Continuous Quantum Walks

Building Your Own Examples

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Introduction

Graph, adjacency matrix A
Continuous time quantum walk

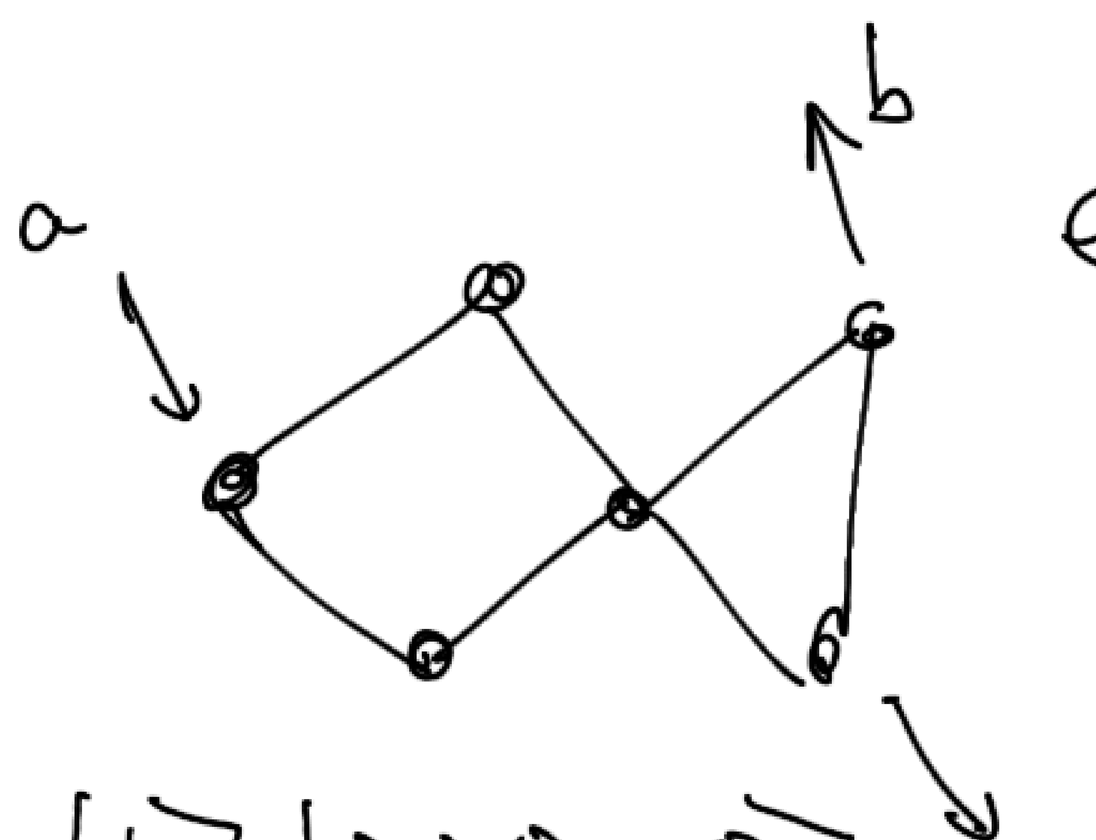
$$e^{-iAt}$$

Spectral decomposition:

$$A = \sum_r \theta_r E^r$$

projectors onto the θ_r subspace

eigenvalues



$$e^{-iAt} = \sum_r e^{-i\theta_r t} E^r$$

$|1\rangle |000\dots0\rangle$

$|\psi\rangle_a |00\dots0\rangle \longrightarrow |00\dots0\rangle |\psi\rangle_b$

Various tasks:

- Perfect state transfer

$$\exists t : \left| \left(e^{-iAt}\right)_{b,a} \right| = 1$$

- Pretty good state transfer

$$\forall \varepsilon > 0, \exists t : \left| \left(e^{-iAt}\right)_{b,a} \right| \geq 1 - \varepsilon$$

- Fractional Revivals

How to understand these tasks?

