TOPOLOGICAL GROUPS

MATH 519

The purpose of these notes is to give a mostly self-contained topological background for the study of the representations of locally compact totally disconnected groups, as in [BZ] or [B, Chapter 4]. These notes have been adapted mostly from the material in the classical text [MZ, Chapters 1 and 2], and from [RV, Chapter 1]. An excellent resource for basic point-set topology is [M].

1. Basic examples and properties

A *topological group* G is a group which is also a topological space such that the multiplication map $(g,h) \mapsto gh$ from $G \times G$ to G, and the inverse map $g \mapsto g^{-1}$ from G to G, are both continuous. Similarly, we can define *topological rings* and *topological fields*.

Example 1. Any group given the discrete topology, or the indiscrete topology, is a topological group.

Example 2. $\mathbb{R}$ under addition, and $\mathbb{R}^\times$ or $\mathbb{C}^\times$ under multiplication are topological groups. $\mathbb{R}$ and $\mathbb{C}$ are topological fields.

Example 3. Let R be a topological ring. Then $\mathrm{GL}(n,R)$ is a topological group, and $M_n(R)$ is a topological ring, both given the subspace topology in R^{n^2}.

If G is a topological group, and $t \in G$, then the maps $g \mapsto tg$ and $g \mapsto gt$ are homeomorphisms, and the inverse map is a homeomorphism. Thus, if $U \subset G$, we have

$$U \text{ is open} \iff tU \text{ is open} \iff Ut \text{ is open} \iff U^{-1} \text{ is open.}$$

A topological space X is called *homogeneous* if given any two points $x, y \in X$, there is a homeomorphism $f : X \to X$ such that $f(x) = y$. A homogeneous space thus looks topologically the same near every point. Any topological group G is homogeneous, since given $x, y \in G$, the map $t \mapsto yx^{-1}t$ is a homeomorphism from G to G which maps x to y.

If X is a topological space, $x \in X$, a *neighborhood* of x is a subset U of X such that x is contained in the interior of U. That is, U is not necessarily open, but there is an open set $W \subset X$ containing x such that $W \subset U$.

If G is a group, and S and T are subsets of G, we let ST and S^{-1} denote

$$ST = \{st \mid s \in S, t \in T\} \quad \text{and} \quad S^{-1} = \{s^{-1} \mid s \in S\}.$$

The subset S is called *symmetric* if $S^{-1} = S$. We will let 1 denote the identity element of a group unless otherwise stated. The following result, although innocent enough looking, will be the most often used in all of the results which follow.

Proposition 1.1. *Let G be a topological group. Every neighborhood U of 1 contains an open symmetric neighborhood V of 1 such that $VV \subset U$.*

Proof. Let U' be the interior of U. Consider the multiplication map $\mu : U' \times U' \to G$. Since μ is continuous, then $\mu^{-1}(U')$ is open and contains $(1,1)$. So, there are open sets $V_1, V_2 \subset U$ such that $(1,1) \in V_1 \times V_2$, and $V_1V_2 \subset U$. If we let $V_3 = V_1 \cap V_2$, then $V_3V_3 \subset U$ and V_3 is an open neighborhood of 1. Finally, let $V = V_3 \cap V_3^{-1}$, which is open, contains 1, is symmetric, and satisfies $VV \subset U$. $\square$

Proposition 1.2. *If G is a topological group, then every open subgroup of G is also closed.*

Proof. Let H be an open subgroup of G. Then any coset xH is also open. So,

$$Y = \bigcup_{x \in G \setminus H} xH$$

is also open. From elementary group theory, $H = G \setminus Y$, and so H is closed. $\square$

Proposition 1.3. *If G is a topological group, and if K_1 and K_2 are compact subsets of G, then K_1K_2 is compact.*

Proof. The set $K_1 \times K_2$ is compact in $G \times G$, and multiplication is continuous. Since the continuous image of a compact set is compact, K_1K_2 is compact. $\square$

If X is a topological space, and A is a subset of X, recall that the *closure* of A, denoted $\overline{A}$, is the intersection of all closed subsets containing A. A necessary and sufficient condition for x to be an element of $\overline{A}$ is for every open neighborhood U of x, $U \cap A$ is nonempty, which may be seen as follows. If $x \notin \overline{A}$, then there is a closed set F which contains A, but $x \notin F$. Then $U = X \setminus F$ is an open neighborhood of x such that $U \cap A = \varnothing$. Conversely, if U is an open neighborhood of x such that $U \cap A = \varnothing$, then $X \setminus U$ is a closed set containing A which does not contain x, so $x \notin \overline{A}$.

