancy vary with the dimension  $s$  for these sequences (for readers who do not have access to the paper [1], we recommend [3, Theorem 3.36] where a detailed proof of [1, Theorem 2.1] is given in a slightly less general form and [23] for a complete investigation of the proofs of Theorem 2.1 and Theorem 2.2 from [1]). For  $(t, s)$ -sequences in dimension  $s \geq 2$ , another recent improvement was provided by Kritzer in [11], who was able to reduce by a factor of about two the leading constant  $c_s$  obtained in [13]. His results are presented in Section 1.3 below. In the same paper, Kritzer also states a conjecture for  $(t, s)$ -sequences in even bases that would substantially improve his own bounds. A careful comparison of these improved bounds — through the constant  $c_s$  — for both Halton sequences and  $(t, s)$ -sequences can be found in [7, Section 2.3].

One of the objectives of our study is to closely approach the conjecture of Kritzer for even bases (within a factor of about  $1/2$ ). To this end, we deepen the process from [8] which consists in using Atanassov’s method for Halton sequences to obtain discrepancy bounds for  $(t, s)$ -sequences. We also provide a new result for the discrepancy of  $(t, 1)$ -sequences and extend to a larger class of  $(t, s)$ -sequences the bounds presented in [8, 11].

The paper is organized as follows. The rest of this first section is used to cover background material on discrepancy bounds for  $(t, s)$ -sequences together with known results on  $c_s$  until now. Section 2 is devoted to the one-dimensional case, where we provide a new result for the discrepancy bound of  $(t, 1)$ -sequences. Section 3 describes how to use Atanassov’s approach to study the discrepancy of  $(t, s)$ -sequences (in the “broad sense”, a terminology to be defined shortly), closely following the recent study [8]. The important improvement mentioned above for even bases  $b$  is given in Section 4, along with some numerical comparisons showing how this result improves upon the previously established best bounds from [11].

## 1.1 Discrepancy

We start with a review of the notion of discrepancy, which will be used throughout the paper. Various types exist but here, for short, we only consider the so-called *extreme discrepancy*, which corresponds to the worst case error in the domain of complexity of multivariate problems. Assume we have a point set  $\mathcal{P}_N = \{X_1, \dots, X_N\} \subseteq I^s$  and denote  $\mathcal{J}$  (resp  $\mathcal{J}^*$ ) the set of intervals  $J$  of  $I^s$  of the form  $J = \prod_{j=1}^s [y_j, z_j)$ , where  $0 \leq y_j < z_j \leq 1$  (resp.  $J = \prod_{j=1}^s [0, z_j)$ ). Then the *discrepancy function* of  $\mathcal{P}_N$  on such an interval  $J$  is the difference  $\boxed{1}$

$$E(J; N) = A(J; \mathcal{P}_N) - NV(J),$$

where  $A(J; \mathcal{P}_N) = \#\{n; 1 \leq n \leq N, X_n \in J\}$  is the number of points in  $\mathcal{P}_N$  that fall in the subinterval  $J$ , and  $V(J) = \prod_{j=1}^s (z_j - y_j)$  is the volume of  $J$ .

Then, the *star (extreme) discrepancy*  $D^*$  and the *(extreme) discrepancy*  $D$  of  $\mathcal{P}_N$  are

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<sup>1</sup>De Henri: je crois qu’il fallait tenir compte de la Rem de l’arbitre pour mcqmc10.

defined by

$$D^*(\mathcal{P}_N) = \sup_{J \in \mathcal{J}^*} |E(J; N)| \quad \text{and} \quad D(\mathcal{P}_N) = \sup_{J \in \mathcal{J}} |E(J; N)|.$$

It is well known that  $D^*(\mathcal{P}_N) \leq D(\mathcal{P}_N) \leq 2^s D^*(\mathcal{P}_N)$ . For an infinite sequence  $X$ , we denote by  $D(N, X)$  and  $D^*(N, X)$  the discrepancies of its first  $N$  points. Note that several authors have a  $1/N$  factor when defining the above quantities.

A sequence satisfying  $D^*(N, X) \in O((\log N)^s)$  is typically considered to be a *low-discrepancy sequence*. But the constant hidden in the  $O$  notation needs to be made explicit to make comparisons possible across sequences. This is achieved in many papers with an inequality of the form

$$D^*(N, X) \leq c_s (\log N)^s + O((\log N)^{s-1}). \quad (1)$$

The constant  $c_s$  in this inequality is the main object of our study, as in many other papers dealing with theoretical comparisons of low-discrepancy sequences.

## 1.2 Review of $(t, s)$ -sequences

The concept of  $(t, s)$ -sequences has been introduced by Niederreiter [13] to give a general framework for various constructions including Sobol' sequences [21], Faure sequences [5], and later a more general class of constructions referred to as Niederreiter-Xing sequences [17].

**Definition 1** Given an integer  $b \geq 2$ , an *elementary interval* in  $I^s$  is an interval of the form  $\prod_{i=1}^s [a_i b^{-d_i}, (a_i + 1) b^{-d_i})$  where  $a_i, d_i$  are nonnegative integers with  $0 \leq a_i \leq b^{d_i}$  for  $1 \leq i \leq s$ .

Given integers  $t, m$  with  $0 \leq t \leq m$ , a  $(t, m, s)$ -net in base  $b$  is an  $s$ -dimensional set with  $b^m$  points such that any elementary interval in base  $b$  with volume  $b^{t-m}$  contains exactly  $b^t$  points of the set.

An  $s$ -dimensional sequence  $(X_n)_{n \geq 1}$  in  $I^s$  is a  $(t, s)$ -sequence if the subset  $\{X_n : kb^m < n \leq (k+1)b^m\}$  is a  $(t, m, s)$ -net in base  $b$  for all integers  $k \geq 0$  and  $m \geq t$ .

In order to give sense to new important constructions, Tezuka [22] and then Niederreiter and Xing [16, 17] introduced a new definition using the so-called truncation operator that we now define.

*Truncation*: Let  $x = \sum_{i=1}^{\infty} x_i b^{-i}$  be a  $b$ -adic expansion of  $x \in [0, 1]$ , with the possibility that  $x_i = b - 1$  for all but finitely many  $i$ . For every integer  $m \geq 1$ , we define the  $m$ -truncation of  $x$  by  $\boxed{2} \quad \boxed{3} \quad [x]_{b,m} = \sum_{i=1}^m x_i b^{-i}$  (depending on  $x$  via its expansion).

**Definition 2** An  $s$ -dimensional sequence  $(X_n)_{n \geq 1}$ , with prescribed  $b$ -adic expansions for each coordinates, is a  $(t, s)$ -sequence (*in the broad sense*) if the subset  $\{[X_n]_{b,m}; kb^m < n \leq (k+1)b^m\}$  is a  $(t, m, s)$ -net in base  $b$  for all integers  $k \geq 0$  and  $m \geq t$ .

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<sup>2</sup>De Christiane: J'ai ajouté un commentaire pour le cas multi-dim quand il se présente.

<sup>3</sup>De Henri: Mais le cas se présente tout de suite dans la Def 2 ! Je mettrais donc la def multi-dim ici.

The former  $(t, s)$ -sequences are now called  $(t, s)$ -sequences *in the narrow sense* and the others just  $(t, s)$ -sequences (Niederreiter-Xing [17], Definition 2 and Remark 1); in this paper, we will sometimes use intentionally the expression (in the broad sense) to emphasize the difference.

