AN INTRODUCTION TO THE THEORY OF TOPOLOGICAL GROUPS AND THEIR REPRESENTATIONS

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Abstract. These notes are intended to give an introduction to the representation theory of finite and topological groups. We assume that the reader is only familar with the basics of group theory, linear algebra, topology and analysis. We begin with an introduction to the theory of groups acting on sets and the representation theory of finite groups, especially focusing on representations that are induced by actions. We then proceed to introduce the theory of topological groups, especially compact and amenable groups and show how the "averaging" technique allows many of the results for finite groups to extend to these larger families of groups. We then finish with an introduction to the Peter-Weyl theorems for compact groups.

1. Review of Groups

We will begin this course by looking at finite groups acting on finite sets, and representations of groups as linear transformations on vector spaces. Following this we will introduce topological groups, Haar measures, amenable groups and the Peter-Weyl theorems.

We begin by reviewing the basic concepts of groups.

Definition 1.1. *A* **group** *is a non-empty set G equipped with a map $p : G \times G \to G$, generally, denoted $p(g,h) = g \cdot h$, called the* **product** *satisfying:*
- *(associativity) $(g \cdot h) \cdot k = g \cdot (h \cdot k)$, for every $g, h, k \in G$,*
- *(existence of identity) there is an element, denoted $e \in G$ such that $g \cdot e = e \cdot g = g$ for every $g \in G$,*
- *(existence of inverses) for every $g \in G$, there exists a unique element, denoted g^{-1}, such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.*

Often we will write a group as, $(G, \cdot)$, to denote the set and the specific product.

Recall that the identity of a group is unique.

Definition 1.2. *If G is a group, then a non-empty subset, H, of G is called a* **subgroup** *provided that:*
- *$e \in H$,*
- *$g, h \in H$, then $g \cdot h \in H$,*

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- *$g \in H$, then $g^{-1} \in H$.*

A subgroup $H \subseteq G$ is called **normal** *provided that the set $g \cdot H \cdot g^{-1} = \{g \cdot h \cdot g^{-1} : h \in H\}$ is always a subset of H, for every $g \in G$.*

A group, G, is called **abelian**, or **commutative**, provided, $g \cdot h = h \cdot g$ for every, $g, h \in G$. When G is abelian, then every subgroup is normal.

When $N \subseteq G$ is a normal subgroup of G then we may define a **quotient group, G/N** whose elements are **left cosets**, i.e., subsets of G of the form $g \cdot N = \{g \cdot h : h \in N\}$, and with the product defined by $(g_1 \cdot N) \cdot (g_2 \cdot N) = (g_1 \cdot g_2) \cdot N$.

Following are a few examples of groups, subgroups and quotient groups, that we assume that the reader is familiar with:

- $(\mathbb{Z}, +)$–the additive group of integers, with identity, e=0,
- $(n\mathbb{Z}, +)$–the normal, subgroup of $\mathbb{Z}$, consisting of all multiples of n,
- $(\mathbb{Z}/(n\mathbb{Z}), +)$–the quotient group, usually denoted, $(\mathbb{Z}_n, +)$ and often called the **cyclic group of order n.**
- $(\mathbb{Q}, +)$–the additive group of rational numbers, with identity, e=0,
- $(\mathbb{R}, +)$–the additive group of real numbers, with identity, e=0,
- $(\mathbb{C}, +)$–the additive group of complex numbers, with identity, e=0,
- $(\mathbb{Q}^*, \cdot)$–the multiplicative group of non-zero rationals, with identity, e=1,
- $(\mathbb{R}^*, \cdot)$–the multiplicative group of non-zero reals, with identity, e=1,
- $(\mathbb{R}^+, \cdot)$–the multiplicative subgroup of $(\mathbb{R}^*, \cdot)$, consisting of positive reals,
- $(\mathbb{C}^*, \cdot)$–the multiplicative group of non-zero complex numbers, with identity, e=1,
- $(\mathbb{T}, \cdot)$–the multiplicative subgroup of complex numbers of modulus 1.

Definition 1.3. *Given two groups, G_1, G_2, a map, $\pi : G_1 \to G_2$ is called a* **homomorphism**, *provided that $\pi(g \cdot h) = \pi(g) \cdot \pi(h)$, for every $g, h \in G_1$. A homomorphism that is one-to-one and onto, is called an* **isomorphism.**

Problem 1.4. *Prove that if G is a group and $g \in G$ satisfies, $g \cdot g = g$, then $g = e$-the identity of G.*

Problem 1.5. *Prove that if $\pi : G_1 \to G_2$ is a homomorphism, and $e_i \in G_i, i = 1, 2$ denotes the respective identities, then*

- *$\pi(e_1) = e_2$,*
- *$\pi(g^{-1}) = \pi(g)^{-1}$,*
- *$N = \{g \in G_1 : \pi(g) = e_2\} \subseteq G_1$ is a normal subgroup. The set N is called the* **kernel of the homomorphism** *and is denoted $ker(\pi)$.*
- *Prove that there is a well-defined homomorphism, $\tilde{\pi} : G_1/N \to G_2$ given by $\tilde{\pi}(g \cdot N) = \pi(g)$. We call $\tilde{\pi}$ the* **induced quotient map.**
- *Prove that if $\pi(G_1) = G_2$, then $\tilde{\pi}$ is an isomorphism.*

Problem 1.6. *Prove that $\mathbb{R}^*/\mathbb{R}^+$ and $\mathbb{Z}_2$ are isomorphic.*

Problem 1.7. *Prove that the map,* $\pi : (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ *given by* $\pi(t) = e^t$, *is an isomorphism.*

Problem 1.8. *Prove that the map* $\pi : (\mathbb{R}, +) \to (\mathbb{T}, \cdot)$ *given by* $\pi(t) = e^{2\pi it} = cos(2\pi t) + isin(2\pi t)$ *is an onto homomorphism with kernel,* $\mathbb{Z}$. *Deduce that* $\mathbb{R}/\mathbb{Z}$ *and* $\mathbb{T}$ *are isomorphic.*

Problem 1.9. *Prove that the map* $\pi : (\mathbb{C}^*, \cdot) \to (\mathbb{R}^+, \cdot)$ *defined by* $\pi(z) = |z| = \sqrt{a^2 + b^2}$, *where* $z = a + ib$ *is a onto, homomorphism with kernel,* $\mathbb{T}$. *Deduce that* $\mathbb{C}^*/\mathbb{T}$ *and* $\mathbb{R}^+$ *are isomorphic.*

Problem 1.10. *Let* G_1, G_2 *be groups and let* $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ *denote their Cartesian product. Show that* $G_1 \times G_2$ *is a group with product,* $(g_1, g_2) \cdot (h_1, h_2) = (g_1h_1, g_2h_2)$ *and identity,* (e_1, e_2). *Show that* $N_1 = \{(g_1, e_2) : g_1 \in G_1\}$ *is a normal subgroup and that* $(G_1 \times G_2)/N_1$ *is isomorphic to* G_2. *Prove a similar result for the other variable.*

We now examine some other ways to get groups.

The Matrix Groups

We let $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the vector spaces of real and complex n-tuples. Recall that, using the canonical basis for $\mathbb{R}^n$, we may identify the (real) linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n)$ with the real $n \times n$ matrices, which we denote, $M_n(\mathbb{R})$. Similarly, the (complex) linear maps, $\mathcal{L}(\mathbb{C}^n)$ can be identified with $M_n(\mathbb{C}^n)$. Under these identifications, composition of linear maps becomes matrix multiplication.

Recall also, that the key properties of the determinant map, are that $det(A) \neq 0$ if and only if the matrix A is invertible and that $det(AB) = det(A)det(B)$.

We let $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : det(A) \neq 0\}$, which by the above remarks is a group under matrix multiplication, with identity the identity matrix, I. This is called the **general linear group.** Using the fact that $det : GL(n, \mathbb{R}) \to \mathbb{R}^*$ is a homomorphism, we see that the kernel of this map is the normal subgroup, denoted $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : det(A) = 1\}$ and called the **special linear group.**

The groups, $GL(n, \mathbb{C}), SL(n, \mathbb{C})$ are defined similarly.

Given a matrix, $A = (a_{i,j})$, we let $A^t = (a_{j,i})$ denote the **transpose** and let $A^* = (\bar{a_{j,i}})$ denote the **conjugate, transpose**, also called the **adjoint**.

The **orthogonal matrices,** $\mathcal{O}(n)$ are defined by $\mathcal{O}(n) = \{A \in M_n(\mathbb{R}) : A^tA = I\}$, which is easily seen to be a subgroup of $GL(n, \mathbb{R})$ and the **special orthogonal** matrices by $\mathcal{SO}(n) = \{A \in \mathcal{O}(n) : det(A) = 1\}$, which can be easily seen to be a normal subgroup of $\mathcal{O}(n)$.

Similarly, the **unitary matrices,** $\mathcal{U}(n)$ are defined by $\mathcal{U}(n) = \{A \in M_n(\mathbb{C}) : A^*A = I\}$ and the **special unitary matrices,** $\mathcal{SU}(n)$ are defined by $\mathcal{SU}(n) = \{A \in \mathcal{U}(n) : det(A) = 1\}$.

Problem 1.11. *Let* $SL(n, \mathbb{Z})$ *denote the set of* $n \times n$ *matrices with integer entries whose determinant is 1. Prove that* $SL(n, \mathbb{Z})$ *is a subgroup of*

$SL(n,\mathbb{R})$, but is not a normal subgroup.(Hint: Cramer's Rule.) Exhibit infinitely many matrices in $SL(2,\mathbb{Z})$.

Problem 1.12. *Prove that $\mathcal{SO}(n)$ is a subgroup of $SL(n,\mathbb{R})$. Is it a normal subgroup?*

Problem 1.13. *Let $H_n = \{A \in GL(n,\mathbb{C}) : |det(A)| = 1\}$. Prove that H_n is a normal subgroup of $GL(n,\mathbb{C})$ and that $SL(n,\mathbb{C})$ is a normal subgroup of H_n. Identify the quotient groups, $GL(n,\mathbb{C})/H_n$ and $H_n/SL(n,\mathbb{C})$, up to isomorphism.*

The Permutation Groups

Let X be any non-empty set. Any one-to-one, onto function, $p : X \to X$, is called a **permutation.** Note that the composition of any two permutations is again a permutation and that every permutation function has a function inverse that is also a permutation. Also if $id_X : X \to X$ denotes the identity map, then $p \circ id_X = id_X \circ p = p$. Thus, the set of permutations of X, with product defined by composition forms a group with identity, $e = id_X$. This group is denoted, **Per(X).**

Note that this group, up to isomorphism, only depends on the cardinality of X. Indeed, if Y is another set of the same cardinality as X and $\phi : X \to Y$ is a one-to-one, onto map, then there is a group isomorphism, $\pi : Per(X) \to Per(Y)$ givien by $\pi(p) = \phi \circ p \circ \phi^{-1}$.

Thus, when X is a set with n elements, Per(X) can be identified with the set of permutations of the set $\{1, \ldots, n\}$ and this group is called the **symmetric group on n elements** and is denoted S_n.