Proposition 1.4. *If G is a topological group, and H is a subgroup of G, then the topological closure of H, $\overline{H}$, is a subgroup of G.*

Proof. Let $g, h \in \overline{H}$. Let U be an open neighborhood of the product gh. Let $\mu : G \times G \to G$ denote the multiplication map, which is continuous, so $\mu^{-1}(U)$ is open in $G \times G$, and contains (g,h). So, there are open neighborhoods V_1 of g and V_2 of h such that $V_1 \times V_2 \subset \mu^{-1}(U)$. Since $g, h \in \overline{H}$, then there are points $x \in V_1 \cap H \neq \varnothing$ and $y \in V_2 \cap H \neq \varnothing$. Since $x, y \in H$, we have $xy \in H$, and since $(x,y) \in \mu^{-1}(U)$, then $xy \in U$. Thus, $xy \in U \cap H \neq \varnothing$, and since U was an arbitrary open neighborhood of gh, then we have $gh \in \overline{H}$. Now let $\iota : G \to G$ denote the inverse map, and let W be an open neighborhood of h^{-1}. Then $\iota^{-1}(W) = W^{-1}$ is open and contains h, so there is a point $z \in H \cap W^{-1} \neq \varnothing$. Then we have $z^{-1} \in H \cap W \neq \varnothing$, and as before this implies $h^{-1} \in \overline{H}$. $\square$

Remark. Note that in the last part of the proof of Proposition 1.4, we have shown that the closure of a symmetric neighborhood of 1 is again symmetric.

Lemma 1.1. *Let G be a topological group, F a closed subset of G, and K a compact subset of G, such that $F \cap K = \varnothing$. Then there is an open neighborhood V of 1 such that $F \cap VK = \varnothing$ (and an open neighborhood V' of 1 such that $F \cap KV' = \varnothing$).*

Proof. Let $x \in K$, so $x \in G \setminus F$, and $G \setminus F$ is open. So, $(G \setminus F)x^{-1}$ is an open neighborhood of 1. By Proposition 1.1, there is an open neighborhood W_x of 1 such that $W_x W_x \subset (G \setminus F)x^{-1}$. Now, $K \subset \cup_{x \in K} W_x x$, and K is compact, so there exists a finite number of points $x_1, \ldots, x_n \in K$, such that $K \subset \cup_{i=1}^n W_i x_i$, where we write $W_i = W_{x_i}$. Now let

$$V = \bigcap_{i=1}^{n} W_i.$$

For any $x \in K$, $x \in W_i x_i$ for some i. Now we have

$$Vx \subset W_i x \subset W_i W_i x_i \subset G \setminus F.$$

In other words, $F \cap Vx = \varnothing$. Since this is true for any $x \in K$, we now have $F \cap VK = \varnothing$. $\square$

Remark. Note that from Proposition 1.1, the neighborhood V in Lemma 1.1 may be taken to be symmetric.

Proposition 1.5. *Let G be a topological group, K a compact subset of G, and F a closed subset of G. Then FK and KF are closed subsets of G.*

Proof. If $FK = G$, we are done, so let $y \in G \setminus FK$. This means $F \cap yK^{-1} = \varnothing$. Since K is compact, yK^{-1} is compact. By Lemma 1.1, there is an open neighborhood V of 1 such that $F \cap VyK^{-1} = \varnothing$, or $FK \cap Vy = \varnothing$. Since Vy is an open neighborhood of y contained in $G \setminus FK$, we have FK is closed. $\square$

2. Separation properties and functions

A topological space X is said to be T_1 if for any two distinct points $x, y \in X$, there is an open set U in X such that $x \in U$, but $y \notin U$. This is equivalent to one-point sets being closed. If G is a topological group, then G being T_1 is equivalent to $\{1\}$ being a closed set in G, by homogeneity.

A topological space X is said to be *Hausdorff* (or T_2) if given any two distinct points $x, y \in X$, there are open sets $U, V \subset X$, $x \in U$, $y \in V$, such that $U \cap V = \varnothing$. Recall the following basic properties of Hausdorff spaces.

Exercise 1. If X is a Hausdorff space, then every compact subset of X is closed.

Exercise 2. Let X be a topological space, and let $\Delta = \{(x, x) \mid x \in X\} \subset X \times X$ be the *diagonal* in $X \times X$. Then X is Hausdorff if and only if Δ is closed in $X \times X$.

Of course, if X is T_2, then X is T_1, but the converse does not hold in general. If G is a topological group however, the converse is true, which we now show.

Proposition 2.1. *Let G be a T_1 topological group. Then G is Hausdorff.*

Proof. Given distinct $g, h \in G$, take an open set U containing 1, such that $gh^{-1} \notin U$, which we may do since G is T_1. Applying Proposition 1.1, let V be an open symmetric neighborhood containing 1, such that $VV \subset U$. Now, Vg is open and contains g, and Vh is open and contains h. We must have $Vg \cap Vh = \varnothing$, otherwise there are $v_1, v_2 \in V$ such that $v_1 g = v_2 h$, which would mean

$$gh^{-1} = v_2 v_1^{-1} \in VV^{-1} = VV \subset U,$$

while gh^{-1} was chosen to be not an element of U. Thus G is Hausdorff. $\square$

We can say even more than Proposition 2.1. A topological space X is called *regular* or T_3 if X is T_1, and for any point $x \in X$ and any closed subset $F \subset X$ such that $x \notin F$, there is an open set U containing x and an open set V containing F such that $U \cap V = \varnothing$. The space X is called *completely regular* or *Tychonoff* or $T_{3\frac{1}{2}}$ if it is T_1 and for any point $x \in X$ and any closed set $F \subset X$ such that $x \notin F$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for every $y \in F$. Every space which is completely regular is also regular, since, for example, $f^{-1}([0,1/3))$ and $f^{-1}((2/3,1])$ are disjoint open sets in X containing x and F, respectively. We now see that any topological group which is T_1 is also completely regular, and thus regular.