### 1.3 Review of bounds for the discrepancy of $(t, s)$ -sequences

Bounds for the discrepancy of  $(t, s)$ -sequences in the narrow sense have been established by Niederreiter in [13] with constant  $c_s$  in (1):

$$c_s^{Ni} = \frac{b^t}{s!} \frac{b-1}{2^{\lfloor \frac{b}{2} \rfloor}} \left( \frac{\lfloor \frac{b}{2} \rfloor}{\log b} \right)^s. \quad (2)$$

In the same paper an improved bound with  $c_s^{Ni} = \frac{b^t}{s} \left( \frac{b-1}{2 \log b} \right)^s$ , is obtained for  $s = 2$ , for  $s = 3$  and  $b = 2$  and for  $s = 4$  and  $b = 2$ .

More recently, Kritzer [11], still for  $(t, s)$ -sequences in the narrow sense and in dimension  $s \geq 2$

$$c_s^{Kr} = \frac{b^t}{s!} \frac{b-1}{2(b+1)} \left( \frac{b}{2 \log b} \right)^s \quad \text{when } b \text{ is an even base and to} \quad (3)$$

$$c_s^{Kr} = \frac{b^t}{s!} \frac{1}{2} \left( \frac{b-1}{2 \log b} \right)^s \quad \text{when } b \text{ is an odd base,} \quad (4)$$

hence improving the constants from [13] by a multiplicative factor  $\frac{1}{2}$  for an odd  $b$  and  $\frac{b}{2(b+1)}$  for an even  $b$  (apart from the special cases where  $b = 2$  and  $b$  even,  $b \geq 4$  when  $s = 2$ ).

Further, in the same paper, Kritzer states the conjecture that, when the base  $b$  is even, the constant  $c_s$  for  $s \geq 2$  should be

$$c_s^{conj} = \frac{b^t}{s!} \frac{b^2}{2(b^2-1)} \left( \frac{b-1}{2 \log b} \right)^s. \quad (5)$$

As for  $(t, s)$ -sequences (in the broad sense), Niederreiter and Xing [16] showed that constant  $c_s^{Ni}$  in (2) is still valid, but Kritzer in [11] did not take into account this generalization.

## 2 The one-dimensional case

In his paper [11], Kritzer obtains improved bounds for dimensions  $s \geq 2$ . His method is based on a result for  $(t, m, 2)$ -nets (shown in [2]), and thus it cannot be used to get corresponding improved bounds in one dimension. Hence, this section is devoted to the special case  $s = 1$ , for which the corresponding bounds are obtained in Corollary 1 (note

the slight subtlety in the case of an even base). Furthermore, we believe that Theorem 1 and Corollary 1 below are likely to be the starting point for proving Kritzer's conjecture. But until now we could not manage to insert our better bounds in one dimension in our proof using the method of Atanassov. See the remarks at the end of Sections 2 and 4 for further details. Before presenting our results (Section 2.2), we must review known facts about the discrepancy of one-dimensional sequences (Section 2.1).

## 2.1 Review of van der Corput sequences

Generalized van der Corput sequences have been introduced in [4] to improve the behavior of the original van der Corput sequence in base 2.

**Definition 3** Let  $b \geq 2$  be an integer. For integers  $n$  and  $N$  with  $n \geq 1$  and  $1 \leq N \leq b^n$ , write  $N - 1 = \sum_{r=0}^{\infty} a_r(N) b^r$  in the  $b$ -adic system (so that  $a_r(N) = 0$  if  $r \geq n$ ) and let  $\Sigma = (\sigma_r)_{r \geq 0}$  be a sequence of permutations of  $\{0, 1, \dots, b-1\}$ . Then the *generalized* van der Corput sequence in base  $b$  associated with  $\Sigma$  is defined by

$$S_b^\Sigma(N) = \sum_{r=0}^{\infty} \frac{\sigma_r(a_r(N))}{b^{r+1}}.$$

If  $\sigma_r = \text{id}$  for all  $r \geq 0$ , we write  $S_b^\Sigma = S_b^\sigma$ . The original van der Corput sequence in base  $b$ ,  $S_b^{\text{id}}$ , is obtained with the identical permutation  $\text{id}$ .

The study of van der Corput sequences involves two more precise notions of discrepancy defined as follows:

$$D^+(N, X) = \sup_{0 \leq \alpha \leq 1} E([0, \alpha); N; X) \text{ and } D^-(N, X) = \sup_{0 \leq \alpha \leq 1} (-E([0, \alpha); N; X)).$$

These discrepancies  $D^+$  and  $D^-$  are linked to the preceding ones by the relations

$$D(N, X) = D^+(N, X) + D^-(N, X) \text{ and } D^*(N, X) = \max(D^+(N, X), D^-(N, X)).$$

**Reminders on asymptotic results for  $S_b^{\text{id}}$ .** We only recall here the results about van der Corput sequences useful for the statement of Corollary 1.

For any permutation  $\sigma$ , there exists an effectively computable constant  $\alpha_b^\sigma$  such that

$$\limsup_{N \rightarrow \infty} \frac{D(N, S_b^\sigma)}{\log N} = \frac{\alpha_b^\sigma}{\log b} \text{ and } D(N, S_b^\sigma) \leq \frac{\alpha_b^\sigma}{\log b} \log N + \alpha_b^\sigma + 2 \text{ for all } N \geq 1.$$

Moreover, for any sequence of permutations  $\Sigma$ , we have

$$D(N, S_b^\Sigma) \leq D(N, S_b^{\text{id}}) = D^*(N, S_b^{\text{id}}) \text{ for all } N \geq 1,$$

i.e., the original van der Corput sequence  $S_b^{\text{id}}$  is the worst distributed among  $S_b^\Sigma$  sequences with respect to both the discrepancy  $D$  and the star discrepancy  $D^*$ . Since

$$\frac{\alpha_b^{\text{id}}}{\log b} = \frac{b-1}{4 \log b} \text{ if } b \text{ is odd and } \frac{\alpha_b^{\text{id}}}{\log b} = \frac{b^2}{4(b+1) \log b} \text{ if } b \text{ is even,}$$

these constants are the worst possible leading constants for the discrepancies  $D$  and  $D^*$  of  $S_b^\Sigma$  sequences. All these results come from [4].

**Reminders on van der Corput sequences towards  $(0, 1)$ -sequences.** These results come from [6]. First, recall the relation between these two families:

Generalized van der Corput sequences  $S_b^\Sigma$  are  $(0, 1)$ -sequences (in the broad sense). Here, the truncation is required for sequences  $S_b^\Sigma$  such that  $\sigma_r(0) = b-1$  for all sufficiently large  $r$  [6, Proposition 3.1].

Next, a result about the star discrepancy: The original van der Corput sequences  $S_b^{\text{id}}$  are the worst distributed with respect to the star discrepancy among all  $(0, 1)$ -sequences  $X_b$  (in the broad sense) and in arbitrary base  $b \geq 2$ , that is [6, Theorem 5.1]

$$D^*(N, X_b) \leq D^*(N, S_b^I) = D(N, S_b^I) \text{ for all } N \geq 1.$$

This theorem generalizes the same result from Kritzer [10] which was restricted to  $(0, 1)$ -sequences (in the narrow sense).

Moreover, in the same paper, we give examples showing that van der Corput sequences  $S_b^{\text{id}}$  are *not* the worst distributed with respect to the discrepancy  $D$  among all  $(0, 1)$ -sequences  $X_b$  (in the broad sense) for any base  $b$  and among all digital  $(0, 1)$ -sequences  $X_2$  (in the broad sense) for base 2, see [6, Theorems 5.3 and 5.2] for more details.