Free Groups with Generators and Relations

The **free group** $\mathbb{F}_2$ on two generators, say a,b, consists of all expressions of the form $a^{i_1}b^{j_1} \cdots a^{i_m}b^{j_m}$, where m is an arbitrary, non-negative integer and $i_1, j_1, \ldots, i_m, j_m$ are arbitrary integers. Such expressions are called **words.** We identify $a^0b^J = b^j$ and the multiplication of two such expressions is called **concatenation,** which rather than trying to define formally, we illustrate with a few examples, $(aba^2)(a^{-3}b) = aba^{-1}b, (ba)(ba^3) = baba^3, (ba)(a^{-1}b^2a^2) = b^3a^2, (ab^2a^{-3})^{-1} = a^3b^{-2}a^{-1}$.

The reason that $\mathbb{F}_2$ is important is that it has the following "universal" property. Given any group G and any two elements, $g, h \in G$ there exists a unique homomorphism $\pi : \mathbb{F}_2 \to G$ defined by setting $\pi(a) = g, \pi(b) = h$. The range of π will be the subgroup generated by g and h.

Similarly, for any $n \in \mathbb{N}$, there is a free group on n generators which is denoted $\mathbb{F}_n$, which consists of words in n letters with a corresponding operation of concatenation. If these generators are denoted $a_1, ..., a_n$, then $\mathbb{F}_n$ enjoys the universal property that given any group G and elements $g_1, ..., g_n \in G$, there will exist a unique homomorphism $\pi : \mathbb{F}_n \to G$ defined by setting $\pi(a_i) = g_i, i = 1, ..., n$, and the range of π will be the subgroup generated by the given set of elements.

It is fairly easy to see that $\mathbb{F}_1$ is just $\mathbb{Z}$.

Other groups are defined as free groups with generators and **relations.** The relations really mean a set of equations that the generators must satisfy. These groups have the universal property that whenever one is given a group G and elements that satisfy the same relations, then there will exist a homomorphism from the free group with generators and relations defined by sending the generators to the given set of elements.

An example of such a group and the notation used, would be, $K =< a, b|a^2 = e >$. This group would behave like the free group on two generators, but when concatenating words, every time an a^2 appears it can be replaced with the identity, so for example, $(ba)(aba) = b^2a$, in this group.

This group has the universal property that whenever we are given a group G and two elements $g, h \in G$, with $g^2 = e$, then there will exist a unique homomorphism $\pi : K \to G$ defined by setting $\pi(a) = g, \pi(b) = h$, and the range of π will be the subgroup generated by g and h.

It is fairly easy to see that $< a, b|ab = ba >$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Some examples of such groups that play a role in wavelet theory, are the **Baumslag-Solitar groups** which are defined for each $K, N \in \mathbb{N}$. For example, $BS(1, N) =< D, T|T^N D = DT >$. Note that $BS(1, 1)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

In the groups, $BS(1, N)$, every word is equivalent to a word of the form T^iD^j. For example, in $BS(1, 2)$ we would have that $DT^2D = (DT)TD = (T^2D)TD = T^2(DT)D = T^4D^2$.

Another important family of free groups with relations are the groups, $D(k, n) =< R, F|R^n = e, F^k = e, RF = FR^{n-1} >$.

New Groups from Old

We've already seen several ways to get new groups from old groups. One way is by taking quotients by normal subgroups. Another is to take the product of two given groups.

Given groups $G_i, i = 1, 2$ with respective identities e_1 and e_2, we let $G_1 \times G_2$ denote the group that as a set is the Cartesian product and has a binary operation defined by $(g_1, g_2) \cdot (h_1, h_2) = (g_1h_1, g_2h_2)$. It is easy to see that this operation is associative that $e = (e_1, e_2)$ is an identity and that (g_1, g_2) has an inverse given by $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$. More, generally, if one is given a collection of groups $G_i, i \in I$, where I is some index set, then this same "entrywise" product makes the Cartesian product $\Pi_{i\in I}G_i$ into a group. This group is called the **direct product** of the groups $G_i, i \in I$.

There is a subgroup of the direct product that often plays a role, especially when all of the groups are abelian. Suppose that we are given abelian groups, $(G_i, +), i \in I$, then we let $\sum_{i\in I} \oplus G_i \subset \Pi_{i\in I}G_i$ denote the subgroup consisting of all elements of the Cartesian product that are non-zero for only finitely many entries. To see that it is a subgroup, note that it contains the identity element. Also, if g is an element that has n_1 non-zero entries and h is an element with n_2 non-zero entries, then $g + h$ has at most $n_1 + n_2$ non-zero entries and $-g$ has n_1 non-zero entries. Thus, the sum of two elements of this subset and the (additive) inverse of an element of this subset are both back in the subset. This group is called the **direct sum** of the groups, $G_i, i \in I$.

We now record a couple of other ways to obtain new groups that will play a role.

First, given any group G, the set $Z(G) = \{g \in G : gh = hg, \text{ for every } h \in G\}$ is a normal, subgroup called the **center of G.**

Also, given any group G, the subgroup generated by all elements of the form, $ghg^{-1}h^{-1}$, is called the **commutator subgroup** of G and is denoted, **[G,G]**.

Given any group, G, an isomorphism, $\pi : G \to G$ is called an **automorphism** of G. Clearly, the composition of automorphisms is again an automorphism, the function inverse of an automorphism is again an automorphism and the identity map on G plays the role of an identity for composition. Thus, the set of all automorphisms of G, **Aut(G)** is a new group obtained from G.

If $g \in G$, then there is an automorphism $\pi_g : G \to G$ defined by $\pi_g(h) = ghg^{-1}$ with inverse, $(\pi_g)^{-1} = \pi_{g^{-1}}$. The map, $\rho : G \to Aut(G)$ defined by $\rho(g) = \pi_g$ is a homomorphism and the subgroup of Aut(G) that is the range of this map, i.e., the set consisting of $\{\pi_g : g \in G\}$, is a subgroup of Aut(G), called the **inner automorphisms** and denoted, $Aut_i(G)$.

Finally, given groups, G and H, and a homomorphism $\theta : G \to Aut(H)$, the **semidirect product** is the group $H \times_\theta G = \{(h, g) : h \in H, g \in G\}$ with product defined by, $(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot \theta(g_1)(h_2), g_1 \cdot g_2)$.

Problem 1.14.
- *Prove that $H \times_\theta G$ is a group,*
- *prove that $K = \{(h, e) : h \in H\}$ is a normal subgroup of $H \times_\theta G$ and that K is isomorphic to H,*
- *prove that the quotient group $(H \times_\theta G)/K$ is isomorphic to G.*

It is known that if L is any group, with a normal subgroup H and if L/H is isomorphic to G, then there exists a homomorphism, $\theta : G \to Aut(H)$, such that L is isomorphic to the semidirect product, $H \times_\theta G$.

2. Groups Acting on Sets

A **(left) action** of a group, G, on a set, X, is a map $\alpha : G \times X \to X$ satisfying $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for every $x \in X$ and every $g, h \in G$. Usually, we will write $\alpha(g, x) = g \cdot x$, so that the first property is that $e \cdot x = x$ and the second property is $g \cdot (h \cdot x) = (gh) \cdot x$ which can be seen to be an associativity property.

Some books also discuss **right actions** of a group G on a set X. This is a map, $\beta : X \times G \to X$ satisfying $\beta(x, e) = x$ and $\beta(\beta(x, h), g) = \beta(x, hg)$ and these are, generally, denoted, $\beta(x, g) = x \cdot g$.

The following exercise shows the correspondence between the theories of left and right actions on sets.

Problem 2.1. *Let* $\beta : X \times G \to X$ *be a right action of* G *on* X *and define* $\alpha : G \times X \to X$, *by* $\alpha(g, x) = \beta(x, g^{-1})$. *Prove that* α *is a left action of* G *on* X. *Conversely, given* $\alpha : G \times X \to X$, *a left action of* G *on* X, *prove that* $\beta(x, g) = \alpha(g^{-1}, x)$ *defines a right action of* G *on* X.

When we only say that G acts on X, we shall always mean that there is a left action of G on X.

The following problem gives an alternative way to define group actions on sets.

Problem 2.2. *Let* $\alpha : G \times X \to X$ *be an action of* G *on* X. *For each* $g \in G$, *define* $\pi(g) : X \to X$ *by* $\pi(g)(x) = g \cdot x$. *Prove that* $\pi(g) \in Per(X)$ *and that the map,* $\pi : G \to Per(X)$ *is a group homomorphism. Conversely, prove that if* $\pi : G \to Per(X)$ *is a group homomorphism and we set* $\alpha(g, x) = \pi(g)(x)$, *then* α *is an action of* G *on* X.

One important group action is the action of a group G on itself by either left or right multiplication.

Problem 2.3. *Let* G *be a group, define* $\alpha_l : G \times G \to G$ *by* $\alpha_l(g, h) = g \cdot h$ *and* $\alpha_r : G \times G \to G$ *by* $\alpha_r(g, h) = h \cdot g^{-1}$. *Prove that* α_l *and* α_r *are both actions of* G *on* G.

Definition 2.4. *Let* G *be a group. We call* α_l *the* **action of G on itself given by left multiplication** *and* α_r *the* **(left) action of G on itself given by right multiplication.**

Note that as in the problem relating right and left actions, to define a left action of G on itself by a right multiplication, i.e., α_r, we had to introduce an inverse.

If G is a finite group, say $|G| = n$, so that $Per(G) = S_n$, then the action α_l gives rise to a homomorphism, $\pi_l : G \to S_n$ that is easily seen to be one-to-one. This homomorphism is generally, called the **Cayley representation** or sometimes the **Jordan representation** of G.(It seems that Jordan actually did it first.)

Several important actions arise when one is given a subgroup H of a group G. First, the map $\alpha_l : H \times G \to G$, defined by $\alpha_l(h, g) = hg$ defines an action of H on G which is the restriction of the action of G on tself given by left multiplication. Similarly, $\alpha_r : H \times G \to G$ defined by $\alpha_r(h, g) = gh^{-1}$ is the restriction to H of the action of G on itself given by right multiplication.

In addition to actions of H on G, there is an action of G on the left cosets of H. Recall that given $k \in G$, then the left coset of H in G generated by k is the set $kH = \{kh : h \in H\}$. Recall also that the left cosets of H in G form a partition G into subsets and that $k_1H = k_2H$ if and only if $k_1 \in k_2H$. We let G/H denote the collection of left cosets of H in G, i.e., $G/H = \{kH : k \in G\}$.