Theorem 2.1. *Let G be a topological group, let 1_G denote the identity element in G, and let F be a closed subset of G such that $1_G \notin F$. Then there is a continuous function $f : G \to [0,1]$ such that $f(1_G) = 0$ and $f(y) = 1$ for every $y \in F$.*

Proof. See Problem Set 1. $\square$

Corollary 2.1. *If G is a topological group which is T_1, then G is completely regular and thus regular.*

Proof. Let $x \in G$ and let F be a closed subset of G such that $x \notin F$. Then $x^{-1}F$ is a closed subset of G not containing 1_G, and from Theorem 2.1, there is a continuous function $f : G \to [0,1]$ such that $f(1_G) = 0$ and $f(y) = 1$ for $y \in x^{-1}F$. Now the function $h(g) = f(x^{-1}g)$ is the desired continuous function, and since G is also T_1, G is completely regular, and so is also regular. $\square$

Let f be an $\mathbb{R}$-valued continuous function on a topological group G (we could also consider $\mathbb{C}$-valued functions). The *left* and *right translates* of f, written $L_h f$ and $R_h f$, respectively, are given by

$$L_h f(g) = f(h^{-1}g) \quad \text{and} \quad R_h f(g) = f(gh).$$

The function f is *left uniformly continuous* if for every $\varepsilon > 0$, there is a neighborhood V of 1 such that

$$h \in V \Longrightarrow ||L_h f - f||_\infty < \varepsilon,$$

where $||f||_\infty$ denotes the supremum norm. We may define a function to be *right uniformly continuous* similarly.

The *support* of a function f on a topological group G, written $\mathrm{supp}(f)$, is defined to be the topological closure of the set of points in G for which f is nonzero. That is,

$$\mathrm{supp}(f) = \overline{\{g \in G \mid f(g) \neq 0\}}.$$

We let $C_c(G)$ denote the set of continuous $\mathbb{R}$-valued functions on G with compact support. That is,

$$C_c(G) = \{f : G \to \mathbb{R} \mid f \text{ is continuous, } \mathrm{supp}(f) \text{ is compact}\}.$$

Proposition 2.2. *Let G be a topological group, and let $f \in C_c(G)$. Then f is left and right uniformly continuous.*

Proof. We will prove that f is right uniformly continuous, as the proof for left uniformly continuous is exactly analogous. Let $K = \mathrm{supp}(f)$, and let $\varepsilon > 0$. Let $g \in G$, and let $B_{\varepsilon/3}(f(g))$ be the open ball of radius $\varepsilon/3$ in $\mathbb{R}$ centered at $f(g)$. Then $f^{-1}(B_{\varepsilon/3}(f(g)))$ is an open neighborhood of g, call it W_g. Let $U_g = g^{-1}W_g$, which is an open neighborhood of 1, and if $h \in U_g$, then $gh \in W_g$. So, we have

$$h \in U_g \Longrightarrow |f(gh) - f(g)| < \varepsilon/3.$$

In other words, $f(g')$ is within $\varepsilon/3$ of $f(g)$ whenever $g^{-1}g' \in U_g$, or

$$g^{-1}g' \in U_g \Longrightarrow |f(g') - f(g)| < \varepsilon/3. \tag{2.1}$$

Applying Proposition 1.1, let V_g be an open symmetric neighborhood of 1 such that $V_gV_g \subset U_g$. K is compact, and

$$K \subset \bigcup_{g\in K} gV_g,$$

so we may take a finite number of $g \in K$, say $g_1, \ldots, g_n$, such that

$$K \subset \bigcup_{j=1}^{n} g_jV_{g_j}.$$

Let us write $V_j = V_{g_j}$ and $U_j = U_{g_j}$. Now let $V = \cap_{j=1}^{n} V_j$, which is an open symmetric neighborhood of 1. This will be the neighborhood which will give right uniform continuity.

Let $g \in K$, so that $g \in g_jV_j$ for some j, and let $h \in V$. Since $V_j \subset U_j$, we have $g_j^{-1}g \in U_j$. Since $h \in V_j$ and $V_jV_j \subset U_j$, we also have $g_j^{-1}gh \in U_j$. From (2.1) and our choice of U_j, we have, for any $h \in V$,

$$|f(g_j) - f(g)| < \varepsilon/3 \quad \text{and} \quad |f(gh) - f(g_j)| < \varepsilon/3.$$

The triangle inequality now gives

$$|f(gh) - f(g)| \le |f(gh) - f(g_j)| + |f(g_j) - f(g)| < 2\varepsilon/3,$$

for any $h \in V$.