## 2.2 On the star discrepancy of $(t, 1)$ -sequences

We extend the above theorem to  $(t, 1)$ -sequences in base  $b$ , hence getting the worst possible leading constants  $c_1$  for  $(t, 1)$ -sequences.

**Theorem 1** *For any base  $b$ , the original van der Corput sequences are the worst distributed with respect to the star discrepancy among all  $(t, 1)$ -sequences  $X_b^t$  (in the broad sense) in the sense that for all  $N \geq 1$*

$$D^*(b^t N, X_b^t) \leq b^t D^*(N, S_b^{\text{id}}) = b^t D(N, S_b^{\text{id}}).$$

*Proof.* The proof closely follows the proof of [6, Theorem 5.1] with the sequence  $S_b^{\text{id}, t}$  consisting of  $b^t$  copies of  $S_b^{\text{id}}$  instead of the sequence  $S_b^{\text{id}}$ . For the sake of completeness, we give the main steps in the following.

Let be given an integer  $m \geq 0$  and the collection of elementary intervals of length  $b^{-m}$ . By construction (see [4] Property 3.1.2 with  $\sigma_{n-1} = \text{id}$ ), the van der Corput sequence  $S_b^{\text{id}}$  is the unique  $(0, 1)$ -sequence whose points are always placed, in increasing order, as left as possible within the given collection of elementary intervals and satisfy the elementary interval property.

Let  $\tau$  be the permutation defined by  $\tau(k) = b-1-k$  for  $0 \leq k \leq b-1$ . Then, by construction (see [4] Property 3.1.2 with  $\sigma_{n-1} = \tau$ ),  $S_b^\tau$  is the unique  $(0, 1)$ -sequence

whose points are always placed, in increasing order, as right as possible within the given collection of elementary intervals and satisfy the elementary interval property.

Therefore, by the elementary interval property for  $X_b^t$ , for any real  $\alpha \in [0, 1]$  and any integer  $N \geq 1$  we have

$$A([0, \alpha); b^t N; X_b^t) \leq A([0, \alpha); b^t N; S_b^{\text{id}, t}) = b^t A([0, \alpha); N; S_b^{\text{id}}) \text{ and}$$

$$A([0, \alpha); b^t N; X_b^t) \geq A([0, \alpha); b^t N; S_b^{\tau, t}) = b^t A([0, \alpha); N; S_b^\tau),$$

where  $S_b^{\tau, t}$  consists of  $b^t$  copies of  $S_b^\tau$ .

Now, from the definitions of  $D^+$  and  $D^-$ , we get

$$D^+(b^t N, X_b^t) \leq b^t D^+(N, S_b^{\text{id}}) \quad \text{and} \quad D^-(b^t N, X_b^t) \leq b^t D^-(N, S_b^\tau).$$

Finally, see [6, proof of Th. 5.1] for details, we have  $D^-(N, S_b^\tau) = D^+(N, S_b^{\text{id}})$ , so that

$$D^*(b^t N, X_b^t) = \max(D^+(b^t N, X_b^t), D^-(b^t N, X_b^t)) \leq b^t D^+(N, S_b^{\text{id}}) = b^t D^*(N, S_b^{\text{id}}).$$

(Notice, from [4], that  $D^-(N, S_b^{\text{id}}) = 0$ ,  $D^+(N, S_b^\tau) = 0$  and hence  $D^*(N, S_b^{\text{id}}) = D(N, S_b^{\text{id}}) = D^*(N, S_b^\tau) = D(N, S_b^\tau)$ ).  $\square$

**Remark 1** The main idea of the proof is in the first sentence; it was already used by Dick and Kritzer [2] in the context of Hammersley two-dimensional point sets and by Kritzer [10]. But in this paper, Kritzer considers only  $(0, 1)$ -sequences in the narrow sense and his proof is longer. Here, the use of  $S_b^\tau$  together with the good control of discrepancy by means of so-called  $\varphi$ -functions permits a shorter proof and a more general result.

**Corollary 1** *For all  $(t, 1)$ -sequences  $X_b^t$  (in the broad sense) and for all  $N \geq 1$  we have*

$$D^*(b^t N, X_b^t) \leq b^t \left( \frac{b-1}{4 \log b} \log N + \frac{b-1}{4} + 2 \right) \text{ if } b \text{ is odd and}$$

$$D^*(b^t N, X_b^t) \leq b^t \left( \frac{b^2}{4(b+1) \log b} \log N + \frac{b^2}{4(b+1)} + 2 \right) \text{ if } b \text{ is even,}$$

Hence, we find that constants  $c_s^{Kr}$  in (4) and  $c_s^{\text{conj}}$  are valid for  $s = 1$ , but not  $c_s^{Kr}$  in (3). Note that for  $s = 2$ ,  $c_s^{\text{conj}} = c_s^{Kr}$  in (3).

**Remark 2** 1. Corollary 1 shows that the conjecture of Kritzer is true for  $s = 1$  whereas the bound (3), valid for  $s \geq 2$ , is false for  $s = 1$ . Indeed this bound is equal to  $\frac{b^t(b^2-b)}{4(b+1) \log b} < \frac{b^{t+2}}{4(b+1) \log b}$  and (5) is reached with  $S_b^{\text{id}, t}$ , a contradiction. But for  $s = 2$  the conjecture is true and until now this is the only dimension for which it is known to hold.

2. As for one-dimensional sequences having the smallest star discrepancy that are currently known, recent progress has been made by Ostromoukhov [19] who improves the preceding ones from the first author [4] cited in [13, Section 8] and [14, Chapter 3.1] (in the same paper, Ostromoukhov also improves the discrepancy  $D$ ).

3. The results evoked at the end of Section 2.1 for  $(0, 1)$ -sequences, [6, Theorems 5.3 and 5.2], should have analogs for  $(t, 1)$ -sequences,  $t > 0$ .

### 3 Discrepancy bound for $(t, s)$ -sequences ( $s \geq 2$ )

In this section, we extend to  $(t, s)$ -sequences (in the broad sense) the discrepancy bound obtained in [8] for  $(t, s)$ -sequences in the narrow sense by using Atanassov's method first initiated for Halton sequences [1]. We also give more details on the end of the proof than what we did for Theorem 1 in [8], where they were omitted due to lack of space. In that way, apart from these slight improvements, we prepare the proof of Section 4 concerning even bases. In what follows,  $\mathcal{P}_N(X)$  denotes the set containing the first  $N$  points of a sequence  $X$  and until the end, we set  $n := \left\lfloor \frac{\log N}{\log b} \right\rfloor$ . Also, several results in this section apply to the *truncated version* of the  $(t, s)$ -sequence under consideration, a concept that we now define.

**Definition 4** Let  $X$  be a  $(t, s)$ -sequence in base  $b$ , with its  $k$ th term defined as  $X_k = (X_k^{(1)}, \dots, X_k^{(s)})$ , for  $k \geq 1$ . Let

$$[\mathcal{P}_N(X)] = \{([X_k^{(1)}]_{b,\Delta}, \dots, [X_k^{(s)}]_{b,\Delta}), 1 \leq k \leq N\}, \text{ where } \Delta = n + 1.$$

We refer to  $[\mathcal{P}_N(X)]$  as the first  $N$  points of a truncated version of the sequence  $X$ .

The next result would be trivial without the truncation operator.