- Strongly cospectral vertices

$$E^r_{a,b} = \pm E^r_{a,a} \qquad \forall r$$

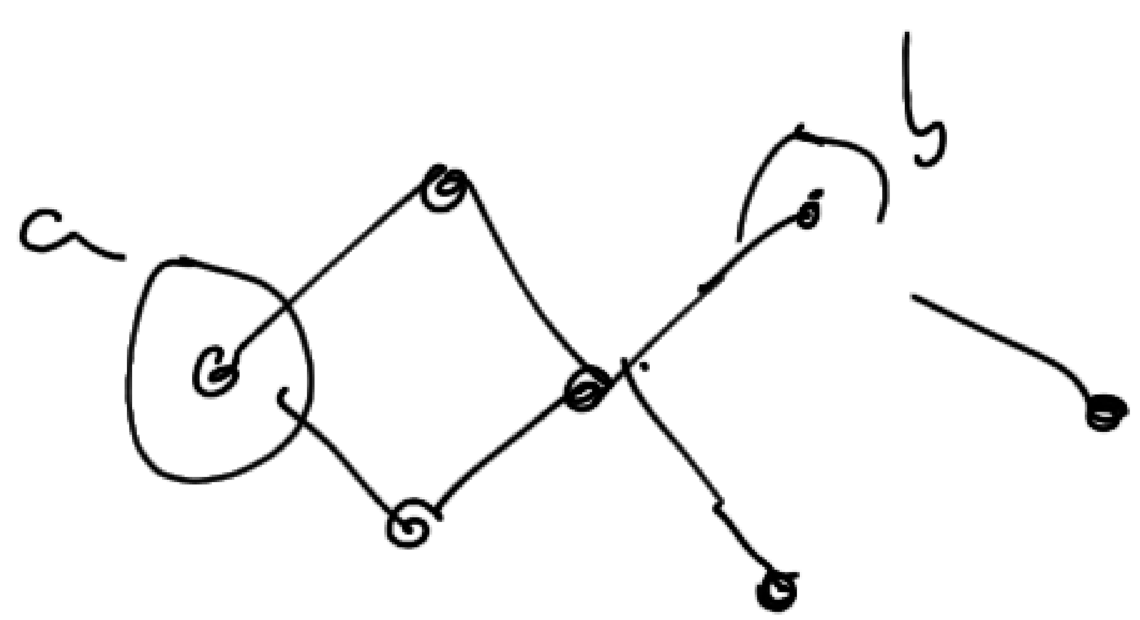
- Spectral properties

e.g. PST: $e^{-i\theta_r t} E^r_{a,b} = e^{i\phi} E^r_{a,a} \qquad \forall r : E^r_{a,a} \neq 0.$

$$\left(e^{-iAt}\right)_{b,a} = \sum_r e^{-i\theta_r t} E^r_{b,a}$$

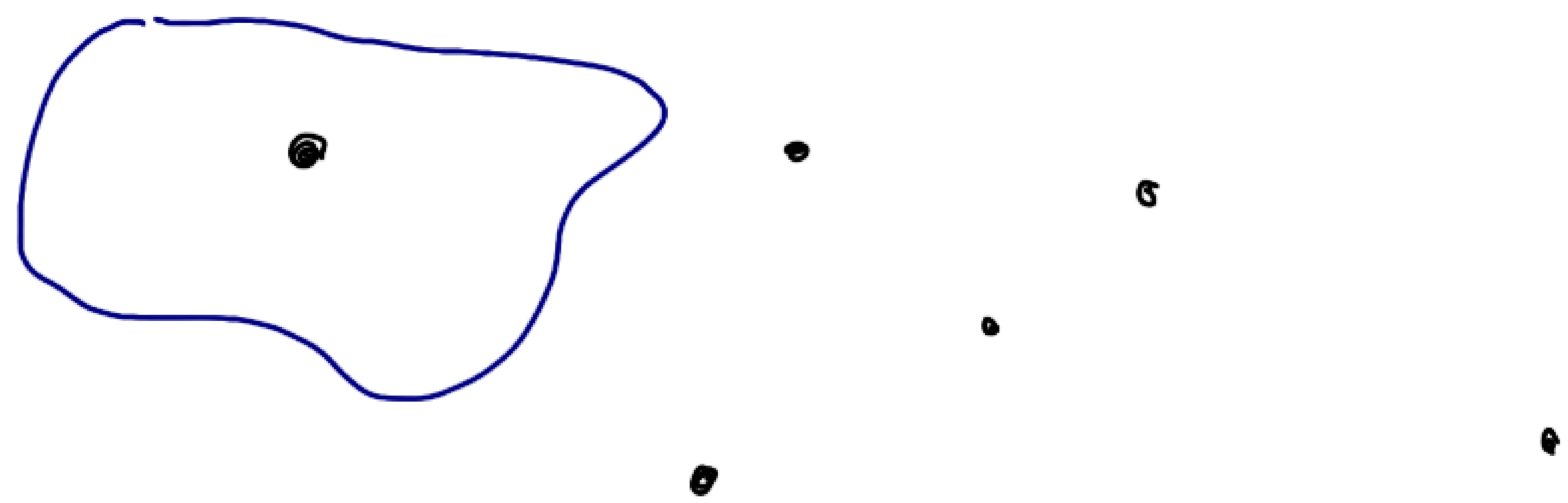
$$= \sum_r e^{i\phi} E^r_{a,a} \qquad \sum E^r_{a,a} = 1$$