Now suppose $g \notin K$, $h \in V$, and that $f(gh) \neq 0$ (otherwise $|f(gh) - f(g)| = 0$). For some j, we have $gh \in g_jV_j$, so $g_j^{-1}(gh) \in V_j \subset U_j$, and by continuity $f(gh)$ is within $\varepsilon/3$ of $f(g_j)$. Now, $h^{-1} \in V_j^{-1} = V_j$, since V_j is symmetric, and so we have

$$g_j^{-1}g = g_j^{-1}ghh^{-1} \in V_jV_j \subset U_j.$$

By (2.1), $f(g_j)$ is within $\varepsilon/3$ of $f(g) = 0$. Finally, we have

$$|f(gh)| \le |f(gh) - f(g_j)| + |f(g_j)| < 2\varepsilon/3.$$

Now, for any $g \in G$, $h \in V$, we have $|f(gh) - f(g)| < 2\varepsilon/3$. So,

$$h \in V \Longrightarrow ||R_hf - f||_\infty \le 2\varepsilon/3 < \varepsilon,$$

and f is right uniformly continuous. □

3. Quotients

If X is a topological space, and $\sim$ is an equivalence relation on X, let $X/\sim$ denote the set of equivalence classes in X under $\sim$, and if $x \in X$, let $[x]$ denote the equivalence class of x under $\sim$. We may give the set $X/\sim$ the *quotient topology* as follows. Let

$$p : X \to X/\sim, \quad p(x) = [x],$$

be the natural projection map. Define $U \subset X/\sim$ to be open if and only if $p^{-1}(U)$ is open in X (forcing p to be continuous). Note that this implies that $F \subset X/\sim$ is closed if and only if $p^{-1}(F)$ is closed in X.

Now let G be a topological group, and H a subgroup of G. We can look at the collection G/H of left cosets of H in G (or the collection $H\backslash G$ of right cosets), which defines an equivalence relation on G. So, we can put the quotient topology on G/H as above. Recall that G/H is not a group under coset multiplication unless H is a normal subgroup of G.

Proposition 3.1. *Let G be a topological group, and H a subgroup of G.*

(1) *G/H is a homogeneous space under translation by G.*
(2) *The map $p : G \to G/H$ is an open map.*
(3) *If H is compact, then $p : G \to G/H$ is a closed map.*
(4) *G/H is a Hausdorff space if and only if H is closed.*
(5) *H is open in G if and only if G/H is a discrete space. If G is compact, then H is open in G if and only if G/H is a finite and discrete space.*
(6) *If H is a normal subgroup of G, then G/H is a topological group.*
(7) *If H is the closure of the trivial subgroup, $H = \overline{\{1\}}$, then H is a normal subgroup of G and G/H is Hausdorff.*

Proof. (1): For $x \in G$, left translation by x on G/H gives a map $gH \mapsto xgH$. The inverse of this map is also a left translation, by x^{-1}, so to show this is a homeomorphism, we just need to show that it maps open sets to open sets, or is an open map. Let $U \subset G/H$ be open. By definition of the quotient topology, $p^{-1}(U) \subset G$ is open. It may be directly checked that we have $p^{-1}(xU) = xp^{-1}(U)$, which is also open. Since $p^{-1}(xU)$ is open, then xU is open by the definition of quotient topology, and so translation is an open map.

(2): Let $V \subset G$ be open. By the definition of quotient topology, $p(V) \subset G/H$ is open if and only if $p^{-1}(p(V)) \subset G$ is open. It may be checked that $p^{-1}(p(V)) = VH$. Since V is open, Vh is open for every $h \in H$. Since $VH = \cup_{h\in H} Vh$, VH is open, and so $p(V)$ is open.

(3): As in the proof of (2), we are reduced to showing that if $F \subset G$ is closed, then FH is closed. But H is compact, and so by Proposition 1.5, FH is closed.

(4): Suppose G/H is Hausdorff, so that it is T_1, and one point sets in G/H are closed. In particular, $\{H\}$ is closed in G/H. By the definition of the quotient topology, $\{H\} \subset G/H$ is closed if and only if $p^{-1}(\{H\}) = H \subset G$ is closed.

From Exercise 2, to show that G/H is Hausdorff, it is enough to show that the diagonal $\Delta = \{(gH, gH) \mid gH \in G/H\}$ is closed in $G/H \times G/H$. Through the natural map $f : (g_1H, g_2H) \mapsto (g_1, g_2)(H\times H)$, the space $G/H\times G/H$ is homeomorphic to $G\times G/H\times H$, and the image of the diagonal Δ under this map is $f(\Delta) = \{(g, g)(H \times H) \mid g \in G\}$. From the definition of the quotient topology, $f(\Delta)$ is closed if and only if

$$p^{-1}(f(\Delta)) = \{(g_1, g_2) \in G \times G \mid g_1g_2^{-1} \in H\}$$

is closed. But this is the inverse image of H of the continuous map from $G \times G$ to G which maps (g_1, g_2) to $g_1g_2^{-1}$. Since H is closed, then this set is closed as well.