**Lemma 1** Let  $X$  be a  $(t, s)$ -sequence in base  $b$  and  $J = \prod_{i=1}^s [b_i b^{-d_i}, c_i b^{-d_i})$  with integers  $b_i, c_i$  satisfying  $0 \leq b_i \leq c_i \leq b^{d_i}$ . Then for  $N \geq b^{d_1} \dots b^{d_s}$ ,  $A(J; [\mathcal{P}_N(X)])$  is an increasing function of  $N$ .

*Proof.* If  $N \geq b^{d_1} \dots b^{d_s}$ , then  $\Delta \geq d_i$  for all  $i$ . Therefore as  $N$  increases, there can only be more points (from the truncated sequence) inside a particular interval  $J$ . The reason why we have to make sure  $\Delta \geq d_i$  for all  $i$  is that otherwise, as  $N$  increases some points could leave the interval  $J$  as more precision is added to their digital expansion, but once the precision  $\Delta$  is greater than the precision  $d_i$  used to define the interval, then this can no longer happen.  $\square$

The following lemma directly follows from the definition of  $(t, s)$ -sequences, but it requires some adaptation with the truncation operator.

**Lemma 2** Let  $X$  be a  $(t, s)$ -sequence. Let  $J = \prod_{i=1}^s [b_i b^{-d_i}, c_i b^{-d_i})$  where  $b_i, c_i$  are integers satisfying  $0 \leq b_i \leq c_i \leq b^{d_i}$ . Then

$$A(J, [\mathcal{P}_N(X)]) = u b^t (c_1 - b_1) \cdots (c_s - b_s) \text{ where } N = u b^t b^{d_1} \dots b^{d_s} \ (u \geq 0),$$

$$|A(J, [\mathcal{P}_N(X)]) - NV(J)| \leq b^t \prod_{i=1}^s (c_i - b_i), \text{ for any integer } N \geq 1, \quad \text{and,}$$

$$A(J, [\mathcal{P}_N(X)]) \leq b^t \prod_{i=1}^s (c_i - b_i), \text{ if } N < b^t b^{d_1} \dots b^{d_s}.$$



*Proof.* To prove the first statement, we first observe that from the definition of a  $(t, s)$ -sequence, the sets  $\{[X_k]_{b,m}, lb^m \leq k < (l+1)b^m\}$  are  $(t, m, s)$ -nets for  $0 \leq l < u$ , where  $u \geq 1$  and the notation  $[X_k]_{b,m}$  means an  $m$ -truncation is applied to each coordinate of  $X_k$   $\begin{bmatrix} 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$ . The sets  $\{[X_k]_{b,M}, lb^m \leq k < (l+1)b^m\}$  are also  $(t, m, s)$ -nets for  $0 \leq l < u$  and  $M > m$ , which proves the result since  $\Delta > m$  in the case  $N = ub^m$ .

For the second statement, let  $l \geq 1$  be such that  $lb^t b^{d_1} \dots b^{d_s} \leq N < (l+1)b^t b^{d_1} \dots b^{d_s}$ . Then, using Lemma 1, we get

$$A(J, [\mathcal{P}_N(X)]) - NV(J) \leq (l+1)b^t \prod_{i=1}^s (c_i - b_i) - lb^t b^{d_1} \dots b^{d_s} \prod_{i=1}^s (c_i - b_i) b^{-d_i} = b^t \prod_{i=1}^s (c_i - b_i).$$

A similar argument shows that  $NV(J) - A(J, [\mathcal{P}_N(X)]) \leq b^t \prod_{i=1}^s (c_i - b_i)$ .

For the third statement, in the case where  $N < b^t b^{d_1} \dots b^{d_s}$ , we define  $\tilde{d}_i = \min(\Delta, d_i)$ , where  $\Delta = n+1$ . Let  $[J]$  be defined as the smallest interval of the form  $\prod_{i=1}^s [\tilde{b}_i b^{-\tilde{d}_i}, \tilde{c}_i b^{-\tilde{d}_i})$  with  $0 \leq \tilde{b}_i \leq \tilde{c}_i \leq b^{\tilde{d}_i}$  and such that  $J \subseteq [J]$ . We can see that  $[J]$  is obtained by using  $\tilde{c}_i = \lceil c_i / b^{\tilde{d}_i - d_i} \rceil$  and  $\tilde{b}_i = \lfloor b_i / b^{\tilde{d}_i - d_i} \rfloor$ . It is easy to see that each interval of the form  $\prod_{i=1}^s [l_i b^{-\tilde{d}_i}, (l_i + 1) b^{-\tilde{d}_i})$  has at most  $b^t$  points from  $[\mathcal{P}_N(X)]$ . Hence

$$A(J; [\mathcal{P}_N(X)]) \leq A([J]; [\mathcal{P}_N(X)]) \leq b^t \prod_{i=1}^s (\tilde{c}_i - \tilde{b}_i) \leq b^t \prod_{i=1}^s (c_i - b_i),$$

where the last inequality follows from the definition of  $\tilde{b}_i$  and  $\tilde{c}_i$ .  $\square$

**Lemma 3** Let  $N$  and  $b \geq 2$  be integers. Let  $c_j^{(i)} \geq 0$  be given numbers, for  $j \geq 0, i = 1 \dots, k$ , satisfying  $c_0^{(i)} \leq 1$  and  $c_j^{(i)} \leq c$  for  $j \geq 1$ , for some fixed number  $c$ . Then

$$\sum_{\{(j_1, \dots, j_k); b^{j_1} \dots b^{j_k} \leq N\}} \prod_{i=1}^k c_{j_i}^{(i)} \leq \frac{1}{k!} \left( c \frac{\log N}{\log b} + k \right)^k. \quad (6)$$

*Proof.* For each  $m \in \{0, 1, \dots, k\}$ , fix a subset  $L = \{i_1, \dots, i_m\}$  of  $\{1, \dots, k\}$  and consider the contributions of all the  $k$ -tuples  $\mathbf{j}$  with  $j_r > 0$  for  $r \in L$ , and  $j_r = 0$  for  $r \notin L$ , with  $\prod_{i=1}^k b^{j_i} = \prod_{i \in L} b^{j_i} \leq N$ . One can verify like in [1, Lemma 3.2] (see also [3, Lemma 3.38]) that there are  $\frac{1}{m!} \left( \frac{\log N}{\log b} \right)^m$  such  $k$ -tuples, each having a contribution of

$$\prod_{i=1}^k c_{j_i}^{(i)} = \prod_{i \in L} c_{j_i}^{(i)} \prod_{i \notin L} c_{j_i}^{(i)} \leq \prod_{i \in L} c \prod_{i \notin L} 1 = c^m.$$

Expanding both sides of (6), the result now follows since  $\frac{1}{m!} \leq \frac{1}{k!} k^{k-m}$ .  $\square$

<sup>4</sup>De Christiane: Ajouté.

<sup>5</sup>De Henri: enlever et mettre un renvoi à la troncation p.3 ?

**Definition 5** Consider an interval  $J \subseteq I^s$ . We call a *signed splitting* of  $J$  any collection of intervals  $J_1, \dots, J_n$  and respective signs  $\epsilon_1, \dots, \epsilon_n$  equal to  $\pm 1$ , such that for any (finitely) additive function  $\nu$  on the intervals in  $I^s$ , we have  $\nu(J) = \sum_{i=1}^n \epsilon_i \nu(J_i)$ .

The following lemma, slightly reformulated, is taken directly from [1, Lemma 3.5] (see also [3, Lemma 3.40]).