Problem 2.5. *Let* G *be a group and let* H *be a subgroup of* G. *Prove that the map* $\alpha : G \times G/H \to G/H$ *given by* $\alpha(g, kH) = (gk)H$, *defines a left* *action of G on G/H. This action is called* **the action of G on G/H given by left multiplication.**

2.1. **Effective and Free Actions.**

Definition 2.6. *An action of a group G on a set X is called* **effective** *if whenever $g \neq e$, then there exists $x \in X$, such that $g \cdot x \neq x$. An action is called* **free** *if for each $x \in X$, $g \cdot x = x$, implies that $g = e$.*

Problem 2.7. *Show that an action is effective if and only if the homomorphism $\pi : G \to Per(X)$, is one-to-one. Show that every free action is effective.*

Problem 2.8. *Let G be a group. Show that the action of G on itself given by left multiplication is free.*

Problem 2.9. *Let G be a group, let H be a subgroup, let $\alpha : G \times G/H \to G/H$ be the action of G on G/H given by left multiplication and let $\pi : G \to Per(G/H)$, be the homomorphism induced by α. Prove that the kernel of π is equal to $\cap_{k \in G} k^{-1}Hk$ and that this set is the largest normal subgroup of G that is contained in H.*

2.2. **Groups Defined by Actions.** Many groups are naturally defined as subgroups of $Per(X)$, or, equivalently, by specifying their actions on a set X. More precisely, the map $\alpha : Per(X) \times X \to X$, defined by $\alpha(p, x) = p(x)$, is a left action of $Per(X)$ on X. To see this note that the identity element of $Per(X)$ is the identity map and hence, $\alpha(e, x) = e(x) = x$. Moreover, given $p, q \in Per(X)$, we have that$\alpha(p, \alpha(q, x)) = \alpha(p, q(x)) = p(q(x)) = p \circ q(x) = \alpha(p \circ q, x)$. Thus, each time that we specify a subgroup of $Per(X)$, we are really defining a group action on X.

One example of a group defined this way is the **dihedral group,** D_4**.** This group is defined as the group of rigid motions of a square. Since a rigid motion of a square is determined by what happens to the four corners of the square, D_4 is really being defined as a group acting on the set X, consisting of the 4 corners. If we label these corners as $X = \{1, 2, 3, 4\}$, then we are really defining D_4 as a subgroup of $Per(X) = S_4$.

We adopt the following notation. A permutation $p : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ will be represented by the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ p(1) & p(2) & p(3) & p(4) \end{pmatrix}$. Thus, the identity is the matrix $e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

Labelling the vertices of the square in counterclockwise notation, beginning with 1 in the Northeast corner. We see that D_4 has 8 elements, since every rigid motion is determined by sending 1 to one of the four corners, say k, and then the corner 2, must either go to k+1(mod 4) or k-1(mod 4) and this choice determines where the remaining corners must be sent. We call the first 4 maps, direction preserving and the other four maps direction reversing.

We have that the rigid direction preserving map, R, of counterclockwise rotation through angle $\pi/4$, is given by $R = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and its powers are $R^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, R^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, R^4 = e$. Thus the direction preserving maps are e, R, R^2, R^3.

To get the direction reversing maps of the square to the square, we can first perform a flip along the axis joining 1 to 3, this is the permutation, $F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ which serves to reverse direction and then rotate to get a map that sends 1 to k and 2 to k-1. Thus, the 4 direction reversing maps are F, RF, R^2F, R^3F, and these last three maps are the permutations, $RF = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, R^2F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, R^2F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

Finally, we can see that $RF = FR^3$.

Similarly, the group D_n is defined to be the rigid motions of a regular n-gon. Again only the corners matter and so D_n is represented as a subgroup of S_n. As above this group has 2n elements, n that are direction preserving and these are given by the rotation, R, through an angle $2\pi/n$, and n that are direction reversing, and these are given by a flip that fixes 1, followed by a rotation. Thus, the elements of D_n are given by, $e, R, \ldots, R^{n-1}, F, RF, \ldots, R^{n-1}F$. The rotation and flip satisfy the relation, $RF = FR^{n-1}$.

Problem 2.10. *Prove that D_n is isomorphic to the free group with generators and relations, $D(2,n)$. HINT: Use the universal properties to show that there is a homomorphism of $D(2,n)$ onto D_n, then show that $D(2,n)$ has at most $2n$ elements.*

A second example of a group defined by actions on a set is the **group of affine maps of** $\mathbb{R}$, also known as the **ax+b group.** For each $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ we define a map, $\phi_{(a,b)} : \mathbb{R} \to \mathbb{R}$ by setting, $\phi_{(a,b)}(x) = ax + b$. Composing two functions, $\phi_{(c,d)} \circ \phi_{(a,b)}(x) = \phi_{(c,d)}(ax+b) = cax + (cb+d) = \phi_{(ca,cb+d)}(x)$, we see that $ca \in \mathbb{R}^*$ and so we obtain another such function. The identity map is given by $\phi_{(1,0)}$. Solving $ax + b = y$ for x, we obtain, $x = (1/a)y + (-b/a)$ and so the inverse satisfies, $(\phi_{(a,b)})^{-1} = \phi_{(1/a,-b/a)}$. This last calculation shows that the set $G = \{\phi_{(a,b)} : a \in \mathbb{R}^*, b \in \mathbb{R}\}$ is a group under composition and since every function in this group has a function inverse, it must be one-to-one and onto, hence $G \subseteq Per(\mathbb{R})$ is a subgroup, and so by our earlier equivalence can be thought of as coming from an action on $\mathbb{R}$. The set $G^+ = \{\phi_{(a,b)} : a \in \mathbb{R}^+, b \in \mathbb{R}\}$ is a subgroup of G, called the **affine direction preserving maps of** $\mathbb{R}$.

Note that we could have defined $G = \mathbb{R}^* \times \mathbb{R}$ with product given by $(c,d) \cdot (a,b) = (ca, cb+d)$ and rather tediously verified that this indeed defines a product making G a group, but then all of the geometric intuition would be lost.

Problem 2.11. *Verify that for $a \in \mathbb{R}^*$, setting $\theta(a)(t) = at$ defines an element, $\theta(a) \in Aut((\mathbb{R}, +))$ and that the map $\theta : (\mathbb{R}^*, \cdot) \to Aut((\mathbb{R}, +))$ is a homomorphism. Prove that G is isomorphic to $\mathbb{R} \times_\theta \mathbb{R}^*$.*

A third example of a group defined by actions on a set is the **group, $E(n)$, of all isometric maps of $\mathbb{R}^n$,** this group is also called the group of **Euclidean motions.** It can be shown that a map $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if and only if there exists, $U \in \mathcal{O}_n$ and a vector $b \in \mathbb{R}^n$ such that $T(x) = Ux + b$. The **group of rigid motions of** $\mathbb{R}^n$ also called the **proper Euclidean group** is the subgroup $E^+(n) \subseteq E(n)$ such that $T(x) = Ux + b$ with $U \in S\mathcal{O}_n$.

Problem 2.12. *Prove that $E(n)$ is a group and that $E^+(n)$ is a normal subgroup.*

2.3. **Orbits, Orbit Equivalence and Transitive Actions.** Given an action of G on X, $\alpha : G \times X \to X$ with $\alpha(g, x) = g \cdot x$, for each $x \in X$, the set $\mathcal{O}_x = \{g \cdot x : g \in G\}$ is called the **orbit of x.**

Proposition 2.13. *Let G act on X and let $x, y \in X$. Then the following are equivalent:*

i) $y \in \mathcal{O}_x$,
ii) $x \in \mathcal{O}_y$,
iii) $\mathcal{O}_x = \mathcal{O}_y$.

Proof. If $y \in \mathcal{O}_x$, then there exists $g \in G$, such that $y = g \cdot x$. But then $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = e \cdot x = x$, by associativity. Hence, $x \in \mathcal{O}_y$, and i) implies ii). the equivalence of i) and ii) follows by reversing the roles of x and y.

Since $y = e \cdot y \in \mathcal{O}_y$, we see that iii) implies i).

Finally, assuming i), we have that $y = h \cdot x$, and hence, $\mathcal{O}_y = \{g \cdot y : g \in G\} = \{gh \cdot x : g \in G\} = \{k \cdot x : k \in Gh\} = \mathcal{O}_x$, since $Gh = G$, and so iii) follows. □

Proposition 2.14. *Let G act on X and let $x, y \in X$. If $\mathcal{O}_x \cap \mathcal{O}_y$ is non-empty, then $\mathcal{O}_x = \mathcal{O}_y$.*

Proof. Let $z \in \mathcal{O}_x \cap \mathcal{O}_y$, then by iii) of the above, $\mathcal{O}_x = \mathcal{O}_z = \mathcal{O}_y$. □

Definition 2.15. *We set $x \sim_G y$ if and only if $\mathcal{O}_x = \mathcal{O}_y$ and call this relation* **orbit equivalence.**

It is easily seen that orbit equivalence is indeed an equivalence relation on X and by the above results that the equivalence class of a point is its orbit.

If H is a subgroup of a group G and we consider the two actions α_l and α_r of H on G given by left and right multiplication, respectively, then the orbit of $k \in G$ under α_l is $\mathcal{O}_k = \{h \cdot k : h \in H\} = Hk$ the right coset generated by k, while the orbit of k under α_r is $\mathcal{O}_k = \{k \cdot h^{-1} : h \in H\} =$

$\{k \cdot h : h \in H\} = kH$ is the left coset generated by k. If one applies the above results about orbit equivalence to the action α_r, then one can deduce the earlier well-known statements about left coset equivalence.

Definition 2.16. *Let G act on the set X. the action is called* **transitive** *if given $x, y \in X$, there is $g \in G$, such that $g \cdot x = y$. The action is called* **n-transitive,** *if given any two sets of n distinct points in X, i.e., $x_1, \ldots, x_n, y_1, \ldots y_n$ with $x_i \neq x_j$ and $y_i \neq y_j$, when $i \neq j$, then there exists $g \in G$, such that for every $i, g \cdot x_i = y_i$.*

Note that the group S_n, which is defined as all permutations of a set of n elements, is n-transitive.

Proposition 2.17. *Let $\alpha : G \times X \to X$ be an action. Then the following are equivalent:*

i) *α is transitive,*
ii) *for every $x \in X, \mathcal{O}_x = X$,*
iii) *there exists an $x \in X$, such that $\mathcal{O}_x = X$.*

Proof. If α is transitive, $x \in X$ is fixed and $y \in X$ is arbitrary, then there exists, $g \in G$ with $g \cdot x = y$. Hence, $\mathcal{O}_x = X$.

Clearly, 2) implies 3). Finally, if $\mathcal{O}_w = X$ and $x, y \in X$ are arbitrary, then there exists, $g, h \in G$ with $g \cdot w = x, h \cdot w = y$. Hence, $(hg^{-1}) \cdot x = h \cdot (g^{-1} \cdot x) = h \cdot w = y$ and so α is transitive. □

Example 2.18. *Let $\alpha : G \times G \to G$ be defined by $\alpha(g, h) = gh$. This action is transitive, since $\mathcal{O}_e = \{ge : g \in G\} = G$. However, if $card(G) > 2$, then this action is not 2-transitive, since given $g_1 \neq g_2, h_1 \neq h_2$, when $gg_1 = h_1$ we have that $gg_2 = h_1 g_1^{-1} g_2 \neq h_2$, in general.*

Example 2.19. *Consider the group $G = \{\phi_{(a,b)}\}$ of affine maps of $\mathbb{R}$. Since, $\phi_{a,b)}(0) = b$, we see that $\mathcal{O}_0 = \mathbb{R}$, and so G acts transitively. It is also 2-transitive. To see this note that given any, $x_1 \neq x_2, y_1 \neq y_2$, we need to be able to find a, b that solve, $ax_i + b = y_i, i = 1, 2$. In matrix form this becomes,*

$$\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Since, $det(\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix}) = x_1 - x_2 \neq 0$, these equations have a solution,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We still need to see that $a \neq 0$, but this follows by either explicitly inverting the matrix and computing a, or more readily by noting that if $a = 0$, then $y_1 = b = y_2$, contradiction.