(5): See Problem Set 1.

(6): Let T_g denote left multiplication by g, so that $T_g(x) = gx$, let ι and ι' denote the group inverse maps in G and G/H, respectively, and let $p : G \to G/H$ be the natural projection map. Since for any $x \in G$ we have

$$(p \circ T_g)(x) = gxH = (gH)(xH) = (T_{p(g)} \circ p)(x) \quad \text{and} \quad (p \circ \iota)(x) = x^{-1}H = (\iota' \circ p)(x),$$

the following diagrams are commutative:

$$\begin{array}{ccc} G & \xrightarrow{T_g} & G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ G/H & \xrightarrow{T_{p(g)}} & G/H \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\iota} & G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ G/H & \xrightarrow{\iota'} & G/H \end{array}$$

Since p is an open map by (2), and T_g and ι are continuous, this means $T_{p(g)}$ and ι' must also be continuous, making G/H a topological group.

(7): By Proposition 1.4, $H = \overline{\{1\}}$ is a subgroup of G. H is then the minimal closed subgroup of G containing 1, while for any $x \in G$, xHx^{-1} is also a closed subgroup of G containing 1. Thus $H \subset xHx^{-1}$, and so $x^{-1}Hx \subset H$ for any $x \in G$, and H is a normal subgroup of G. Now, G/H is a topological group by (6), and G/H is Hausdorff by (4). $\square$

4. Local compactness and connectedness

A topological space X is called *locally compact* if for every $x \in X$, there is a compact neighborhood $U \subset X$ of x. Before proving a few basic properties of locally compact spaces, recall the following, which we will need.

Exercise 3. If X is a compact space, then every closed subset of X is compact.

Lemma 4.1. *Let X be a locally compact Hausdorff space. Then X is regular, and any neighborhood V of any point $x \in X$ contains a compact neighborhood K of x.*

Proof. Let $x \in X$, and let F be a closed subset in X such that $x \notin F$. Let K be a compact neighborhood of x. Since K is closed (by Exercise 1), then $M = K \cap F$ is closed, and so $M \subset K$ is compact (by Exercise 3). For every $y \in M$, choose an open neighborhood U_y of y, and an open neighborhood U_x^y of x such that $U_y \cap U_x^y = \varnothing$, which we can do since X is Hausdorff. Since M is compact, there are a finite number of points $y_1, \ldots, y_n \in M$ such that $M \subset \cup_{i=1}^n U_i$, where $U_i = U_{y_i}$. Writing $U_x^{y_i} = U_x^i$, we have

$$x \in W = \bigcap_{i=1}^n U_x^i, \quad \text{and} \quad W \cap \bigcup_{i=1}^n U_i = \varnothing,$$

where W is an open neighborhood of x. Now, if $K' = K \setminus (\cup_{i=1}^n U_i)$, then K' is a closed subset of K, and so is compact, and it is a neighborhood of x since $W \subset K'$. In particular, K' is disjoint from F. Now let $V = \operatorname{int}(K')$ be the interior of K', which is an open neighborhood of x. Then $V' = X \setminus K'$ is an open set containing F, and $V \cap V' = \varnothing$. Thus X is regular.

For the second statement, let V be any neighborhood of $x \in X$, and let $V' = \operatorname{int}(V)$ be the interior of V. Let C be a compact neighborhood of x, and let $U' = \operatorname{int}(C)$. Then $U = U' \cap V$ is an open neighborhood of x. Now $X \setminus U$ is closed, and since we have shown that X is regular, then there are open sets V_1, V_2 such that $X \setminus U \subset V_1$, $x \in V_2$, and $V_1 \cap V_2 = \varnothing$. Now let $K = \overline{V_2}$. Since $V_2 \subset X \setminus V_1$, which is closed, then $K \subset X \setminus V_1$.

Then $K \subset U$ since $X \setminus U \subset V_1$, and so $K \subset C$. Since C is compact and K is closed, K is also compact. Now K is a compact neighborhood of x and $K \subset U \subset V$. $\square$

A topological group G is called a *locally compact group* if it is a locally compact space and it is Hausdorff.

Proposition 4.1. *Let G be a Hausdorff topological group. Any subgroup H of G which is locally compact (in the subspace topology) is closed.*

Proof. Let K be a compact neighborhood of 1 in H. Then K is closed in H by Exercise 1. By the definition of subspace topology, there is a closed neighborhood F of 1 in G such that $K = F \cap H$. Since K is compact in H, it is compact in G, and so K is closed in G. Applying Proposition 1.1, let V be an open neighborhood of 1 such that $VV \subset F$.