**Lemma 4** Let  $J = \prod_{i=1}^s [0, z^{(i)})$  be an  $s$ -dimensional interval and, for each  $1 \leq i \leq s$ , let  $n_i \geq 0$  be given integers. Set  $z_0^{(i)} = 0$ ,  $z_{n_i+1}^{(i)} = z^{(i)}$  and, if  $n_i \geq 1$ , let  $z_j^{(i)} \in [0, 1]$  be arbitrary given numbers for  $1 \leq j \leq n_i$ . Then the collection of intervals  $\prod_{i=1}^s [\min(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}), \max(z_{j_i}^{(i)}, z_{j_i+1}^{(i)})]$ , with signs  $\epsilon(j_1, \dots, j_s) = \prod_{i=1}^s \text{sgn}(z_{j_i+1}^{(i)} - z_{j_i}^{(i)})$ , for  $0 \leq j_i \leq n_i$ , is a signed splitting of  $J$ .

Now we can prove the result announced at the beginning of Section 3:

**Theorem 2** For any  $(t, s)$ -sequence  $X$  (in the broad sense) in any base  $b$  and for any  $N \geq 1$  we have

$$D^*(N, X) \leq \frac{b^t}{s!} \left( \left\lfloor \frac{b}{2} \right\rfloor \frac{\log N}{\log b} + s \right)^s + b^t \sum_{k=1}^{s-1} \frac{b}{k!} \left( \left\lfloor \frac{b}{2} \right\rfloor \frac{\log N}{\log b} + k \right)^k \cdot \boxed{6} \quad (7)$$

*Proof.* As in [5] and [1], we will use special numeration systems in base  $b$  — using signed digits  $a_j$  bounded by  $\lfloor \frac{b}{2} \rfloor$  — to expand reals in  $[0, 1)$ . That is, we write  $z \in [0, 1)$  as

$$z = \sum_{j=0}^{\infty} a_j b^{-j} \begin{cases} \text{with } |a_j| \leq \frac{b-1}{2} \text{ if } b \text{ is odd} \\ \text{with } |a_j| \leq \frac{b}{2} \text{ and } |a_j| + |a_{j+1}| \leq b-1 \text{ if } b \text{ is even.} \end{cases} \quad (8)$$

The existence and unicity of such expansions are obtained by induction, see [1, p. 21–22] or [23, p. 12–13] where more details are given. For later use, it is worth pointing out that the expansion starts at  $b^0$  and as a result, it is easy to see that  $a_0$  is either 0 or 1.

Now we can begin the proof: Let  $\mathbf{z} = (z^{(1)}, \dots, z^{(s)}) \in [0, 1)^s$  and consider its  $b$ -adic expansion  $z^{(i)} = \sum_{j=0}^{\infty} a_j^{(i)} b^{-j}$  according to our numeration systems (8) above. Recall that  $n = \left\lfloor \frac{\log N}{\log b} \right\rfloor$  and define  $z_0^{(i)} = 0$  and  $z_{n_i+1}^{(i)} = z^{(i)}$ . If  $n \geq 1$ , consider the numbers  $z_k^{(i)} = \sum_{j=0}^{k-1} a_j^{(i)} b^{-j}$  for  $k = 1, \dots, n$ . Applying Lemma 4 with  $n_i = n$  for all  $1 \leq i \leq s$ , we split up  $J = \prod_{i=1}^s [0, z^{(i)})$  using the numbers  $z_j^{(i)}$ , with  $0 \leq j \leq n+1$ , and obtain the signed splitting

$$I(\mathbf{j}) = \prod_{i=1}^s [\min(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}), \max(z_{j_i}^{(i)}, z_{j_i+1}^{(i)})], \quad 0 \leq j_i \leq n, \quad (9)$$

with signs  $\epsilon(\mathbf{j}) = \prod_{i=1}^s \text{sgn}(z_{j_i+1}^{(i)} - z_{j_i}^{(i)})$ , where  $\mathbf{j} = (j_1, \dots, j_s)$ .

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<sup>6</sup>De Henri: Enlevé +s: voir la fin de la dém.

Since  $V$  and  $A(\cdot; [\mathcal{P}_N(X)])$  are both additive,  $A(J; [\mathcal{P}_N(X)]) - NV(J)$  may be expanded as

$$A(J; [\mathcal{P}_N(X)]) - NV(J) = \sum_{j_1=0}^n \cdots \sum_{j_s=0}^n \epsilon(\mathbf{j}) (A(I(\mathbf{j}); [\mathcal{P}_N(X)]) - NV(I(\mathbf{j}))) =: \Sigma_1 + \Sigma_2 \quad (10)$$

where we rearrange the terms so that in  $\Sigma_1$  we put the terms  $\mathbf{j}$  such that  $b^{j_1} \dots b^{j_s} \leq N$  (that is  $j_1 + \dots + j_s \leq n$ ) and in  $\Sigma_2$  the rest. Notice that in  $\Sigma_1$ , the  $j'_i$ 's are small, so the corresponding  $I(\mathbf{j})$  is bigger. Hence,  $\Sigma_1$  deals with the coarser part whereas  $\Sigma_2$  deals with the finer part.

Thanks to Lemmas 2 and 3, it is easy to deal with  $\Sigma_1$ : from Lemma 2, we have that

$$|A(I(\mathbf{j}); [\mathcal{P}_N(X)]) - NV(I(\mathbf{j}))| \leq b^t \prod_{i=1}^s |z_{j_i+1}^{(i)} - z_{j_i}^{(i)}| b^{j_i} = b^t \prod_{i=1}^s |a_{j_i}^{(i)}|. \quad (11)$$

Then, applying Lemma 3 with  $k = s$ ,  $c_j^{(i)} = |a_j^{(i)}|$  and  $c = \lfloor \frac{b}{2} \rfloor$ , we obtain

$$|\Sigma_1| \leq \sum_{\{\mathbf{j}; b^{j_1} \dots b^{j_s} \leq N\}} |A(I(\mathbf{j}); [\mathcal{P}_N(X)]) - NV(I(\mathbf{j}))| \leq \frac{b^t}{s!} \left( \left\lfloor \frac{b}{2} \right\rfloor \frac{\log N}{\log b} + s \right)^s$$

which is the first part of the bound of Theorem 2.

The terms gathered in  $\Sigma_2$  give the second part of the bound of Theorem 2, i.e., the part in  $O((\log N)^{s-1})$ . The idea of Atanassov for his proof of Theorem 2.1 on Halton sequences is to divide the set of  $s$ -tuples  $\mathbf{j}$  in  $\Sigma_2$  into  $s$  disjoint sets included in larger ones for which Lemma 3 applies and gives the desired upper bound. The adaptation to  $(t, s)$ -sequences follows the same way. For the sake of completeness, we survey the proof in the following, something we could not achieve in [8] due to lack of space.

Recall that in  $\Sigma_2$ , we are summing over the set of all  $s$ -tuples  $\mathbf{j}$  such that  $b^{j_1} \dots b^{j_s} > N$  with  $0 \leq j_i \leq n$  for all  $1 \leq i \leq s$ . We split this set a priori into  $s$  disjoint sets  $B_0, \dots, B_{s-1}$ , where  $B_0 = \{\mathbf{j}; b^{j_1} > N\}$  and

$$B_k = \{\mathbf{j}; b^{j_1} \dots b^{j_k} \leq N < b^{j_1} \dots b^{j_k} b^{j_{k+1}}\} \text{ for } k \geq 1.$$

But we immediately observe that  $B_0 = \emptyset$ , since  $b^{j_1} > N$  implies  $j_1 > n$  which is in contradiction with the definition of the signed splitting defined in (9). For the same reason, the corresponding set  $B_0$  in the proof for Halton sequences is also empty, a remark that was skipped in papers dealing with Atanassov's Theorem 2.1 [1, 23, 3].