If we consider instead the subgroup of order preserving affine transforms, then it is transitive, but not 2-transitive, since if $\phi_{(a,b)}(x_i) = y_i, i = 1, 2$ and $x_1 < x_2$, then necessarily, $y_1 < y_2$.

Example 2.20. *The actions of the dihedral groups, D_n on the n-gon are transitive for every n. But they are not 2-transitive, when $n > 3$. To see this, note that if $n > 3$, then we may take $x_1 = 1, x_2 = 2, y_1 = 1, y_2 = 3$, and then the only group elements such that $gx_1 = y_1$ are the identity which has $gx_2 = 2$, or the flip which has $gx_2 = n \neq y_2$, since $n > 3$.*

Note that $D_3 = S_3$ which is 2-transitive and 3-transitive.

2.4. **Stabilizer Subgroups.** Given $x \in X$ the set $G_x = \{g \in G : g \cdot x = x\}$ is called the *stabilizer subgroup.* In other texts, this is sometimes called the *isotropy subgroup* or *stationary subgroup.*

Proposition 2.21. *Let G be a group acting on X.*

(1) *G_x is a subgroup of G.*
(2) *If $y = h \cdot x$, then $G_y = h \cdot G_x \cdot h^{-1}$.*
(3) *If G acts transitively, then G_x is isomorphic to G_y, for all $x, y \in X$.*

Problem 2.22. *Prove the above proposition.*

Example 2.23. *When G acts on G by left multiplication, then for any $g \in G, G_g = \{e\}$.*

Example 2.24. *When the dihedral group, D_4 acts on the four vertices of the square, $\{1, 2, 3, 4\}$, then $(D_4)_1 = \{e, F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}\} \simeq \mathbb{Z}_2$. Since D_4 acts transitively, we have that $(D_4)_i \simeq \mathbb{Z}_2$ for every corner of the square, but in general, these are not the same subsets of D_4, in fact, $(D_4)_1 = (D_4)_3 \neq (D_4)_2 = (D_4)_4$.*

Example 2.25. *Let G be the group of affine transformations acting on $\mathbb{R}$, then $G_0 = \{\phi_{(a,0)} : a \neq 0\} \simeq \mathbb{R}^*$. Note that $G_x = \{\phi_{(a,b)} : ax + b = x\}$, but since G acts transitively, this subgroup is also isomorphic to $\mathbb{R}^*$.*

Problem 2.26. *Prove that $E(n)$ and $E^+(n)$ act transitively, but not 2-transitively, on $\mathbb{R}^n$. Find the (unique up to isomorphism) stabilizer subgroup for $E(n)$ and $E^+(n)$ of any point.*

Let G be a group, $H \subseteq G$, be a subgroup(not necessarily normal). For $g \in G$ let $g \cdot H$ be the (left) coset generated by g. Recall that defining $g_1 \sim_H g_2$ if and only if $g_1 H = g_2 H$ defines an equivalence relation on G called *(left) coset equivalence(mod H).*

Note that $card(gH) = card(H)$.

Proposition 2.27 (Lagrange)**.** *Let G be a finite group, $H \subseteq G$ be a subgroup, then $card(H)$ divides $card(G)$ and the number of coset equivalence classes is equal to $\frac{card(G)}{card(H)}$.*

Proof. Assume that there are k coset equivalence classes, and choose $g_1, \ldots, g_k$ such that, $g_1 H, \ldots, g_k H$, are the equivalence classes. Since G is the disjoint union of these sets, and $card(g_i H) = card(H)$, for all i, we have that

$card(G) = card(g_1H) + \cdots + card(g_kH) = k \cdot card(H)$, and the result follows. $\square$

Proposition 2.28. *Let G be a finite group, acting on a finite set, X and let $x \in X$. Then:*

- $g_1 \cdot x = g_2 \cdot x$ *if and only if* $g_1G_x = g_2G_x$,
- $card(\mathcal{O}_x) \cdot card(G_x) = card(G)$.

Proof. We have that $g_1 \cdot x = g_2 \cdot x$ if and only if $g_2^{-1}g_1 \in G_x$ if and only if $g_1 \in g_2G_x$ if and only if $g_1G_x = g_2G_x$, and the first result follows.

Let $k = \frac{card(G)}{card(G_x)}$, which is equal to the number of cosets of G_x. Hence, we may pick $g_1, \ldots, g_k$, such that G is the disjoint union of g_iG_x. Hence, $\mathcal{O}_x = \{g \cdot x : g \in G\} = \bigcup_{j=1}^{k}\{g \cdot x : g \in g_jG_x\} = \{g_1 \cdot x\} \cup \cdots \cup \{g_k \cdot x\}$. But, $g_i \cdot x \neq g_j \cdot x$ when $i \neq j$.

Thus, $card(\mathcal{O}_x) = k$. $\square$

Example 2.29. *Look at D_4 acting on the corners of a square. We've seen that $card((D_4)_1) = 2$, and $card(\mathcal{O}_1) = 4$. Thus, $card(D_4) = 2 \cdot 4 = 8$.*

Example 2.30. *Look at the symmetric group, S_n acting on $\{1, \ldots, n\}$. Since, $(S_n)_n \simeq S_{n-1}$ and $card(\mathcal{O}_n) = n$, it follows that $card(S_n) = n \cdot card(S_{n-1})$, and so we have another way to see that $card(S_n) = n!$.*

Problem 2.31. *Let G_3 be the group of rigid motions of a cube, viewed as acting on its 8 corners, so that $G_3 \subseteq S_8$. Compute $card(G_3)$. (Careful: many permutations of the vertices can not be done on an actual cube.)*

Problem 2.32. *Let $C_n = \{(x_1, \ldots, x_n) : 0 \leq x_i \leq 1\}$ denote the cube in n-dimensions. How many corners does it have? The group of rigid motions of C_n is defined to be the subgroup of $E^+(n)$ that maps C_n back onto itself. Compute the order of this group.*

2.5. **The Action of Conjugation.** Define, $\alpha : G \times G \to G$ by $\alpha(g, h) = ghg^{-1}$. Since, $\alpha(g_1, \alpha(g_2, h)) = \alpha(g_1, g_2hg_2^{-1}) = g_1g_2hg_2^{-1}g_1^{-1} = \alpha(g_1g_2, h)$, we see that this defines an action of G on G. This action is called *the action of conjugation.*

Two elements, h_1, h_2 of G are called *conjugate* if there exists $g \in G$, such that $h_2 = gh_1g^{-1}$, which is equivalent to h_2 being in the orbit of h_1 under the action of conjugation. Thus, conjugacy is an equivalence relation on G. The conjugacy orbit of $h \in G, \mathcal{O}_h = \{ghg^{-1} : g \in G\}$, is called the *conjugacy class of h* and $c(h) = card(\mathcal{O}_h)$ is called the *conjugacy order of h.*

The set $C_G(h) = \{g \in G : ghg^{-1} = h\}$ is called the *centralizer of h.* Note that $C_G(h)$ is the stabilizer subgroup of h, for the action of conjugacy, which is one way to see that it is a subgroup. Also, by the above results, $c(h) = card(\mathcal{O}_h) = \frac{card(G)}{card(C_G(h))}$.

Problem 2.33. *For the dihedral group D_4 find the conjugacy classes.*

Problem 2.34. *For the dihedral groups, D_n find the conjugacy classes.*

Problem 2.35. *For the groups, $GL(n,\mathbb{C})$ and $\mathcal{U}_n$, find the conjugacy classes.*

Problem 2.36. *For the group of affine maps of $\mathbb{R}$, find the conjugacy classes.*

Problem 2.37. *For the groups, $E(n)$ and $E^+(n)$ find the conjugacy classes.*

Problem 2.38. *For the group of rigid motions of a cube, find the conjugacy classes.*

3. Representation Theory of Finite Groups

We will assume throughout this section that all vector spaces are over either the field, $\mathbb{R}$ or $\mathbb{C}$. When we wish to state a result that is true for either field, we say that it is a vector space over, $\mathbb{F}$. Mnay of the results that we prove are true over any field of characteristic 0.

Definition 3.1. *Let V be a vector space over the field $\mathbb{F}$ and let $\mathcal{L}(V)$ denote the linear transformations of V into V. We let $GL(V)$ denote the group of invertible linear maps. If G is a group, then a* **representation of G on V***, is a homomorphism, $\pi : G \to GL(V)$. A representation is called* **faithful** *if π is one-to-one.*

When $V = \mathbb{F}^n$, we have that $\mathcal{L}(V) = M_n(\mathbb{F})$ the set of $n \times n$ matrices with entries from $\mathbb{F}$, and $GL(V) = GL(n,\mathbb{F})$.

For a basic example, let V be any vector space and let $A \in GL(V)$. Then $\pi : \mathbb{Z} \to GL(V)$ given by $\pi(n) = A^n$, defines a representation of $\mathbb{Z}$ on V. If $A^n = I_V$ for some positive integer n, then by quotienting out the kernel of π, one obtains an induced homomorphism, $\tilde{\pi} : \mathbb{Z}_n \to GL(V)$, and hence a representation of $\mathbb{Z}_n$ on V.

Thus, we have a representation of $\mathbb{Z}_2$ on $\mathbb{R}^2$ given by $\pi(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Similarly, we have a representation of$\mathbb{Z}_4$ on $\mathbb{R}^2$ given by $\pi(k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Example 3.2. *Let $G \subseteq (\mathbb{F}, +)$ be any additive subgroup. For example, we could have, $G = \mathbb{Z}, \mathbb{Q}, \mathbb{Z} + i\mathbb{Z}, \mathbb{R}$. Then we have a representation of G on $\mathbb{F}^2$ defined by $\pi(g) = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$.*

Example 3.3. *Let $G = \{\phi_{(a,b)} : a \neq 0, b \in \mathbb{R}\}$ denote the group of affine maps of $\mathbb{R}$. Then we have a representation of G on $\mathbb{R}^2$ defined by $\pi(\phi_{(a,b)}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.*

Example 3.4. *If we regard the square as the subset of $\mathbb{R}^2$ given by the set $\{(x,y) : -1 \leq x, y \leq +1\}$, then each of the rigid motions of the square, naturally extends to define an invertible linear map on all of $\mathbb{R}^2$. In this* *manner we obtain a representation π of D_4 on $\mathbb{R}^2$. For example, the map R of the square naturally extends to the counterclockwise rotation through $\pi/2$. This map sends the basis vector e_1 to e_2 and the basis vector e_2 to $-e_1$. Thus, we have that $\pi(R) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The flip F of the square, can be extended to the linear map of $\mathbb{R}^2$ that is the reflection about the line $x = y$, this sends e_1 to e_2 and e_2 to e_1 and so has matrix given by $\pi(F) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

Problem 3.5. *For the above representation of D_4 on $\mathbb{R}^2$, write down the matrices of all 8 elements of D_4. Deduce that this representation is faithful.*

Problem 3.6. *Find a representation of $E(n)$ on $\mathbb{R}^{n+1}$.*

3.1. **Free Vector Spaces.** Given a set X, we can form a vector space of dimension card(X) with a basis, $\{e_x : x \in X\}$. A vector in this space is just a finite linear combination of the form, $\sum_i \lambda_i e_{x_i}$, where two such sums are equal if and only if the set of x's(with non-zero coeficients) appearing in the sums are the same and the coefficients of the corresponding e_x's are the same. This is often called the *free vector space over X* and is denoted $\mathbb{F}(X)$. Another, concrete way, to present this space, is to regard it as the set of all functions, $f : X \to \mathbb{F}$ which are *finitely supported,* i.e., such that the set of $x \in X$, with $f(x) \neq 0$ is finite. Clearly, the usual sum of two finitely supported functions is finitely supported and a scalar multiple of a fintely supported function will be finitely supported.