Now, $\overline{H}$ is a subgroup of G by Proposition 1.4. Let $x \in \overline{H}$. To show H is closed, it is enough to show that $x \in H$. Now $x^{-1} \in \overline{H}$, and so every neighborhood of x^{-1} intersects H. In particular, Vx^{-1} is a neighborhood of x^{-1}, so take some point $y \in Vx^{-1} \cap H$. Now consider yx, and let W be a neighborhood of yx. Now $y^{-1}W$ and xV are neighborhoods of x, and so $y^{-1}W \cap xV$ is a neighborhood of x. Since $x \in \overline{H}$, there is a point

$$z \in (y^{-1}W \cap xV) \cap H.$$

Now, $y \in Vx^{-1}$ and $z \in xV$, so $yz \in (Vx^{-1})(xV) = VV \subset F$. Also $yz \in W \cap H$, since $z \in y^{-1}W$, and both y and z are in the subgroup H. Therefore we have

$$yz \in W \cap (F \cap H) = W \cap K,$$

which is thus nonempty. Since K is closed and W was an arbitrary neighborhood of yx, then we must have $yx \in K \subset H$. Since $y, yx \in H$, then $x \in H$, and so H is closed. $\square$

We will need to apply the following technical lemma later.

Lemma 4.2. *Let G be a locally compact group, K a compact subset of G, and U an open neighborhood of 1 in G. Then there is a neighborhood V of 1 in G such that $x^{-1}Vx \subset U$ for every $x \in K$.*

Proof. Let $x \in K$. Then xUx^{-1} is an open neighborhood of 1. Since G is a locally compact group, there is a compact neighborhood V_x of 1 such that $V_x \subset xUx^{-1}$ by Lemma 4.1. Let $F = G \setminus U$, then $x^{-1}V_x x \cap F = \varnothing$. From Lemma 1.1, there is a neighborhood W_x' of 1, which may be chosen to be compact by Lemma 4.1, and symmetric by Proposition 1.1 and the remark after Proposition 1.4 (along with Exercise 3), such that

$$(x^{-1}V_x x)W_x' \cap F = \varnothing.$$

The fact that W_x' is symmetric implies that we also have

$$x^{-1}V_x x \cap FW_x' = \varnothing.$$

From Proposition 1.5, FW_x' is closed, since F is closed and W_x' is compact. So, again by Lemma 1.1 and Proposition 1.1, there is a symmetric neighborhood W_x'' of 1 such that

$$W_x''(x^{-1}V_x x) \cap FW_x' = \varnothing. \tag{4.1}$$

Now let $W_x = W_x' \cap W_x''$, which is a symmetric neighborhood of 1. Then we must have

$$W_x(x^{-1}V_x x)W_x \cap F = \varnothing, \tag{4.2}$$

otherwise (4.1) would be violated (since W_x and W_x' are symmetric).

For each $x \in K$, let $U_x = \operatorname{int}(W_x)$ be the interior of W_x. Then, the collection of all xU_x, $x \in K$, constitutes an open cover of K, and so there is a finite number of points, say $x_1, x_2, \ldots, x_n$, such that, writing $W_{x_i} = W_i$, and $U_{x_i} = U_i$,

$$K \subset \bigcup_{i=1}^{n} x_i U_i \subset \bigcup_{i=1}^{n} x_i W_i.$$

Now let $V = \cap_{i=1}^{n} V_i$, where $V_i = V_{x_i}$. If $x \in K$, then $x \in x_i W_i$ for some i, and so $x^{-1} \in W_i x_i^{-1}$, since W_i is symmetric. Now, by (4.2), we have

$$x^{-1} V x \subset W_i x_i^{-1} V_i x_i W_i \subset G \setminus F = U. \quad \square$$

A topological space X is *connected* if whenever $X = U \cup V$ where U and V are nonempty open sets, then $U \cap V \neq \varnothing$. That is, X is connected when X has no nonempty proper subsets which are both closed and open (or *clopen*). A maximal connected subset of X is called a *connected component* of X. The space X is *totally disconnected* if each one-point subset in X is its own connected component. Of course, every discrete space is totally disconnected. One familiar example of a totally disconnected space which is not discrete is the Cantor middle-thirds set.

Exercise 4. If $A \subset X$ is connected, then $\overline{A}$ is connected. That is, connected components are closed sets.

If G is a topological group, then G is totally disconnected if and only if $\{1\}$ is a connected component, by homogeneity. The connected component of 1 in G will be denoted G°, and its basic properties are as follows.

Proposition 4.2. *If G is a topological group, then G° is a normal subgroup of G, the connected components of G are all of the form xG° for $x \in G$, and G/G° is a totally disconnected group.*

Proof. See Problem Set 1. $\square$

Finally, we turn to the study of spaces which are locally compact and totally disconnected. Before proving the main statements, we first need a few more preliminary lemmas.

A topological space X is called *normal* or T_4 if it is T_1 (one point sets are closed) and for any disjoint closed subsets E and F of X, there are open sets U and V such that $E \subset U$, $F \subset V$, and $U \cap V = \varnothing$.

Exercise 5. Every compact Hausdorff space is normal. Note that from Lemma 4.1, we already know compact Hausdorff spaces are regular.