$$= e^{i\phi}$$



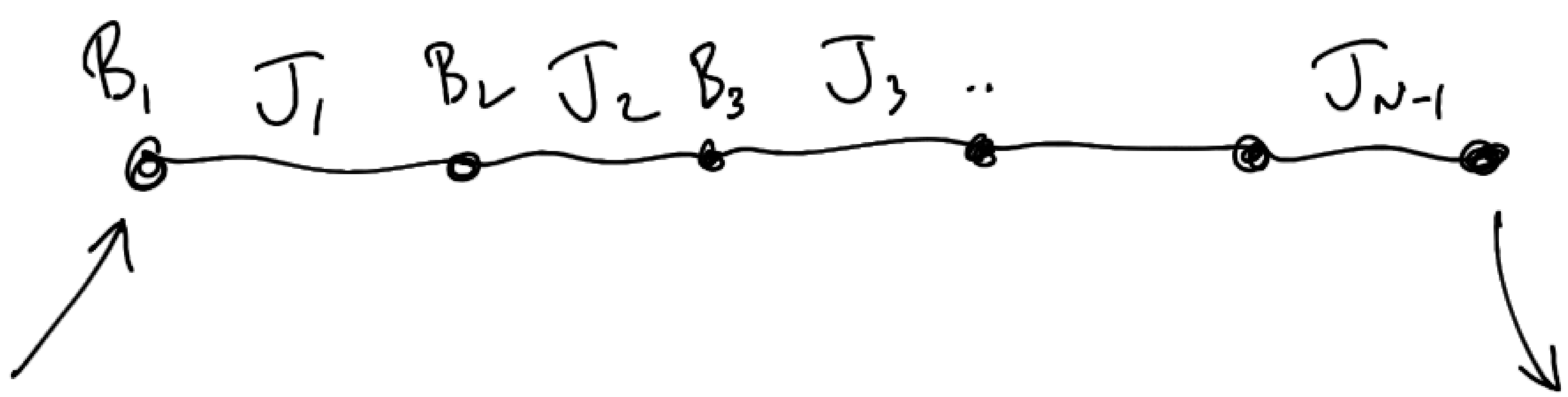
How to tinker?

- Graphs are discrete points in phase space.



Jacobi / tridiagonal

- Solution: weighted graphs.
- Specifically: paths (length N).



$$A = \begin{pmatrix} B_1 & J_1 & & & & 0 \\ J_1 & B_2 & J_2 & & & \\ & J_2 & B_3 & J_3 & & \\ & & J_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & J_{N-1} \\ 0 & & & & J_{N-1} & B_N \end{pmatrix}$$

Properties of weighted paths

- Eigenvalues are unique & all eigenvectors have support on first site.

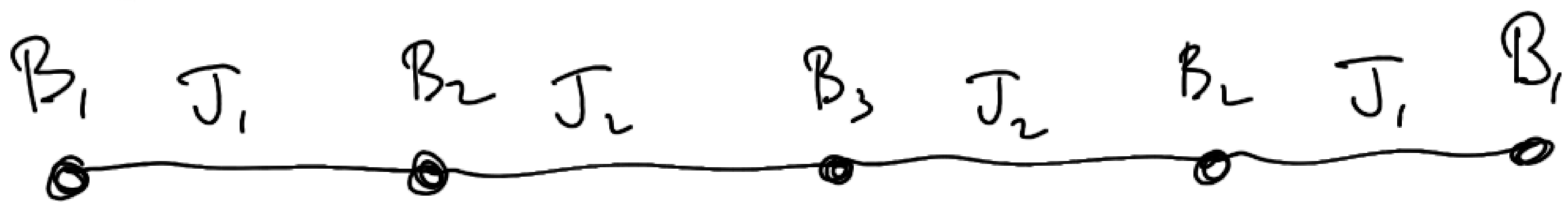
$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_N \end{pmatrix} = \theta \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{pmatrix}$$