These two different representations of $\mathbb{F}(X)$ are identified in the following way. If we let δ_x be the function that is 1 at x and 0 elsewhere, then if f is any finitely supported function, say f is non-zero at $\{x_1, \ldots, x_k\}$, then as functions, $f = \sum_{i=1}^{n} \lambda_i \delta_{x_i}$, where $\lambda_i = f(x_i)$. Clearly, the functions, δ_x are linearly independent. Thus, $\{\delta_x : x \in X\}$ is a basis for the space of finitely supported functions. Clearly, then the map $\delta_x \to e_x$ defines a vector space isomorphism between the space of finitely supported functions on X and the free vector space over X.

We will identify these two different presentations of $\mathbb{F}(X)$ and sometimes we will use one presentation and sometimes we will use the other.

3.2. **The Representation Induced by an Action.** Let G be a group, acting on a set X. We have seen that each element of G induces a permutation of the elements of x, via $x \to g \cdot x$. This permutation extends to a linear map, $\pi(g) : \mathbb{F}(X) \to \mathbb{F}(X)$ by setting $\pi(g)(\sum_i \lambda_i e_{x_i}) = \sum_i \lambda_i e_{gx_i}$.

It is easy to see that $\pi(e)$ is the identity map on $\mathbb{F}(X)$ and that $\pi(g)\pi(h) = \pi(gh)$. Thus, each $\pi(g)$ is invertible and the map, $\pi : G \to GL(\mathbb{F}(X))$ is a homomorphism.

Definition 3.7. *Let G act on a set X. Then the representation of G on $\mathbb{F}(X)$ as above is called the* **permutation representation induced by the action.**

For a first example, consider the action of D_4 on the four vertices of the square and we look at the induced permutation action. If we keep our earlier notation and let R denote the rotation, counterclockwise through angle $\pi/2$ and F denote the flip about the line x=y, then for the representation, $\pi : D_4 \to GL(\mathbb{F}^4)$, we have, $\pi(R) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \pi(F) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

The matrices of the other six elements of D_4 given by this representation can be computed by taking appropirate products of these matrices or from their actions on the vertices.

Problem 3.8. *Find the matrices of the other six elements of D_4 given by this representation.*

Problem 3.9. *The group D_4 can also be regarded as acting on the four edges of the square. Number the edges so that the i-th edge is between the i-th and (i+1)-th vertices, modulo four. Let $\pi : D_4 \to GL(\mathbb{F}^4)$ be the permutation representation induced by this action. Find the matrices for $\pi(R)$ and $\pi(F)$ in this case.*

Definition 3.10. *Let G act on itself via left multiplication α_l and consider the induced permutation representation on $\mathbb{F}(G)$. This representation is denoted $\lambda : G \to GL(\mathbb{F}(G))$ and is called the (left) regular representation.*

Thus, we have that $\lambda(g)e_h = e_{gh}$. Note that this representation is faithful, since $\lambda(g_1)e_h = \lambda(g_2)e_h$ if and only if $g_1 = g_2$. Also every vector in the canonical basis is *cyclic* since $\lambda(G)e_h$ spans $\mathbb{F}(G)$. Algebraists sometimes refer to the left regular representation as the *Cayley representation.*

Definition 3.11. *Let G be a group and let $\alpha_r(g,h) = hg^{-1}$ be the left action of G on G given by right multiplication. The permutation representation induced by this action is denoted $\rho : G \to GL(\mathbb{F}(G))$ and is called the right regular representation.*

Thus, we have that $\rho(g)e_h = e_{hg^{-1}}$. This action is also faithful.

When we discuss the "regular representation" of a group, we will always mean the left regular representation.

It is also valuable to see what the induced permutation representation looks like when we regard $\mathbb{F}(X)$ as functions on X. Since $\pi(g)\delta_x = \delta_{gx}$, we have that $(\pi(g)\delta_x)(y) = \begin{cases} 1 & \text{iff y=gx,} \\ 0 & \text{iff } y \neq x \end{cases} = \begin{cases} 1 & \text{iff } g^{-1}y = x \\ 0 & \text{iff } g^{-1}y \neq x \end{cases} = \delta_x(g^{-1}y)$. Since every finitely supported function is a linear combination of such functions, we have that $(\pi(g)f)(y) = f(g^{-1}y)$. Thus, from the point of view of functions, the induced permutation representation is translation of the variable by g^{-1}.

Many texts simple regard $\mathbb{F}(X)$ as finitely supported functions on X and define the induced representation by $(\pi(g)f)(y) = f(g^{-1}y)$, with no motivation for the appearance of the inverse.

We now have five representations of the group D_4, the left and right regular, which are on 8 dimensional vector spaces, the permutation representation induced by its action on the vertices of the square, the permutation representation induced by its action on the edges of the square, which are both on a four dimensional vector space and the representation on $\mathbb{R}^2$ described in an earlier example.

One of the key problems in representation thoery is determining the relationships between these various representations and determining when one has a complete set of "building block" representations that can be used to construct all representations. These building block representations are called *irreducible representations* and they serve as the building blocks for all representations in much the same way that the prime numbers can be used to construct all integers. The study of irreducible representations will be the topic of the next section. For now we content ourselves with constructing a few more examples of representations of groups.

Problem 3.12. *Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ denote the cyclic group of order 4. Write out the 4×4 matrices of each of these 4 elements for the left and right regular representations.*

Problem 3.13. *Let S_3 denote the group of all permutations of 3 objects. Regard S_3 as actomg on $X = \{1, 2, 3\}$ and write down the matrices of the induced permutation representation for the group elements, $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, k = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.*

Problem 3.14. *We have that $D_n \subseteq S_n$. Show that in fact $D_3 = S_3$.*

For the next problem, we recall that the matrix for the linear transformation on $\mathbb{R}^2$ of counterclockwise rotation through angle θ is given by

$$R(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

Problem 3.15. *Let $X = \{v_1, v_2, v_3\}$ be the three vertices of the equilateral triangle in $\mathbb{R}^2$, where $v_1 = (1, 0), v_2 = (cos(2\pi/3), sin(2\pi/3)), v_3 = (cos(4\pi/3), sin(4\pi/3))$. Regard $D_3 = S_3$ as acting on this set. This induces a 2 dimensional representation of S_3. Find the 2×2 matrices for this representation for the elements g,h and k given in the above problem.*

3.3. **A Representation from P.D.E..** One of the important reasons for wanting to understand group representations, is that often solutions to many problems have a group acting on them and an understanding of the representations of this group often gives rise to a deeper understanding of the set of all solutions. We illustrate this with an example from P.D.E. Let $u : \mathbb{R}^2 \to \mathbb{R}$, be a function. Given a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we set $A \cdot u$ to be the new function, $(A \cdot u)(x, y) = u(ax + cy, bx + dy) = u((x, y)A)$.

Note that $A \cdot (u_1 + u_2) = A \cdot u_1 + A \cdot u_2$, and given another matrix B, $[A \cdot (B \cdot u)](x, y) = (B \cdot u)((x, y)A) = u([(x, y)A]B) = [(AB) \cdot u](x, y)$. Thus, this operation behaves associatively. Hence if we restrict, $A \in GL(2, \mathbb{R})$ then we have an action of this group on the vector space of functions from $\mathbb{R}^2$ to $\mathbb{R}$.

A C^∞-function, $u : \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic*, if $u_{xx} + u_{yy} = 0$ where the subscripts denote partial derivatives. These functions form a vector space, which we will denote by V_H. We will show that if $A \in \mathcal{O}_2$, and $u \in V_H$, then $A \cdot u \in V_H$.

To see this, first note that $(A \cdot u)_x(x_0, y_0) = au_x(ax_0 + cy_0, bx_0 + dy_0) + bu_y(ax_0 + cy_0, bx_0 + dy_0)$. Hence,

$$(A \cdot u)_{xx}(x_0, y_0) = a^2 u_{xx}(ax_0 + cy_0, bx_0 + dy_0) + ab u_{xy}(ax_0 + cy_0, bx_0 + dy_0) + ba u_{yx}(ax_0 + cy_o, bx_0 + dy_0) + b^2 u_{yy}(ax_0 + cy_0, bx_0 + dy_0).$$

Similar calculations, show that

$$(A \cdot u)_{yy}(x_0, y_0) = c^2 u_{xx}(ax_0 + cy_0, bx_0 + dy_0) + cd u_{xy}(ax_0 + cy_0, bx_0 + dy_0) + dc u_{yx}(ax_0 + cy_0, bx_0 + dy_0) + d^2 u_{yy}(ax_0 + cy_0, bx_0 + dy_0).$$

Since A is an orthogonal matrix, $a^2 + c^2 = b^2 + d^2 = 1$, while the first and second columns of A must be orthogonal and hence, $ab + cd = 0$. Thus, recalling that $u_{xy} = u_{yx}$, we see that the $(A \cdot u)_{xx} + (A \cdot u)_{yy} = 1u_{xx} + 0u_{xy} + 1u_{yy} = 0$, since u was harmonic.

Hence, we have a representation, $\pi : \mathcal{O}_2 \to GL(V_H)$ and a deeper understanding of how to decompose representations on $\mathcal{O}_2$ into smaller representations, would lead to a decomposition theory for harmonic functions.

It is also worth remarking that since derivatives remain unchanged under translations by constants, that we actually have a representation of the group $E(2)$ on V_H.

3.4. **One-Dimensional Representations.** Note that $\mathcal{L}(\mathbb{F}) \simeq \mathbb{F}$ and that $GL(\mathbb{F}) \simeq \mathbb{F}^*$, the multiplicative group of $\mathbb{F}$. Thus, one-dimensiona representations of a group are nothing more than homomorphisms into the multiplicative group of the underlying field.

The homomorphism, $\pi(g) = 1$ for all $g \in G$, is called the *trivial representation.*

If G is a finite group and $\pi : G \to \mathbb{R}^*$, then since every element of G is of finite order and the only elements of $\mathbb{R}^*$ of finite order are $\{\pm 1\}$, we have that $\pi(G) \subseteq \{\pm 1\} \simeq \mathbb{Z}_2$. Suppose that an element of $g \in G$ has odd order, say $g^l = e$ with l odd, the necessarily, $\pi(g) = 1$ for if $\pi(g) = -1$ then $1 = \pi(e) = \pi(g^l) = \pi(g)^l = (-1)^l = -1$, a contradiction. These observations show that for a general finite group, there are generally a very limited set of homomorphisms into $\mathbb{R}^*$. For example, since every element of $\mathbb{Z}_3$ has odd order the only homomorphism is $\pi(\mathbb{Z}_3) = \{1\}$.