Lemma 4.3. *Let X be a compact Hausdorff space, and let $x \in X$. Let $\mathcal{U}_x$ denote the collection of compact open neighborhoods of x. Then $\cap_{U \in \mathcal{U}_x} U$ is the connected component of x.*

Proof. Let $F = \cap_{U \in \mathcal{U}_x} U$, which is a nonempty closed set since X itself is a compact open neighborhood of x, and each $U \in \mathcal{U}_x$ is compact and thus closed (by Exercise 3). Suppose that V' and W' are closed and open subsets of F (in the subspace topology of F) such that

$$F = V' \cup W', \quad \text{and} \quad V' \cap W' = \varnothing.$$

Since F is closed, then V' and W' are closed subsets in X. Since X is normal by Exercise 5, then there are disjoint open sets V and W of X such that $V' \subset V$ and $W' \subset W$. To show F is connected, we must show that one of V' or W' is empty.

Now, $B = X \setminus (V \cup W)$ is closed, and thus compact, and does not intersect F. So, the sets $X \setminus U$, $U \in \mathcal{U}_x$, cover B, and are all open (and closed) since each U is compact (thus closed) and open. Since B is compact, there are a finite number of neighborhoods of x, $U_1, \ldots, U_n \in \mathcal{U}_x$, such that $B \subset \cup_{i=1}^n (X \setminus U_i)$. In other words, if we let $A = \cap_{i=1}^n U_i$, then $A \cap B = \varnothing$, $x \in A$, and A is compact and open. Now $A \subset X \setminus B = V \cup W$, and so

$$A = (A \cap V) \cup (A \cap W),$$

where $A \cap V$ and $A \cap W$ are disjoint open sets. Since A is closed, $A \cap V$ and $A \cap W$ are also both closed (and thus compact). So, x can only be an element of one of them, say $x \in A \cap V$, which means that $F \subset A \cap V$ (since $A \cap V$ is a compact open neighborhood of x), while $F \cap (A \cap W) = \varnothing$. This means we must have $F = V'$ and $W' = \varnothing$, so that F is connected.

Now let C be the connected component of x, so that $F \subset C$. Suppose that $F \neq C$, so that there is a point $y \in C \setminus F$. Then there must be a compact open neighborhood M of x such that $y \notin M$. Now $M \cap C$ is closed and open in C, while $(X \setminus M) \cap C$ contains y, contradicting the fact that C is connected. Thus F is the connected component of x. $\square$

Lemma 4.4. *Let X be a compact Hausdorff space, let C be a connected component of X, and let F be a closed subset of X such that $F \cap C = \varnothing$. Then there is a compact open set V such that $C \subset V$ and $F \cap V = \varnothing$.*

Proof. We have F is compact (Exercise 3), and if $x \in C$, then $C = \cap_{U \in \mathcal{U}_x} U$, where $\mathcal{U}_x$ is the collection of compact open neighborhoods of x, by Lemma 4.3. The open sets $X \setminus U$, $U \in \mathcal{U}_x$ cover F, and so for a finite number of sets in $\mathcal{U}_x$, say $U_1, \ldots, U_n$, F is covered by $\cup_{i=1}^n (X \setminus U_i)$. If we let $V = \cap_{i=1}^n U_i$, we have $F \cap V = \varnothing$, and $C \subset V$, as desired. $\square$

Theorem 4.1. *Let X be a Hausdorff space. Then X is locally compact and totally disconnected if and only if every neighborhood of every point $x \in X$ contains a compact open neighborhood of x.*

Proof. ($\Rightarrow$): Let $x \in X$, and let U be the interior of an arbitrary neighborhood of x. By Lemma 4.1, there is a compact neighborhood K of x contained in U. Now let V be an open neighborhood of x, $V \subset K$, and let $F = K \setminus V$. If $F = \varnothing$, then V is open and compact, and we are done. The set F is closed, and $\{x\}$ is a connected component of X since X is totally disconnected, and so $\{x\}$ is a connected component of the compact subset K. Since $F \cap \{x\} = \varnothing$, then by Lemma 4.4, there is a compact open set W containing x such that $F \cap W = \varnothing$. That is, $W \subset V \subset U$, and W is a compact open neighborhood of x.

($\Leftarrow$): First, since every point $x \in X$ has a compact neighborhood, then X is locally compact. Now let S be the connected component of $x \in X$. Suppose that $y \neq x$ and $y \in S$. Since X is Hausdorff, it is T_1, and so x has an open neighborhood W such that $y \notin W$. Let U be a compact open neighborhood of x which is contained in W. Since X is Hausdorff, U is closed. So, $U' = U \cap S$ is closed and open in S. But $y \notin U'$, and so U' is a proper nonempty clopen subset of S, contradicting the fact that S is connected. Thus, $S = \{x\}$, and X is totally disconnected. $\square$

Corollary 4.1. *Let G be a locally compact totally disconnected group, and H a subgroup of G. Then H is closed if and only if H is a locally compact totally disconnected group, if and only if G/H is a locally compact totally disconnected Hausdorff space.*