Hence we only have to deal with  $1 \leq k \leq s-1$ . For such a fixed  $k$  and for a fixed  $k$ -tuple  $(j_1, \dots, j_k)$  with  $b^{j_1} \dots b^{j_k} \leq N$ , define  $r$  as the largest integer such that  $b^{j_1} \dots b^{j_k} b^{r-1} \leq N$ . Then the  $s$ -tuple  $\mathbf{j} = (j_1, \dots, j_k, j_{k+1}, \dots, j_s) \in B_k$  if and only if  $j_{k+1} \geq r$ , the remaining integers  $j_{k+2}, \dots, j_s$  being arbitrary in the interval  $[0, n]$ . Still for fixed  $k$  and  $(j_1, \dots, j_k)$  as above, define  $K_1 = \prod_{i=1}^k [\min(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}), \max(z_{j_i}^{(i)}, z_{j_i+1}^{(i)})]$  and

$$K = K_1 \times [\min(z_r^{(k+1)}, z_r^{(k+1)}), \max(z_r^{(k+1)}, z_r^{(k+1)})] \times \prod_{i=k+2}^s [0, z_{j_i}^{(i)}].$$

Coming back to the sum of terms in  $\Sigma_2$  that belong to  $B_k$ , we obtain by a simple case analysis

$$\pm(A(K; [\mathcal{P}_N(X)]) - NV(K)) = \sum_{\{(j_{k+1}, \dots, j_s); \mathbf{j} \in B_k\}} \epsilon(\mathbf{j})(A(I(\mathbf{j}); [\mathcal{P}_N(X)]) - NV(I(\mathbf{j})))$$

where  $\pm = \text{sgn}(z^{(k+1)} - z_r^{(k+1)})$ . Now, since

$$|z^{(k+1)} - z_r^{(k+1)}| = \left| \sum_{j=r}^{\infty} a_j^{(k+1)} b^{-j} \right| \leq \left\lfloor \frac{b}{2} \right\rfloor \frac{1}{b^r} \frac{b}{b-1} \leq \frac{1}{b^{r-1}},$$

it follows that  $[\min(z_r^{(k+1)}, z^{(k+1)}), \max(z_r^{(k+1)}, z^{(k+1)})] \subseteq [m_1 b^{-r}, m_2 b^{-r}]$  for some non-negative integers  $m_1, m_2$  satisfying  $0 \leq m_2 - m_1 < b$ . Hence  $K$  is contained in the interval  $K' = K_1 \times [m_1 b^{-r}, m_2 b^{-r}] \times [0, 1)^{s-k-1}$ . Thanks to this inclusion we can bound  $A(K; [\mathcal{P}_N(X)])$  using Lemma 2 (observe that  $N < b^{j_1} \dots b^{j_k} b^r \leq b^t b^{j_1} \dots b^{j_k} b^r$ ):

$$A(K; [\mathcal{P}_N(X)]) \leq A(K'; [\mathcal{P}_N(X)]) \leq b^t (m_2 - m_1) \prod_{i=1}^k b^{j_i} |z_{j_{i+1}}^{(i)} - z_{j_i}^{(i)}| \leq b^t \cdot b \prod_{i=1}^k |a_{j_i}^{(i)}|.$$

But on the other hand,

$$NV(K) \leq b^t b^{j_1} \dots b^{j_k} b^r \cdot V(K') = b^t (m_2 - m_1) \prod_{i=1}^k |a_{j_i}^{(i)}| \leq b^t \cdot b \prod_{i=1}^k |a_{j_i}^{(i)}|,$$

so that  $|A(K; [\mathcal{P}_N(X)]) - NV(K)| \leq b^t \cdot b \prod_{i=1}^k |a_{j_i}^{(i)}|$ .

Finally, collecting everything in a unique sum, we get

$$\begin{aligned} |\Sigma_2| &= \left| \sum_{k=1}^{s-1} \sum_{\{(j_1, \dots, j_k); b^{j_1} \dots b^{j_k} \leq N\}} \sum_{\{(j_{k+1}, \dots, j_s); \mathbf{j} \in B_k\}} \epsilon(\mathbf{j})(A(I(\mathbf{j}); N) - NV(I(\mathbf{j}))) \right| \\ &= \left| \sum_{k=1}^{s-1} \sum_{\{(j_1, \dots, j_k); b^{j_1} \dots b^{j_k} \leq N\}} \pm(A(K; N) - NV(K)) \right| \\ &\leq b^t \sum_{k=1}^{s-1} b \sum_{\{(j_1, \dots, j_k); b^{j_1} \dots b^{j_k} \leq N\}} \prod_{i=1}^k |a_{j_i}^{(i)}| \\ &\leq b^t \sum_{k=1}^{s-1} \frac{b}{k!} \left( \left\lfloor \frac{b}{2} \right\rfloor \frac{\ln N}{\log b} + k \right)^k, \end{aligned}$$

where we applied Lemma 3 with  $c = \lfloor \frac{b}{2} \rfloor$  and  $1 \leq k \leq s-1$ . This is the second part of the bound of Theorem 2.

So far, we have proved (7) only for the truncated version of the sequence but, as shown in [15, Lemma 4.2], when we go from the truncated to the untruncated version

of the sequence, the bound for the discrepancy remains the same. <sup>[7]</sup> <sup>[8]</sup> <sup>[9]</sup> Thus if a bound of the form (1) applies to the truncated version of a  $(t, s)$ -sequence, it applies to the untruncated version as well with the same constant  $c_s$ . Notice that [15, Lemma 4.2] is a nice improvement on [16, Rem. 2] where going from the truncated to the untruncated version adds a term at most  $s$  to the discrepancy. <sup>[10]</sup>  $\square$

From Theorem 2 we can derive the following result for the constant  $c_s$  which for the case where  $b$  is odd is the same as constant  $c_s^{Ni}$  and in case of an even  $b$  is larger than  $c_s^{Ni}$  by a factor  $b/(b-1)$ .

**Corollary 2** *The discrepancy of a  $(t, s)$ -sequence  $X$  in base  $b$  satisfies (1) with*

$$c_s = \begin{cases} \frac{b^t}{s!} \left( \frac{b-1}{2 \log b} \right)^s & \text{if } b \text{ is odd} \\ \frac{b^t}{s!} \left( \frac{b}{2 \log b} \right)^s & \text{if } b \text{ is even.} \end{cases}$$

**Remark 3** Until now, we have not been able to include in our proof the improvements in one dimension from Section 2 and hence we are not able to reach the constants  $c_s^{Kr}$  obtained by Kritzer [11]. But, using an idea from Atanassov in [1] in the case of an even base in Halton sequences, we have been able to substantially improve  $c_s^{Kr}$  in (3), not far from  $c_s^{conj}$  conjectured in (5), still for the reason above. That's the matter of the next section.

## 4 Discrepancy bound for $(t, s)$ -sequences in an even base

The idea of Atanassov in case of an even base in Halton sequences can be applied to  $(t, s)$ -sequences as well but it needs a different treatment developed in the following. As a result, we obtain an improvement of previous upper bounds for  $(t, s)$ -sequences in even bases.