If G is a finite group and $\pi : G \to \mathbb{C}^*$, then $\pi(G) \subseteq \mathbb{T}$, since the only elements of finite order in $\mathbb{C}^*$ are the roots of unity, which all lie on the unit circle. In fact, for every $g \in G, \pi(g)$ must be a root of unity. This allows for a considerably larger number of one-dimensional complex representations.

For example, if $G = \mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$, and ω is any n-th root of unity, then setting $\pi([k]) = \omega^k$, uniquely defines a representation of $\mathbb{Z}_n$. Thus, there are exactly n different one-dimensional complex representations.

Things are very different when the group is infinite. For example, if $a \in \mathbb{C}^*$ is any number then one obtains, $\pi : \mathbb{Z} \to \mathbb{C}^*$ by setting $\pi(n) = a^n$. Similarly, if G is the "ax+b" group, then one has an onto homomorphism $\pi : G \to \mathbb{R}^*$, by setting $\pi(\phi_{(a,b)}) = a$.

Note that since $\mathbb{F}^*$ is abelian, for any group G, if $\pi : G \to \mathbb{F}^*$ is a homomorphism, then for any, $g, h \in G, ghg^{-1}h^{-1}$ is in the kernel of π. Recall that the normal subgroup of G generated by all such elements is called the *commutator subgroup* and is denoted by [G,G]. Thus, every one-dimensional representation π of a group G, has $[G, G] \subseteq ker(\pi)$.

Problem 3.16. *For the dihedral group D_4, find the commutator subgroup.*

Problem 3.17. *For the group of affine maps of $\mathbb{R}$, find the commutator subgroup.*

3.5. **Subrepresentations.** Let $\pi : G \to GL(V)$ be a representation. A vector subspace, $W \subseteq V$ is called *invariant* or sometimes, $\pi(G)$-*invariant*, provided that $\pi(g)W \subseteq W$, i.e., for any $w \in W$ and any $g \in G$, we have that $\pi(g)w \in W$. In this case we define $\pi_W(g) : W \to W$ to be the restriction of the map $\pi(g)$ to W. It is easy to see that $\pi_W(g_1)\pi_W(g_2) = \pi_W(g_1g_2)$ and that $\pi_W(e) = I_W$. From these facts it follows that $\pi_W(g) \in GL(W)$ and that $\pi_W : G \to GL(W)$ is a representation. This representation is called a *subrepresentation* of π.

Example 3.18. *Look at the group, G, of affine maps of $\mathbb{R}$. We have seen that setting, $\pi(\phi_{(a,b)}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ defines a representation of G on $\mathbb{R}^2$. If we let $W \simeq \mathbb{R}$ denote the subspace spanned by the first basis vector, then W is invariant, and the induced representation, is the map, $\pi_W(\phi_{(a,b)}) = a \in \mathbb{R}^*$. Note that $\pi_W(G) \simeq \mathbb{R}^*$ and $ker(\pi_W) = \{\phi_{(0,b)}\} \simeq (\mathbb{R}, +)$. From these facts and general group theory it follows that G is a semidirect product, $\mathbb{R}^* \times_\theta \mathbb{R}$.*

Example 3.19. *Let G act on a finite set X and let, $\pi : G \to GL(\mathbb{F}(X))$ be the induced permutation representation. Let $w = \sum_{x \in X} e_x$ and let W be the one-dimensional space spanned by w. Then $\pi(g)w = \sum_{x \in X} e_{gx} = w$, since*

$x \to gx$ is a permutation of X. Thus, W is invariant and π_W is the trivial representation.

Thus, the trivial representation is a subrepresentation of every induced permutation when the set X is finite.

Example 3.20. *Assume that G is acting on a set X and let $\pi : G \to GL(\mathbb{F}(X))$ be the induced permutation representation. For $x \in X$, let $W_x = span(\{e_y : y \in \mathcal{O}_x\}$. Then W_x is invariant and the subrepresentation π_{W_x} is the permutation representation that we would get by resticting the action of G to the invariant subset, $\mathcal{O}_x \subseteq X$.*

A useful way to determine if a subspace is invariant, involves the notion of a generating set for a group.

Definition 3.21. *A subset S of a group G, is said to* **generate G,** *or be a* **generating set for G,** *provided that every element of $G/\{e\}$ can be written as a finite product, allowing repetitions, of elements of S.*

The phrase "allowing repetitions" means that if $g, h \in S$, then g^2h^3g would be considered a finite product of elements in S. For example, $\{R, F\}$ is a generating set for D_4, and $\{1, -1\}$ is a generating for the (additive) group $\mathbb{Z}$, but $\{1\}$ is not a generating set for $\mathbb{Z}$.

Proposition 3.22. *Let $\pi : G \to GL(V)$ be a representation, let S be a generating set for G, and let $W \subseteq V$ be a subspace. If $\pi(g)W \subseteq W$, for every $g \in S$, then W is $\pi(G)$-invariant.*

Proof. If $h \in G$, then $h = g_1 \cdot g_k$, for some finite set of elements of S. Hence, $\pi(h)W = [\pi(g_1) \cdot \pi(g_k)]W = [\pi(g_1) \cdot \pi(g_{k-1})]\pi(g_k)W \subseteq [\pi(g_1) \cdot \pi(g_{k-1})]W$, and we are done by an inductive argument. ☐

Example 3.23. *Let $\lambda : \mathbb{Z} \to GL(\mathbb{F}(\mathbb{Z}))$ be the left regular representation and let $W = span(\{e_n : n \geq 0\})$, then $\lambda(1)W \subseteq W$, but W is not $\lambda(\mathbb{Z})$-invariant, since $\lambda(-1)e_0 \notin W$. Also, the restriction of $\lambda(1)$ to W is not onto, and so not invertible.*

Problem 3.24. *Let $\pi : \mathbb{Z} \to GL(V)$ be a representation and let $W \subseteq V$ be a subspace. Prove that if $\pi(1)W = W$, then W is $\pi(\mathbb{Z})$-invariant.*

Problem 3.25. *Let $\pi : \mathbb{Z} \to GL(V)$ be a representation with V a finite dimensional vector space. Prove that if $\pi(1)W \subseteq W$, then W is $\pi(\mathbb{Z})$-invariant.*

3.6. **Internal and External Direct Sums.** There are two kinds of direct sums of vector spaces. Given vector spaces V and W, their Cartesian product, $V \times W = \{(v, w) : v \in V, w \in W\}$ is a vector space, with operations, $\lambda(v, w) = (\lambda v, \lambda w)$ and $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$. This vector space is called the *(external) direct sum of V and W* and is denoted by $V \oplus W$.

Note that $\tilde{V} = \{(v, 0) : v \in V\}$ and $\tilde{W} = \{(0, w) : w \in W\}$ are vector subspaces of $V \oplus W$ that are isomorphic to V and W, respectively. These subspaces satisfy, $\tilde{V} \cap \tilde{W} = \{0\}$ and $\tilde{V} + \tilde{W} = V \oplus W$.

Given a vector space, Z and two subspaces $V, W \subseteq Z$, if $V + W = Z$ and $V \cap W = \{0\}$, then we say that *Z is the (internal) direct sum of V and W.*

Note that if Z is the internal direct sum of V and W, then the map, $T : V \oplus W \to Z$ defined by $T((v, w)) = v + w$ is a vector space isomorphism between this internal direct sum of V and W and the external sum of V and W. For this reason many authors do not distinguish between these two objects.

There are also many reasons to want distinguish between these objects. For example, when one identifies, $\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$, one usually thinks of the two one-dimensional subspaces as perpendicular. But if one takes any two one dimensional subspaces, i.e., lines, $V, W \subseteq \mathbb{R}^2$, such that $\mathbb{R}^2 = V + W$, then these subspaces need not be perpendicular lines.

When Z is the internal direct sum of subspaces V and W, then we say that *W is a complement of V* and that *V and W are a complementary pair of subspaces of Z.* As the example of $\mathbb{R}^2$ shows, if V is any one-dimensional subspace, then W could be *any* other line, except V. Thus, the complement of a subspace is not unique.

When Z is the internal direct sum of V and W, then every $z \in Z$ has a unique decomposition as $z = v + w$ with $v \in V, w \in W$. Hence, we have well-defined maps, $P_V : Z \to Z$ and $P_W : Z \to Z$, defined by $P_V(z) = v$ and $P_W(z) = w$. Note that to define the map P_V we needed both V and a complementary subspace W.

Example 3.26. *Let $Z = \mathbb{R}^2, V = span\{(1, 0)\}, W_a = span\{(a, 1)\}$ for some fixed $a \in \mathbb{R}$. Then these subspaces are complementary, i.e., Z is their internal direct sum. Given any $z = (z_1, z_2)$ we have that it's unique decomposition is given by, $z = z_2(a, 1) + (z_1 - az_2)(1, 0)$. Thus, $P_V(z) = (z_1 - az_2)(1, 0)$ and $P_W(z) = z_2(a, 1)$. Which shows clearly the dependence of the maps, P_V, P_W on the pair of subspaces.*

The following result summarizes the properties of the maps, P_V and P_W.

Proposition 3.27. *Let Z be the internal direct sum of subspaces, V and W. Then:*

(i) $P_V, P_W \in \mathcal{L}(Z)$,
(ii) $P_V + P_W = I_Z$,
(iii) $P_V^2 = P_V, P_W^2 = P_W$,
(iv) $P_V z = z$ *if and only if* $z \in V, P_V z = 0$ *if and only if* $z \in W$.

Definition 3.28. *A map $P \in \mathcal{L}(Z)$ is called a* **projection** *or* **idempotent**, *if $P \neq 0, P \neq I_Z$ and $P^2 = P$.*

Proposition 3.29. *Let $P \in \mathcal{L}(Z)$, be a projection, let $V = \{z \in Z : Pz = z\}, W = \{z \in Z : Pz = 0\}$, then Z is the internal direct sum of V and W and for this decomposition, $P = P_V, P_W = I_Z - P$.*

Thus, given an idempotent P we have $P = P_V$ for some subspace V and complement W. For this reason we shall refer to the idempotent, P, as a *projection onto* V. Note that we always have that, $V = range(P)$.

Proposition 3.30. *Every subspace of a vector space is complemented.*

Proof. Let V be a subspace of Z. Choose a basis, $\{e_\alpha : \alpha \in A\}$ for V. By Zorn's lemma, there is a linearly independent set, $\{f_\beta : \beta \in B\}$, such that the union of the two sets is a maximal independent set and hence a basis for Z.