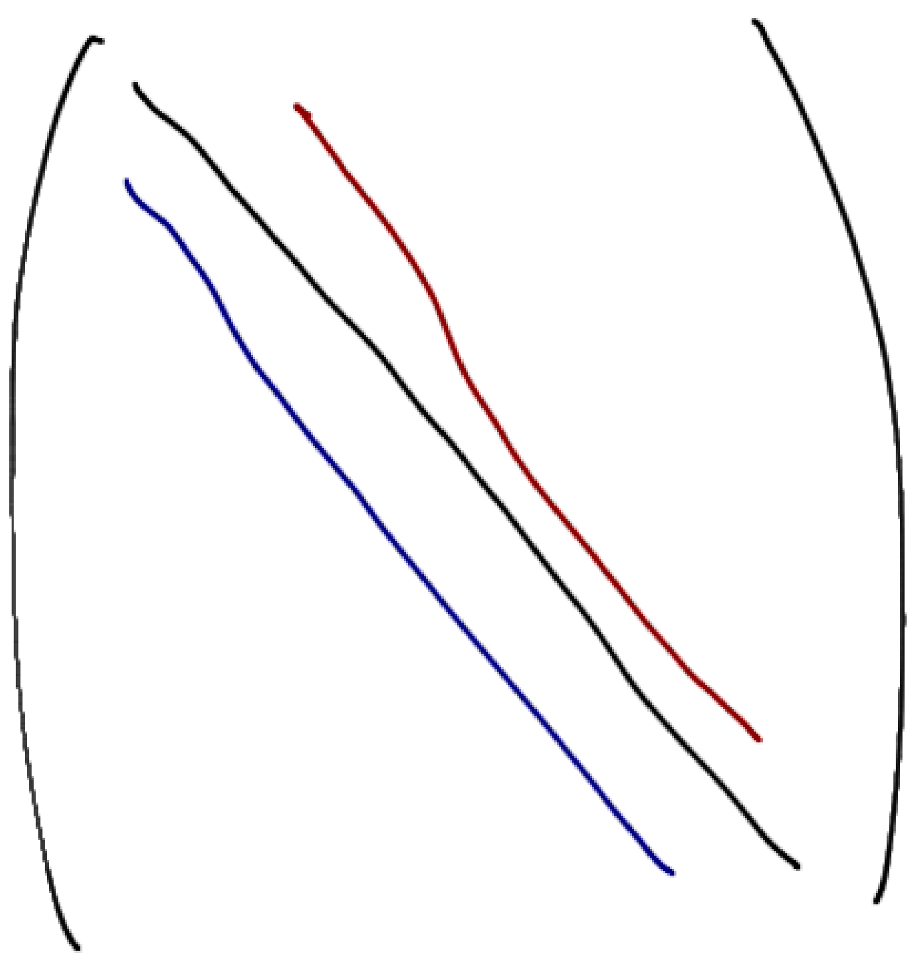
$$B_1\lambda_1 + J_1\lambda_2 = \theta\lambda_1$$
$$B_2\lambda_2 + J_1\lambda_1 + J_2\lambda_3 = \theta\lambda_2$$
$$B_3\lambda_3 + J_2\lambda_2 + J_3\lambda_4 = \theta\lambda_3$$

Consider $\lambda_1 = 0$
$\Rightarrow (J_1 = 0 \text{ or})\ \lambda_2 = 0$
$\Rightarrow \lambda_3 = 0$

pick a value $\lambda_1 = 1 \Rightarrow \lambda_2 \Rightarrow \lambda_3 \ldots\ldots$
for a fixed θ, eigenvector is uniquely determined

- (strongly) cospectral extremal vertices $\iff$ symmetric chain (proof using strong cospectrality is equivalent to a proof I'll do later)





$J_1^2 = J_{N-1}^2$

$J_n > 0$

persymmetric
centrosymmetric

Properties of weighted paths

- Symmetric chain $\implies$ alternating symmetry,

r = 1 largest eigenvalue

$$E^r_{1,1} = (-1)^{r+1} E^r_{1,N}.$$

r = N smallest eigenvalue

(proof not given, but relates to orthogonal polynomials. r^{th} eigenvector has $r-1$ zeros.)

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \cdots & \theta_N \\ \theta_1^2 & \theta_2^2 & \cdots & \theta_N^2 \\ & \vdots & & \\ \theta_1^{N-1} & \theta_2^{N-1} & \cdots & \theta_N^{N-1} \end{pmatrix} \begin{pmatrix} E^1_{1,N} \\ E^2_{1,N} \\ E^3_{1,N} \\ \vdots \\ E^N_{1,N} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \gamma \end{pmatrix}$$

Vandermonde matrix invertible.

- Symmetric chain $\implies$

$$E^r_{1,1} \propto \frac{(-1)^{r+1}}{p'(\theta_r)} \qquad \sum_r E^r_{1,1} = 1$$

where $p(x)$ is characteristic polynomial of A.

$\theta_1, \theta_2, \ldots$

$p(x) = (x-\theta_1)(x-\theta_2)\ldots$

$p'(x) = \frac{dp}{dx}$

$$A^k_{1,N} = \sum \theta_r^k E^r_{1,N}$$

total weight of paths of length k from 1 to N.