Proof. First, suppose H is closed, and let $x \in H$, and U any neighborhood of x in H. Then $U = H \cap V$, where V is a neighborhood of x in G. By Theorem 4.1, V contains a compact open neighborhood of x, say F. By definition, $H \cap F$ is an open neighborhood of x in H, and it is contained in U. Since F is compact and G is Hausdorff, then F is closed, and so $F \cap H$ is closed in G. Moreover, $F \cap H$ is compact in G since it is closed and contained in F, which is compact. Thus $F \cap H$ is compact in H. Now $F \cap H$ is a compact open neighborhood of x in H which is contained in U, and H is locally compact and totally disconnected by Theorem 4.1. Conversely, if H is locally compact in the subspace topology, then it is automatically closed by Proposition 4.1.

For the second part, suppose that H is closed. Since $p : G \to G/H$ is an open map by part (2) of Proposition 3.1, and is continuous by definition, then the image under p of compact open sets of G are compact open sets of G/H. If U is an open neighborhood of $xH \in G/H$, then $p^{-1}(U)$ is an open neighborhood of $x \in G$, which contains a compact open neighborhood K of x, by Theorem 4.1. Then $p(K)$ is a compact open neighborhood of xH contained in $p(p^{-1}(U)) = U$. Thus G/H is locally compact and totally disconnected by Theorem 4.1. Since H is assumed to be closed, then G/H is Hausdorff by part (4) of Proposition 3.1. Conversely, if G/H is Hausdorff, then H is automatically closed also by part (4) of Proposition 3.1. $\square$

Theorem 4.2. *Let G be a locally compact totally disconnected group. Every neighborhood of 1 contains a compact open subgroup of G. If G is a compact totally disconnected group, then every neighborhood of 1 contains a compact open normal subgroup of G.*

Proof. Since G is a locally compact totally disconnected group, each neighborhood of 1 contains a compact open neighborhood V of 1, from Theorem 4.1. Let us denote $V^n = VV^{n-1}$ for $n \geq 2$. Let $F = (G \setminus V) \cap V^2$. Since V is open, $G \setminus V$ is closed, and since G is Hausdorff and V is compact, V is closed and so V^2 is closed, by Proposition 1.5. Thus F is closed.

We have $V \cap F = \varnothing$, where V is compact and F is closed. By Lemma 1.1 and Proposition 1.1, there is an open symmetric neighborhood W of 1, $W \subset V$, such that $VW \cap F = \varnothing$. Since $W \subset V$, then $VW \subset V^2$. Because $F = (G \setminus V) \cap V^2$, and $VW \cap F = \varnothing$, then we must have $VW \subset V$. Now we have

$$VW^2 \subset VW \subset V,$$

and by induction we must have $VW^n \subset V$ for every integer $n \geq 0$. Since W was chosen to be symmetric, then in fact $VW^n \subset V$ for every integer n. In particular, since $1 \in V$, we have

$$\bigcup_{n \in \mathbb{Z}} W^n \subset V.$$

Now, $H = \cup_{n \in \mathbb{Z}} W^n$ is a subgroup of G contained in V. Since each W^n is open, then H is an open subgroup, and is thus closed by Proposition 1.2. Since $H \subset V$ and V is compact and H is closed, then H must be compact. Thus, H is a compact open subgroup of G.

Suppose now that G is compact and totally disconnected. Since G is locally compact and totally disconnected, then any neighborhood of 1 contains a compact open subgroup

H', as we have just shown. Now consider

$$H = \bigcap_{x \in G} xH'x^{-1}.$$

By Lemma 4.2, there is a neighborhood U of 1 such that $U \subset xH'x^{-1}$ for every $x \in G$ (since G is compact). In other words, H contains an open neighborhood of 1, and is thus open. H is a subgroup, since it is the intersection of subgroups, and is normal by construction. Since $xH'x^{-1}$ is closed (since it is compact) for every $x \in G$, then H is closed, and is thus compact since G is compact. So, H is a compact open normal subgroup of G. □

The following characterization of locally compact totally disconnected groups follows immediately from Theorems 4.1 and 4.2.

Corollary 4.2. *A Hausdorff topological group G is locally compact and totally disconnected if and only if every neighborhood of* 1 *contains a compact open subgroup.*

References

[BZ] I.N. Bernstein and A.V. Zelevinskii, Representations of the group $GL(n, F)$ where F is a non-archimedean local field, *Russian Math. Surveys* **31:3** (1976), 1-68.

[B] D. Bump, Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics, 55. *Cambridge University Press, Cambridge*, 1997.

[MZ] D. Montgomery and L. Zippin, Topological Transformation Groups. *Interscience Publishers, New York-London*, 1955.

[M] J.R. Munkres, Topology: a first course. *Prentice-Hall, Inc., Englewood Cliffs, N.J.*, 1975.

[RV] D. Ramakrishnan and R.J. Valenza, Fourier Analysis on Number Fields. Graduate Texts in Mathematics, 186. *Springer-Verlag, New York*, 1999.