Let W be the span of $\{f_\beta : \beta \in B\}$, and check that Z is the internal direct sum of V and W. □

Definition 3.31. *Let* $\pi : G \to GL(Z)$ *be a representation and let* $V \subseteq Z$ *be a* $\pi(G)$*-invariant subspace. We say that* V *is* **G-complemented** *if there is a* $\pi(G)$*-invariant subspace,* $W \subseteq Z$ *that is a complement for* V*.*

Example 3.32. *Not every* $\pi(G)$*-invariant subspace is* $\pi(G)$*-complemented. Consider the representation of the affine maps of* $\mathbb{R}$ *on* $\mathbb{R}^2$ *as the matrices,* $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ *and let* V *be the span of* e_1*, which is* $\pi(G)$*-invariant. Then every complement of* V*, is a vector space of the form,* $W_c = span\{\begin{pmatrix} c \\ 1 \end{pmatrix}\}$*. But it is easy to check that none of these spaces are* $\pi(G)$*-invariant. For if* $\pi(G)W_c \subseteq W_c$*, then for each matrix, we would have,* $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} ac+b \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} c \\ 1 \end{pmatrix}$*. Equating the second components, forces* $\lambda = 1$*, and so,* $ac + b = c$*, which clearly cannot hold for all choices of* a *and* b*.*

This phenomena does not happen for finite groups.

Theorem 3.33. *Let* G *be a finite group and let* $\pi : G \to GL(Z)$ *be a representation. If* $V \subseteq Z$ *is a* $\pi(G)$*-invariant subspace, then* V *is* $\pi(G)$*-complemented.*

Proof. Let $|G| = card(G)$. Pick any complementary subspace, W for V, and let P_V be the projection onto V obtained from this decomposition. Set

$$P = \frac{1}{|G|} \sum_{g \in G} \pi(g) P_V \pi(g^{-1}).$$

If $z \in Z$, then $\pi(g)(P_V \pi(g^{-1})z) = \pi(g)v$, for some $v \in V$, and hence, $\pi(g)P_V\pi(g^{-1})z \in V$ since V is $\pi(G)$-invariant. Thus, $Pz \in V$, for any $z \in Z$.

Moreover, if $v \in V$, then $\pi(g)P_V\pi(g^{-1})v = \pi(g)\pi(g^{-1})v = v$, and hence, $Pv = v$, for every $v \in V$.

Hence, $Pv = v$ if and only if $v \in V$. Thus, we have that $P^2 z = P(Pz) = Pz$, since $Pz \in V$. Therefore, P is a projection onto V.

We claim that for any $h \in G, \pi(h)P\pi(h^{-1}) = P$. To see this, note that,

$$\pi(h)P\pi(h^{-1}) = \frac{1}{|G|}\sum_{g\in G}\pi(h)\pi(g)P_V\pi(g^{-1})\pi(h^{-1}) = \frac{1}{|G|}\sum_{\tilde{g}\in G}\pi(\tilde{g})P_V\pi(\tilde{g}^{-1}) = P.$$

Thus, P is a projection that commutes with each of the matrices, $\pi(h)$. Using P, we get a (possibly) new complement for V, by setting, $W = \{z \in Z : Pz = 0\}$. Now we claim that W is $\pi(G)$-invariant. To see this, for $w \in W$, we have $P\pi(h)w = \pi(h)Pw = \pi(h)0 = 0$, and hence, $\pi(h)W \subseteq W$ for any $h \in G$. □

The technique used in the above theorem of summing a formula over all elements in the group is called *averaging over the group.*

Definition 3.34. *Let G be a group and let,* $\pi_i : G \to GL(W_i), i = 1, 2,$ *be representations. The map,* $\pi : G \to GL(W_1 \oplus W_2)$ *defined by* $\pi(g)(w_1, w_2) = (\pi_1(g)w_1, \pi_2(g)w_2)$ *is easily seen to be a representation. We let* $\pi_1 \oplus \pi_2$ *denote* π *and we call this representation the* **direct sum of the representations,** π_1 **and** π_2**.**

Definition 3.35. *Let G be a group and let* $\pi_i : G \to GL(W_i), i = 1, 2$ *be representations. If there exists an invertible linear map,* $T : W_1 \to W_2$ *such that* $T^{-1}\pi_2(g)T = \pi_1(g)$ *for all* $g \in G$*, then we say that* π_1 *and* π_2 *are* **equivalent representations** *and we write,* $\pi_1 \sim \pi_2$ *to denote that* π_1 *and* π_2 *are equivalent.*

We leave it to the reader to check that the above definition really is an equivalence relation on the set of representations of G.

Proposition 3.36. *Let G be a group, let* $\pi : G \to GL(V)$ *be a representation, and let* $W_i \subseteq V, i = 1, 2$ *be a complementary pair of* $\pi(G)$*-invariant subspaces. Then* $\pi \sim \pi_{W_1} \oplus \pi_{W_2}$.

Proof. Let $T : W_1 \oplus W_2 \to V$ be defined by $T((w_1, w_2)) = w_1 + w_2$, then T is one-to-one and onto. We will show that $\pi(g)T = T(\pi_{W_1} \oplus \pi_{W_2})$ for every $g \in G$.

Now, $\pi(g)T((w_1, w_2)) = \pi(g)(w_1+w_2) = \pi(g)(w_1)+\pi(g)(w_2) = \pi_{W_1}(g)(w_1)+\pi_{W_2}(g)(w_2)$, while, $T(\pi_{W_1}(g)\oplus\pi_{W_2}(g))((w_1, w_2)) = T((\pi_{W_1}(g)(w_1), \pi_{W_2}(g)(w_2))) = \pi_{W_1}(g)(w_1) + \pi_{W_2}(g)(w_2)$, and so we have shown the claimed equality. □

Irreducible Representations

Definition 3.37. *A representation,* $\pi : G \to GL(V)$ *is* **irreducible** *if the only* $\pi(G)$*-invariant subspaces of* V *are* V *and* (0).

Remark 3.38. *This terminology is not absolutely standard. Some authors define a representation to be reducible if V is the internal direct sum of two non-zero* $\pi(G)$*-invariant subspaces. They then define irreducible to mean not reducible. Thus, the representation of the group of affine transformations of* $\mathbb{R}$ *as* 2×2 *matrices is* **not** *irreducible in our sense, but is irreducible in this* *other sense. The difference is that while it has non-trivial $\pi(G)$-invariant subspaces it does not have any $\pi(G)$-complemented subspaces. For finite groups, we have shown that every $\pi(G)$-invariant subspace is complemented, so these two definitions coincide in that case.*

Note that every 1-dimensional representation is irreducible. The following result shows that the irreducible representations are the "building blocks" of all representations in much the same way that prime numbers are the building blocks of the integers.

Theorem 3.39. *Let G be a finite group and let $\pi : G \to GL(V)$ be a finite dimensional representation of G. Then there exists an integer k and $\pi(G)$-invariant subspaces, $W_1, \ldots, W_k$ of V, such that:*

(i) $V = W_1 + \cdots + W_k$, *and* $W_i \cap (\sum_{j \neq i} W_j) = (0)$, *for* $i \neq j$,
(ii) *the subrepresentations,* $\pi_{W_i} : G \to GL(W_i), 1 \leq i \leq k$, *are irreducible,*
(iii) $\pi \sim \pi_{W_1} \oplus \cdots \oplus \pi_{W_k}$.

Proof. The proof of (i) and (ii) is by induction on the dimension of V. When $dim(V) = 1$, then π is irreducible and (i) and (ii) are met by setting $k = 1$ and $W_1 = V$. Now assume that (i) and (ii) hold for any representation of G on a vector space Z with $dim(Z) \leq n$, and let $dim(V) = n + 1$.

If π is irreducible we are done. Otherwise, there exists a $\pi(G)$-invariant subspace, Z_1, with $(0) \neq Z_1 \neq V$. By Theorem 3.33, Z_1 possesses a $\pi(G)$-complement, Z_2. Since $0 < dim(Z_1) < dim(V)$, we have that $dim(Z_1) \leq n$ and $dim(Z_2) \leq n$. Hence, by the inductive hypothesis, $Z_1 = W_1 + \cdots + W_k$ and $Z_2 = W_{k+1} + \cdots + W_m$, where each subrepresentation π_{W_i} is irreducible, for $1 \leq i \leq k, W_i \cap (\sum_{j=1, j\neq i}^{k} W_j) = (0)$, and for $k + 1 \leq i \leq m, W_i \cap (\sum_{j=k+1, j\neq i}^{m} W_j) = (0)$.

Moreover, $V = Z_1 + Z_2 = W_1 + \cdots + W_m$, and so it remains to show that $W_i \cap (\sum_{j=1, j\neq i}^{m}) = (0)$. We first consider the case that $1 \leq i \leq k$. If a vector v is in the intersection, then we have that $v = w_i = \sum_{j=1, j\neq i}^{m} w_j$, with $w_j \in W_j$. Hence, $w_i - \sum_{j=1, j\neq i}^{k} = \sum_{j=k+1}^{m}$. The vector on the left belongs to Z_1, while the vector on the right belongs to Z_2. Since $Z_1 \cap Z_2 = (0)$, they must both be 0. Because (i) holds for $W_1, ..., W_k$ and for $W_{k+1}, ..., W_m$, separately, we have that $0 = w_1 = \cdots = w_m$, and it follows that $W_i \cap (\sum_{j=1, j\neq i}^{m} W_j) = (0)$. The proof for the case that $k + 1 \leq j \leq m$, is identical.

This proves (i) and (ii).

To prove (iii), assume that we have (i) and (ii) and consider the map $T : W_1 \oplus \cdots \oplus W_k \to V$ defined by $T((w_1, \ldots, w_k)) = w_1 + \cdots + w_k$. Clearly T is onto. If $(w_1, \ldots, w_k)$ is in the kernel of T, then $-w_1 = w_2 + \cdots + w_k \in W_1 \cap (\sum_{j=2}^{k} W_j)$, and hence is 0. Thus, $w_1 = 0$, and $-w_2 = w_3 + \cdots + w_k$, from which it follows that $w_2 = 0$. Inductively, one finds that $w_1 = \cdots = w_k = 0$, and so T is invertible. It is easily checked that T implements the similarity that proves (iii). $\square$

The decomposition obtained in the above theorem is not unique. For example, if $\rho : G \to \mathbb{F}^*$ is 1-dimensional and we define, $\pi : G \to GL(\mathbb{F}^n)$ by $\pi(g) = \rho(g)I_n$, where I_n denotes the identity matrix, then every one-dimensional subspace of $\mathbb{F}^n$ will be irreducible and the above decomposition can be obtained by taking *any* decomposition of $\mathbb{F}^n$ into n one-dimensional subspaces.

Note that while the subspaces are not unique, if $W \subseteq \mathbb{F}^n$ is any one-dimensional subspace, then $\pi_W = \rho$.

We will prove that, in general, while the subspaces are not unique, the irreducible representations that one obtains are unique, up to similarity.

First, we will need some further characterizations of irreducible representations.

Definition 3.40. *Let $\mathcal{S} \subseteq \mathcal{L}(V)$ be any set. Then the* **commutant** *of $\mathcal{S}$, is the set $\mathcal{S}' = \{T \in \mathcal{L}(V) : TS = ST \text{ for every } S \in \mathcal{S}\}$.*

Note that since the scalar multiples of the identity commute with every linear transformation, these always belong to the commutant of $\mathcal{S}$. If these are the only linear transformations in the commutant of $\mathcal{S}$, then we say that *$\mathcal{S}$ has trivial commutant.*

The following result is the first result where it really matters if the field is $\mathbb{R}$ or $\mathbb{C}$.