$k=0$: $A^k_{1,N} = 0$

$k=1$: $= 0$

$\vdots$ 0

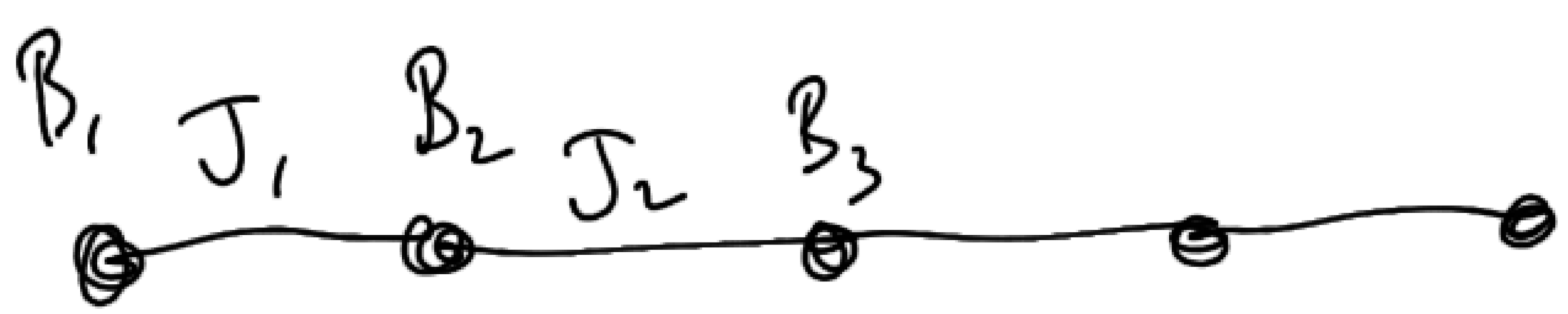
$k = N-2$

$$A^{N-1}_{1,N} = \underbrace{J_1 J_2 \ldots J_{N-1}}_{\gamma}$$

Inverse Eigenvalue Problem

$E^r_{1,1}$

- Given a target set of $\{\theta_r\}$, can construct a symmetric weighted path with those eigenvalues.
- Interpret $A^k_{1,1}$ as weight of paths of length k starting from and returning to site 1.



$$A^k_{1,1} = \sum_r \theta_r^k E^r_{1,1} \leftarrow \text{entirely known.}$$

$$A^k_{N,N} = \sum \theta_r^k E^r_{N,N}$$

$$= \sum \theta_r^k E^r_{1,1}$$

$k=1 \qquad A^1_{1,1} = B_1$

$k=2 \qquad A^2_{1,1} = B_1^2 + J_1^2 \Rightarrow J_1^2 \quad (J_1 > 0)$

$k=3 \qquad A^3_{1,1} = B_1^3 + J_1^2(2B_1 + B_2) \Rightarrow B_2$

Iterative scheme to find weights for A. All distinct spectra yield a solution.

Simplification

If θ_r occur in $\pm\theta$ pairs, (if N is odd, there's a 0 eigenvalue)

$$E^r_{1,1} = E^{N+1-r}_{1,1}$$

and thus

$$A^{2k-1}_{1,1} = 0 \implies B_k = 0$$

for all k. No self-weights.

Example: Perfect State Transfer

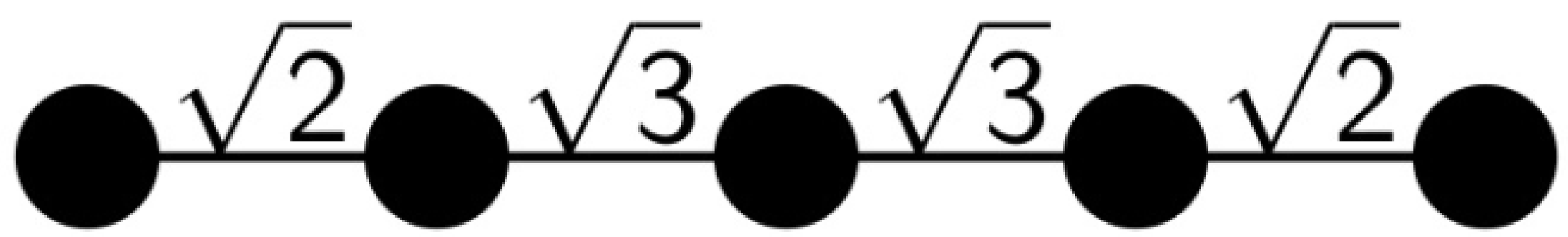
$J_n = \sqrt{n(N-n)}$

$$e^{-i\theta_r t_0} = e^{i\phi}(-1)^{r+1}$$

θ_r (up to a multiplicative factor) should have odd integer gaps.
e.g. $\{-2, -1, 0, 1, 2\}$.