**Theorem 3** *For any  $(t, s)$ -sequence  $X$  (in the broad sense) in an even base  $b$  and for any  $N \geq b^s$  we have*

$$D^*(N, X) \leq \frac{b^t}{s!} \left( \frac{b-1}{2} \frac{\log N}{\log b} + s \right)^s + sb^t \left( \frac{b}{2} \right)^s \left( \frac{\log N}{\log b} \right)^{s-1} + b^t \sum_{k=1}^{s-1} \frac{b}{k!} \left( \frac{b}{2} \frac{\log N}{\log b} + k \right)^k. \quad (12)$$

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<sup>7</sup>De Henri: Après lecture de NX Finite Fields Appl. 2, 1996, 241–273, je mettrais cette référence avec mention de leur Rem 2. Par ailleurs, il faut voir comment intégrer le  $O(\log N)$  ou comment l'éviter dans nos énoncés des Thm 2 et 3.

<sup>8</sup>De Christiane: Dis-moi ce que tu en penses.

<sup>9</sup>De Henri: En fait, le Lemme 4.2 de N-O (p.13-14 du preprint) montre qu'il n'y a rien à ajouter. C'est une bonne amélioration de [16, Rem. 2] qui permet d'enlever le  $+s$  dans l'énoncé. Vérifie et dis moi si tu es d'accord.

<sup>10</sup>De Henri: Texte à améliorer éventuellement.

*Proof.* At the outset, the proof is the same as for Theorem 2 until equation (10) for the discrepancy function  $A(J; [\mathcal{P}_N(X)]) - NV(J)$ . The end of the proof, concerning  $\Sigma_2$ , is the same too and gives the last term in the bound (12). Hence, what remains to be done is to deal with  $\Sigma_1$ . To this end, we split up the set  $\{(j_1, \dots, j_s); b^{j_1} \dots b^{j_s} \leq N\}$  in two parts:

$$S' = \left\{ (j_1, \dots, j_s); b^{j_1} \dots b^{j_s} \leq \frac{N}{b^s} \right\} \text{ and } S'' = \left\{ (j_1, \dots, j_s); \frac{N}{b^s} < b^{j_1} \dots b^{j_s} \leq N \right\}.$$

The part  $S'$  provides the biggest contribution to the bound and hence gives the leading term in (12): using (11), we need to prove

$$\sum_{(j_1, \dots, j_s) \in S'} \prod_{i=1}^s |a_{j_i}^{(i)}| \leq \frac{1}{s!} \left( \frac{(b-1) \log N}{2 \log b} + s \right)^s.$$

To do so, we proceed in three steps:

First, we define integers  $c_j^{(i)} = |a_{2j-1}^{(i)}| + |a_{2j}^{(i)}|$  for  $j \geq 1$  and  $c_0^{(i)} = 1$ . We have that for  $j \geq 1$ ,

$$|a_j^{(i)}| \leq \begin{cases} |a_{j-1}^{(i)}| + |a_j^{(i)}| = c_h^{(i)} = c_{j/2}^{(i)} & \text{if } j = 2h \text{ is even} \\ |a_j^{(i)}| + |a_{j+1}^{(i)}| = c_h^{(i)} = c_{(j+1)/2}^{(i)} & \text{if } j = 2h - 1 \text{ is odd,} \end{cases}$$

and for  $j = 0$ ,  $a_j^{(i)}$  is either 0 or 1 so that  $a_0^{(i)} \leq c_0^{(i)}$ . Hence

$$\prod_{i=1}^s |a_{j_i}^{(i)}| \leq \prod_{i=1}^s |c_{j'_i}^{(i)}|$$

with

$$j'_i = \begin{cases} j_i/2 & \text{if } j_i \text{ is even} \\ (j_i + 1)/2 & \text{if } j_i \text{ is odd.} \end{cases} \quad (13)$$

Secondly, we apply Lemma 3 to the numbers  $c_j^{(i)}$  with  $k = s$ ,  $\tilde{b} = b^2$  and  $c = \sqrt{\tilde{b}} - 1 = b - 1$  (observe that, according to the numeration system (8) in case of  $b$  even, we have  $c_j^{(i)} = |a_{2j-1}^{(i)}| + |a_{2j}^{(i)}| \leq b - 1$ ). Hence we get (observe that  $\log \tilde{b} = 2 \log b$ )

$$\sum_{\{(l_1, \dots, l_s); \tilde{b}^{l_1} \dots \tilde{b}^{l_s} \leq N\}} \prod_{i=1}^s c_{l_i}^{(i)} \leq \frac{1}{s!} \left( \frac{(b-1) \log N}{2 \log b} + s \right)^s. \quad (14)$$

Finally, we show that for  $(j_1, \dots, j_s) \in S'$  the  $s$ -tuple  $(j'_1, \dots, j'_s)$  defined in (13) verifies  $\tilde{b}^{j'_1} \dots \tilde{b}^{j'_s} \leq N$ . Indeed, we have

$$\tilde{b}^{j'_1} \dots \tilde{b}^{j'_s} = b^{2(\sum_{\{j_i; j_i=2h\}} \frac{j_i}{2} + \sum_{\{j_i; j_i=2h-1\}} \frac{j_i+1}{2})} = b^{\sum j_i} \cdot b^{\sum_{\{j_i; j_i=2h-1\}} 1} \leq \frac{N}{b^s} b^s = N.$$

(Recall that we are estimating the sum for  $(j_1, \dots, j_s) \in S'$ , i.e. such that  $b^{\sum j_i} \leq \frac{N}{b^s}$ ). Hence, the  $s$ -tuples  $(j'_1, \dots, j'_s)$  derived from the  $s$ -tuples  $(j_1, \dots, j_s) \in S'$  as in (13) belong

to the set  $\{(l_1, \dots, l_s); \tilde{b}^{l_1} \dots \tilde{b}^{l_s} \leq N\}$ . Therefore, according to the first step and to (14), we can assert that

$$\sum_{(j_1, \dots, j_s) \in S'} \prod_{i=1}^s |a_{j_i}^{(i)}| \leq \sum_{\{(l_1, \dots, l_s); \tilde{b}^{l_1} \dots \tilde{b}^{l_s} \leq N\}} \prod_{i=1}^s c_{l_i}^{(i)} \leq \frac{1}{s!} \left( \frac{(b-1) \log N}{2 \log b} + s \right)^s.$$

(observe that two reasons prevail for the first inequality: one from the inequality in the first step and the other from the fact that  $(j'_1, \dots, j'_s) \in \{(l_1, \dots, l_s); \tilde{b}^{l_1} \dots \tilde{b}^{l_s} \leq N\}$ ). This concludes the study of  $S'$ .