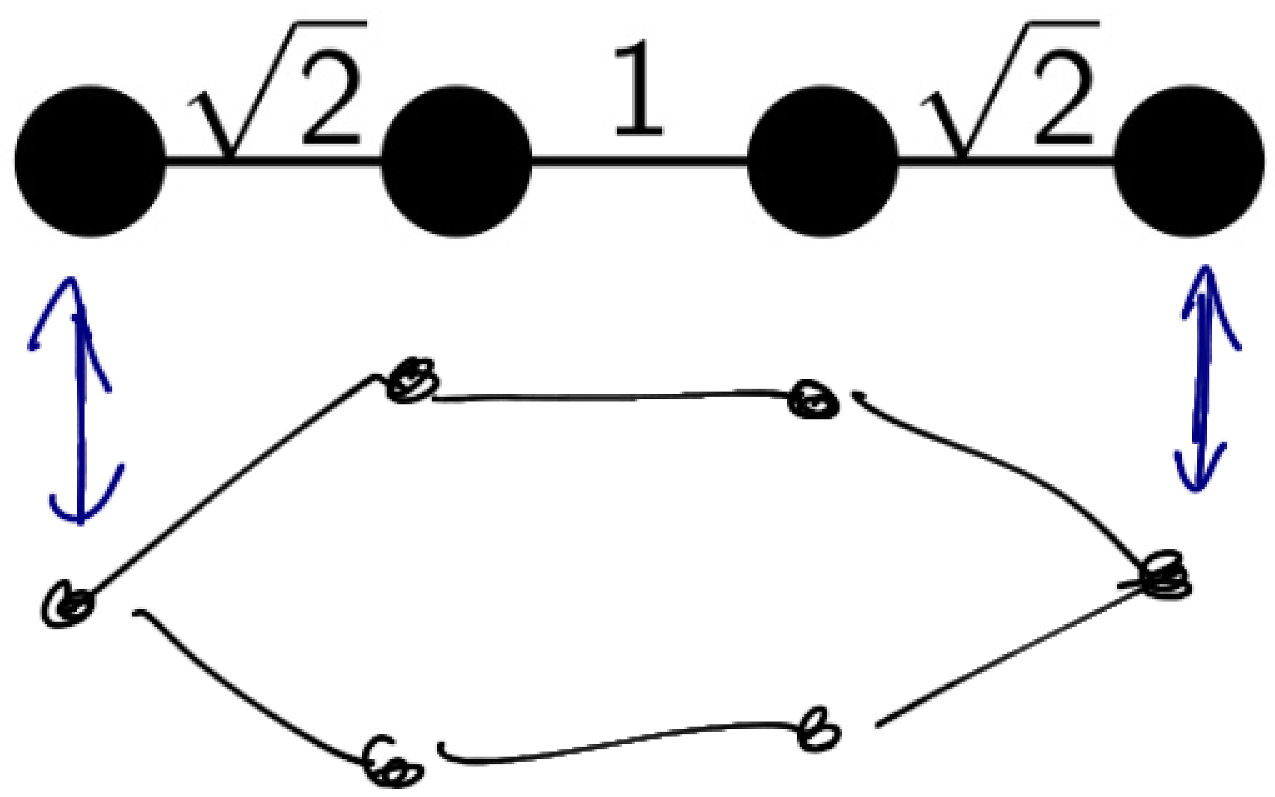
$t_0 = \pi$

$e^{i\pi} = -1$



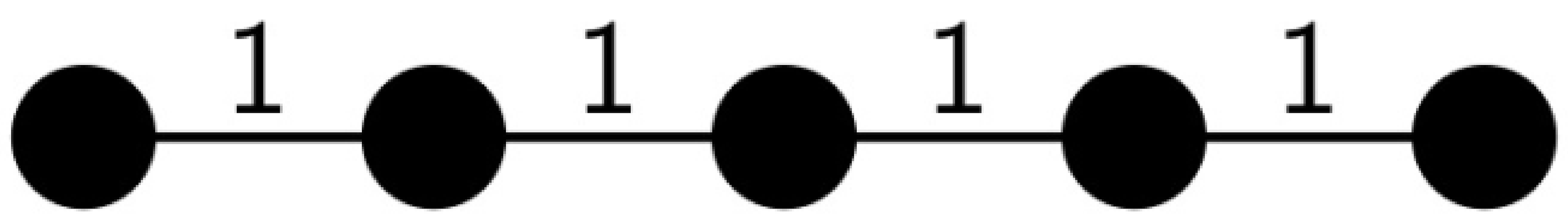
Whereas spectrum $\{\cancel{-}2, -1, 1, 2\}$ fails.

$t_0 = \pi$

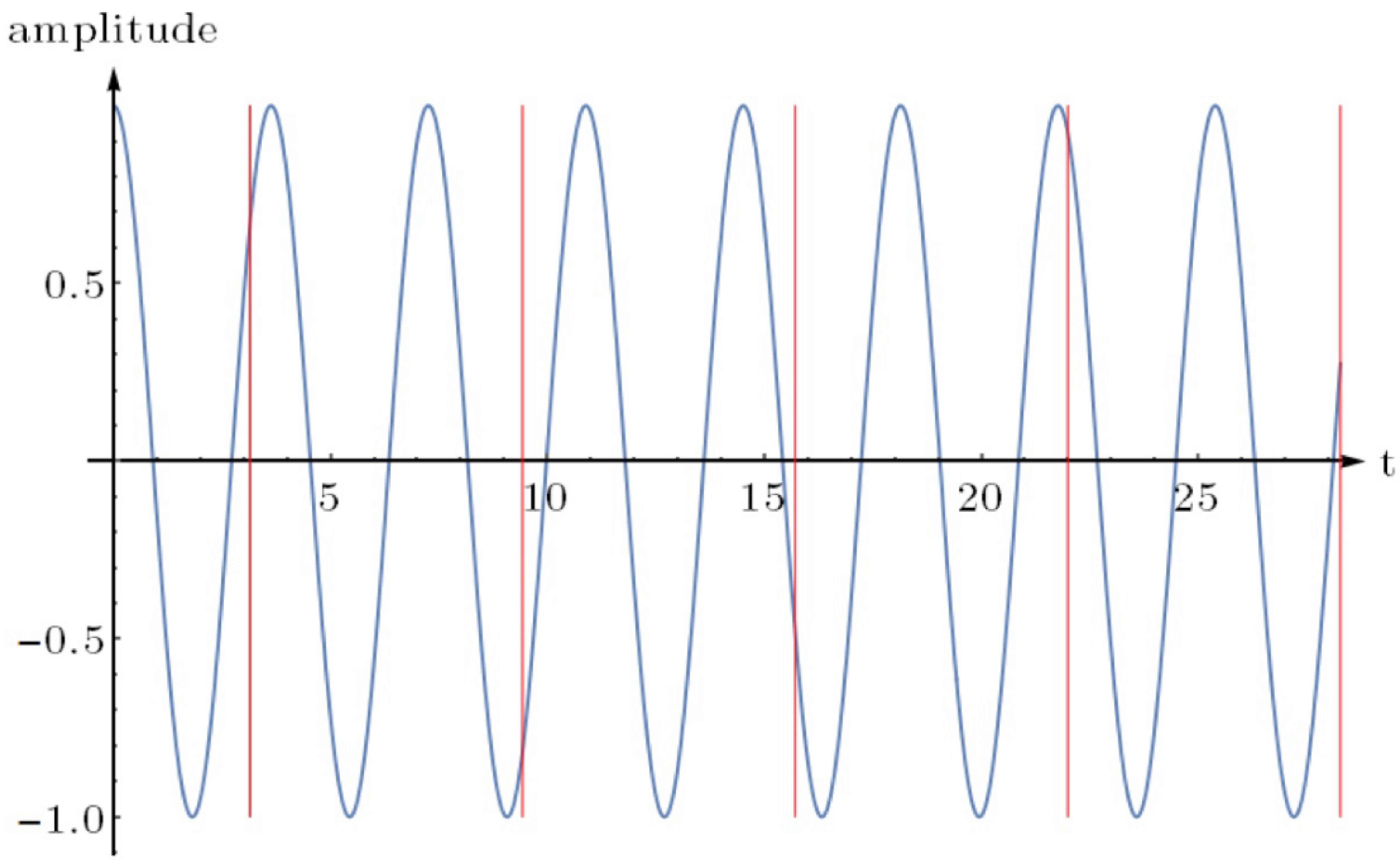


Example: Pretty Good Transfer

Spectrum: $\{-\sqrt{3}, -1, 0, 1, \sqrt{3}\}$. $\{-5, -4, -3, -\sqrt{3}, -1, 0, 1, \sqrt{3}, 3, 4, 5\}$



At all times $(2k+1)\pi$, the eigenvalues $\{-1, 0, 1\}$ are 'right'.
Monitor values $\cos((2k+1)\pi\sqrt{3})$.



Pretty Good Transfer: Time prediction?

- Continued fractions approximations of $\cos(\sqrt{3}t)$

$$\sqrt{3} = (1, 1, 2, 1, 2, \cancel{1, 2, 1, \ldots}).$$

- Approximate e.g. $\sqrt{3} \approx \frac{26}{15}$. Let $t_0 = 15\pi$.
- Amplitude ≈ 0.9997.