Now we deal with  $S'' = \{(j_1, \dots, j_s); \frac{N}{b^s} < b^{j_1} \dots b^{j_s} \leq N\}$ . Taking the logarithm,  $(j_1, \dots, j_s) \in S''$  is equivalent to

$$n - s < \sum_{i=1}^s j_i \leq n \text{ with } n = \left\lfloor \frac{\log N}{\log b} \right\rfloor.$$

Similarly to what is done in the proof of [1, Lemma 3.2] (see also [3, Lemma 3.38]), we can show that the number of such  $s$ -tuples  $(j_1, \dots, j_s)$  is at most  $sn^{s-1}$  since we have  $s$  relations  $\sum_{i=1}^s j_i = m$  with  $n - s < m \leq n$ . On the other hand, each  $s$ -tuple  $\mathbf{j} = (j_1, \dots, j_s)$  contributes a term that is at most  $b^t \left(\frac{b}{2}\right)^s$  in the estimate of the discrepancy function  $A(I(\mathbf{j}); N) - NV(I(\mathbf{j}))$ , as can be seen in (11). Finally, the total contribution from  $S''$  is at most

$$b^t \left(\frac{b}{2}\right)^s s \left(\frac{\log N}{\log b}\right)^{s-1}.$$

This is the second term in (12), which ends the main part of the proof of Theorem 3 according to the outline given at the beginning. Going from the truncated to the untruncated version of the sequence results from [15, Lemma 4.2], as explained at the end of the proof of Theorem 2. 11 □

**Corollary 3** *The discrepancy of a  $(t, s)$ -sequence  $X$  in an even base  $b$  satisfies (1) with*

$$c_s^{FL} = \frac{b^t}{s!} \left( \frac{b-1}{2 \log b} \right)^s. \quad (15)$$

**Remark 4** (1). Comparing to the constant  $c_s^{Kr}$  of Kritzer (3), we observe a significant improvement with  $\frac{c_s^{FL}}{c_s^{Kr}} = \frac{2(b+1)}{b} \left(\frac{b-1}{b}\right)^{s-1}$ , the most striking one being obtained with  $b = 2$  where the ratio is  $\frac{3}{2^{s-1}}$ , hence tending to 0 as  $s$  tends to infinity. But we have to take into account the quality parameter  $t$  to compare sequences in different bases (see the numerical results below).

(2). Comparing to the conjecture of Kritzer (5), we still have the gap  $\frac{c_s^{conj}}{c_s^{FL}} = \frac{b^2}{2(b^2-1)}$ . As we already noted, this gap comes from our inability to insert in our proof — using Atanassov's method — the better bounds in one dimension deduced from our knowledge

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<sup>11</sup>De Henri: Corrigé comme pour la fin du Thm 2. Faut-il en dire plus ?

of van der Corput sequences. Until now, these better bounds — translated in terms of two-dimensional Hammersley point sets — can only be inserted in the method of Sobol' [21], used later on in [5, 13, 11], thanks to the double recursion on  $m$  and  $s$  involving  $(t, m, s)$ -nets. But on the other hand, the ideas in the proof of Theorem 3 seem difficult to transpose in the double recursion method. We view this as a challenge for future research.

**Numerical results** At the end of [11], Kritzer devotes a section to numerical results concerning the smallest possible constants  $c_s$  obtained in each dimension with specific sequences taking into account the best possible values of the quality parameter  $t$ . He gives two tables for dimensions between 2 and 20:

- The first one (Table 2) uses values of  $t$  from [14] and logically, for  $s \geq 3$ , he obtains the same constants as [14, Table 4.4] divided by 2 since all bases are odd primes, except  $b = 9$  for  $s = 8, 9$ . If  $s = 2$  then  $b = 2$  and the constant from Table 4.4 is multiplied by  $\frac{b^2}{2(b^2-1)} = \frac{2}{3}$ .

- The second one (Table 3) uses best known values of  $t$  from the database MinT [20], a remarkable software containing updated information on the best available constructions for a given set of parameters. For base  $b = 2$ , these values already show a drastic improvement over Table 2, but the best constants  $c_s^{Kr}$  are obtained in base  $b = 3$  for all dimensions  $3 \leq s \leq 20$ , these constants being again half of the constants deduced from constant  $c_s^{Ni}$  in (2) with  $t$ -values from MinT.

In the following, we only give one table, corresponding to [11, Table 3] above, where the smallest achievable constant  $c_s^{FL}$  — over all  $(t, s)$ -sequences — for each dimension is given. The difference between our Table 1 and Table 3 from [11] is that we use  $c_s^{FL}$  from formula (15) in the case of even bases. Of course, we also use MinT to find the smallest possible value of  $t$  for each base. We see that we are able to improve upon the results of Kritzer for all dimensions listed there except  $s = 2, 3, 6, 50$  and that base 2 wins on base 3 except in three cases. Note that if Kritzer's conjecture is true, base 2 also wins for  $s = 6$  but not for  $s = 3$  and 50. We observe too that the best sequence in dimension 2 is a  $(0, 2)$ -Sobol' sequence and the best one in dimension 3 is a  $(0, 3)$ -Faure sequence, all others being  $(t, s)$ -Niederreiter-Xing sequences with  $t > 0$ . Although it is not shown here, we performed calculations for all  $s$  between 20 and 50, and observed that for these 24 additional cases, there were only two cases (in dimension 24 and 49) where base 3 won over base 2.

12			
13	14	15	

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<sup>12</sup>De Henri: Est-tu d'accord avec les changements dans la table ? Sinon tu peux arranger autrement. Si oui, il faut enlever les (-) dans les colonnes 4 et 9.

<sup>13</sup>De Henri: faut-il mettre le lien Internet pour [23] ?

<sup>14</sup>De Christiane: Je crois que c'est assez facile à trouver avec un engin de recherche, alors je dirais non.

<sup>15</sup>De Henri: Je repose la Q. au vu de la suggestion de l'arbitre pour mcqmc10.



$s$	best $c_s^{FL}$	$t$	best $c_s^{Kr}$	$t$	$s$	best $c_s^{FL}$	$t$	best $c_s^{Kr}$	$t$
2	2.60e-1	0	<b>0.173</b> (2)	0	15	<b>1.87e-10</b>	15	2.04e-10 (3)	7
3	1.25e-1	1	<b>6.28e-2</b> (3)	0	16	<b>8.42e-12</b>	15	3.48e-11 (3)	8
4	<b>2.26e-2</b>	1	4.29e-2 (3)	1	17	<b>2.86e-12</b>	18	5.59e-12 (3)	9
5	<b>6.51e-3</b>	2	7.81e-3 (3)	1	18	<b>2.29e-13</b>	19	2.83e-13 (3)	9
6	1.57e-3	3	<b>1.18e-3</b> (3)	1	19	<b>8.69e-15</b>	19	4.06e-14 (3)	10
7	<b>3.23e-4</b>	4	4.62e-4 (3)	2	20	<b>1.25e-15</b>	21	1.66e-14 (3)	12
8	<b>5.82e-5</b>	5	1.58e-4 (3)	3	25	<b>3.93e-20</b>	31	4.41e-20 (3)	15
9	<b>9.33e-6</b>	6	1.60e-5 (3)	3	30	<b>1.15e-25</b>	39	1.30e-25 (3)	19
10	<b>2.69e-6</b>	8	4.36e-6 (3)	4	35	<b>1.48e-31</b>	47	1.52e-30 (3)	25
11	<b>3.53e-7</b>	9	3.61e-7 (3)	4	40	<b>4.67e-38</b>	54	9.77e-37 (3)	29
12	<b>4.24e-8</b>	10	8.21e-8 (3)	5	45	<b>1.28e-43</b>	65	3.37e-43 (3)	33
13	<b>4.71e-9</b>	11	1.72e-8 (3)	6	50	4.01e-49	77	<b>6.72e-50</b> (3)	37
14	<b>9.71e-10</b>	13	3.36e-9 (3)	7					

Table 1: The best value of  $c_s$  is indicated in boldface. For  $c_s^{FL}$  the base is always  $b = 2$  and for  $c_s^{Kr}$  the base is always  $b = 3$  except for  $s = 2$  where  $b = 2$ .

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