$\cos(\sqrt{3}t) \approx \cos\left(\frac{26}{15}t\right) = \cos\left(\frac{26}{\cancel{15}} \cdot \cancel{15}\pi\right)$

$= \cos(26\pi)$

$= 1$

$\frac{n}{m}$ $\quad t_0 = m\pi$

$\pm\sqrt{3}$, $\pm 2\sqrt{3}$

$\pm\sqrt{3}$, $\pm\sqrt{5}$

$$\left|\sqrt{3} - \frac{n}{m}\right| \leq \frac{1}{2m^2}$$

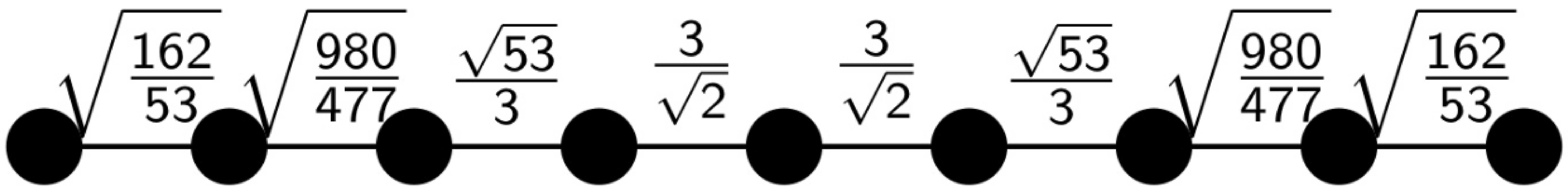
- Add further irrational, incommensurate, frequencies. Need to coordinate all coincidences. Takes (exponentially) longer.

Encoded Inputs and Outputs

What if I control a set of sites S_{in} and S_{out}?

Encoded Inputs and Outputs

Potential understanding: input state is orthogonal to "bad" eigenvectors.
Test: spectrum $\{-4, -3, -2, -\sqrt{2}, 0, \sqrt{2}, 2, 3, 4\}$.



- $1 \to N$ transfer, $t_0 = \pi$: $\frac{1}{106}(61 - 45\cos(\sqrt{2}\pi)) \approx 0.69$
- use 3 sites to make state orthogonal to two eigenvectors $\pm\sqrt{2}$. PST!

Summary

- Weighted paths give a continuous space in which to test spectrum-based insights on continuous walks by solving Inverse Eigenvalue Problem.

$|J - 1| \leq \delta$

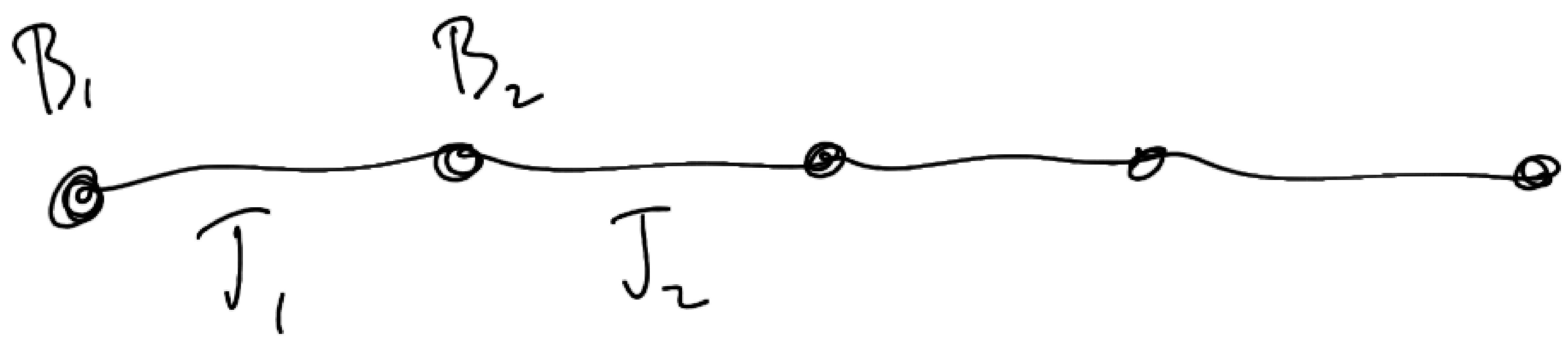
$\Rightarrow$ accuracy of eigenvalues

- What else?
  - New families of PST.
  - Solutions of PST arbitrarily close to uniform paths.

$\Rightarrow$ state transfer time

$J_n < 1$

$\left\{ \frac{J_n}{J_{max}} \right\}$ $t_0 \longrightarrow J_{max}\, t_0$



Apollaro

$$\frac{1}{2}\sum B_n Z_n + \frac{1}{2}\sum_{n,m} J_{n,m}\left(X_n X_m + Y_n Y_m \right) + \left(\gamma Z_n Z_m \right)$$

$$|00\ldots 0\,1\,000\ldots 0\rangle$$

n

1-excitation $H \longrightarrow A$