Theorem 3.41. *Let G be a finite group and let $\pi : G \to GL(V)$ be a finite dimensional representation of G.*

(i) *If $\pi(G)' = \{\lambda I_V : \lambda \in \mathbb{F}\}$, i.e., if $\pi(G)$ has trivial commutant, then π is irreducible.*
(ii) *When $\mathbb{F} = \mathbb{C}$, then π is irreducible if and only if $\pi(G)$ has trivial commutant.*

Proof. If π is not irreducible, then there is a subspace, W, with $0 \neq W \neq V$ that is $\pi(G)$-invariant. In the proof of Theorem 3.33, we constructed a projection, P, onto W that commutes with $\pi(G)$. Thus, $P \in \pi(G)'$. (To see how this follows from the statement of the theorem, instead of the proof, take a $\pi(G)$-complement for W and construct a projection onto W using this complementary subspace and then show that such a projection is in the commutant.)

Hence, for any field, if the commutant is trivial, then the representation is irreducible.

Next, assume that the field is $\mathbb{C}$ and that the commutant is non-trivial. Let $T \in \pi(G)'$ be an operator that is not a scalar multiple of the identity. In this case there exists an eigenvalue, λ of T, and necessarily, $T - \lambda I_V \neq 0$. Let $W = ker(T - \lambda I_V)$, then $0 \neq W \neq V$. But, if $w \in W$, then $(T - \lambda I_V)(\pi(g)w) = \pi(g)(T - \lambda I_V)w = 0$ and so W is $\pi(G)$-invariant. Hence, V is not irreducible. Therefore, when $\mathbb{F} = \mathbb{C}$, if V is irreducible, then the commutant of $\pi(G)$ is trivial. $\square$

Problem 3.42. *Let $\pi : \mathbb{Z}_4 \to GL(\mathbb{R}^2)$ be defined by $\pi(k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^k$. Prove that π is irreducible, but $\pi(\mathbb{Z}_4)' = span\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$. Thus, the assumption that $\mathbb{F} = \mathbb{C}$ is essential for Theorem 3.41(ii).*

Problem 3.43. *Let $\pi : \mathbb{Z}_4 \to GL(\mathbb{C}^2)$ be given by the same matrices as in the last problem. Find a one-dimensional $\pi(\mathbb{Z}_4)$-invariant subspace and its $\pi(\mathbb{Z}_4)$-complement. This determines a decomposition of π into two one-dimensional subrepresentations, describe these representations explicitly as functions from $\mathbb{Z}_4$ into the unit circle.*

Problem 3.44. *Prove that the representation of D_4 on $\mathbb{F}^2$ given by $\pi(R) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pi(F) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is irreducible, whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.*

Problem 3.45. *Let $\pi : D_4 \to GL(\mathbb{C}(X))$ be the permutation representation induced by the action of D_4 on the four vertices of the square, $X = \{1, 2, 3, 4\}$. Find a decomposition of this representation into irreducible subrepresentations as given by Theorem 3.39.*

Problem 3.46. *Let $\lambda : D_4 \to GL(\mathbb{C}(D_4))$ be the left regular representation(so on an 8-dimensional space). Find a decomposition of λ into irreducible subrepresentations as given by Theorem 3.39.*

Problem 3.47. *Let G be a finite group and let $\pi : G \to GL(V)$ be a representation. Prove that if $dim(V) > card(G)$, then π is not irreducible.*

Definition 3.48. *Let $\pi_i : G \to GL(V_i), i = 1, 2$ be representations. The set $\mathcal{I}(\pi_1, \pi_2) = \{T \in \mathcal{L}(v_1, V_2) : \pi_2(g)T = T\pi_1(g), for\ everyg \in G\}$ is called the* **space of intertwining maps between π_1 and π_2.**

Note that when $\pi_1 = \pi_2 = \pi$, then $\mathcal{I}(\pi, \pi) = \pi(G)'$.

Proposition 3.49. *Let G be a group and let $\pi_i : G \to GL(V_i), i = 1, 2$ be representations. Then $\mathcal{I}(\pi_1, \pi_2)$ is a vector subspace of $\mathcal{L}(V_1, V_2)$, and $\pi_1 \sim \pi_2$ if and only if there exists an invertible linear transformation in $\mathcal{I}(\pi_1, \pi_2)$.*

Proof. Let $T_i \in \mathcal{I}(\pi_1, \pi_2)$ and let $\lambda \in \mathbb{F}$. Then for any $g \in G$, $\pi_2(g)(\lambda T_1 + T_2) = \lambda\pi_2(g)T_1 + \pi_2(G)T_2 = (\lambda T_1 + T_2)\pi_1(g)$, and so $\mathcal{I}(\pi_1, \pi_2)$ is a vector subspace.

If $T \in \mathcal{I}(\pi_1, \pi_2)$, is invertible, then $T^{-1}\pi_2(g)T = \pi_1(g)$, or every $g \in G$, and so $\pi_2 \sim \pi_1$. Conversely, if $\pi_2 \sim \pi_1$ and this similarity is implemented by T, then $T \in \mathcal{I}(\pi_1, \pi_2)$. □

Theorem 3.50 (Schur's Lemma)**.** *Let G be a finite group and let $\pi_i : G \to GL(V_i), i = 1, 2$ be irreducible representations. Then $\pi_1 \sim \pi_2$ if and only if $\mathcal{I}(\pi_1, \pi_2) \neq (0)$. In the case that $V_i, i = 1, 2$ are vector spaces over $\mathbb{C}$, then $dim(\mathcal{I}(\pi_1, \pi_2))$ is either 0 or 1.*

Proof. Assume that $\pi_1 \sim \pi_2$ and let $\pi_2(g)T = \pi_1(g)T$ with T invertible. Then $T \in \mathcal{I}(\pi_1, \pi_2)$ and so this set is not equal to (0).

Conversely, if $\mathcal{I}(\pi_1, \pi_2) \neq (0)$, then there exists $T \neq 0$ in this set. Since $T \neq 0$, we have that $W_1 = ker(T) \subsetneq V_1$. But for $w_1 \in W_1, T\pi_1(g)w_1 = \pi_2(g)Tw_1 = 0$, and so $W_1$is $\pi_1(G)$-invariant. Since π_1 is irreducible, $W_1 = (0)$ and T is one-to-one. Similarly, let $W_2 = ran(T) \neq (0)$, since $T \neq 0$ and check that W_2 is $\pi_2(G)$-invariant. Since, π_2 is irreducible, $W_2 = V_2$ and T is onto.

Thus, T is invertible and hence $\pi_1 \sim \pi_2$.

Now assume that we are dealing with vector spaces over $\mathbb{C}$ and that $\mathcal{I}(\pi_1, \pi_2) \neq (0)$. Then by the first result, $\pi_1 \sim \pi_2$. Let $T \in \mathcal{I}(\pi_1, \pi_2)$ be any invertible that implements the similarity. Note that $T^{-1} \in \mathcal{I}(\pi_2, \pi_1)$. Now suppose that $S \in \mathcal{I}(\pi_1, \pi_2)$ is any map. Then $T^{-1}S \in \pi_1(G)$ and since π_1 is irreducible, there exists a scalar, λ such that $T^{-1}S = \lambda I_{V_1}$, and so, $S = \lambda T$. Thus, $\mathcal{I}(\pi_1, \pi_2) = span\{T\}$, and so is one-dimensional. $\square$

Thus, by Schur's Lemma, when π_1, π_2 are irreducible representations over $\mathbb{C}$, then $dim(\mathcal{I}(\pi_1, \pi_2))$ is either 0 or 1 and these determine whether $\pi_1 \nsim \pi_2$ or $\pi_1 \sim \pi_2$. The following example, motivates our next theorem.

Example 3.51. *Let $\pi : G \to GL(V)$ and $\rho_i : G \to GL(W_i), i = 1, 2$ be representations and let $\rho = \rho_1 \oplus \rho_2 : G \to GL(W_1 \oplus W_2)$. Given $T \in \mathcal{L}(V, W_1 \oplus W_2)$ there exists, $T_i \in \mathcal{L}(V, W_i), i = 1, 2$ such that $T(v) = (T_1v, T_2v)$.*

Hence, $T\pi(g)v = (T_1\pi(g)v, T_2\pi(g)v)$, while $\rho(g)Tv = (\rho_1(g)T_1v, \rho_2(g)T_2v)$. Thus, $T \in \mathcal{I}(\pi, \rho)$ if and only if $T_i \in \mathcal{I}(\pi, \rho_i), i = 1, 2$.

Thus, as vector spaces, $\mathcal{I}(\pi, \rho) = \mathcal{I}(\pi, \rho_1) \oplus \mathcal{I}(\pi, \rho_2)$, which shows, in particular, that $dim(\mathcal{I}(\pi, \rho)) = dim(\mathcal{I}(\pi, \rho_1)) + dim(\mathcal{I}(\pi, \rho_2))$.

Problem 3.52. *Let $\pi : G \to GL(V)$ and $\rho_i : G \to GL(W_i), i = 1, 2$ be representations and let $\rho = \rho_1 \oplus \rho_2 : G \to GL(W_1 \oplus W_2)$. Given $T \in \mathcal{L}(W_1 \oplus W_2, V)$ prove that there exists, $T_i \in \mathcal{L}(W_i, V), i = 1, 2$ such that $T((w_1, w_2)) = T_1w_1 + T_2w_2$.*

Prove that $T \in \mathcal{I}(\rho, \pi)$ if and only in $T_i \in \mathcal{I}(\rho_i, \pi), i = 1, 2$. Prove that as vector spaces, $\mathcal{I}(\rho, \pi)$ and $\mathcal{I}(\rho_1, \pi) \oplus \mathcal{I}(\rho_2, \pi)$ are isomorphic and deduce the corresponding result for the dimensions.

Problem 3.53. *Let $\pi_i : G \to GL(V_i), \rho_i : G \to GL(W_i), i = 1, 2$ be representations, with $\pi_1 \sim \pi_2$ and $\rho_1 \sim \rho_2$. Prove that $dim(\mathcal{I}(\pi_1, \rho_1)) = dim(\mathcal{I}(\pi_2, \rho_2))$.*

Theorem 3.54. *Let G be a finite group, let V, W be finite dimensional vector spaces over $\mathbb{C}$, let $\pi : G \to GL(V)$ be an irreducible representation, let $\rho : G \to GL(W)$ be a representation and let $W = W_1 + \cdots + W_m$ be an internal direct sum decomposition of W into $\rho(G)$-invariant subspaces such that each subrepresentation, ρ_{W_i} is irreducible. Then $dim(\mathcal{I}(\pi, \rho)) = card(\{i : \rho_{W_i} \sim \pi\})$, and consequently, this number is independent of the particular decomposition of ρ into irreducible subrepresentations.*

Proof. We have that $\rho \sim \rho_{W_1} \oplus \cdots \oplus \rho_{W_m}$ where the later representation is on the space, $W_1 \oplus \cdots \oplus W_m$. By the above example and problems, $dim(\mathcal{I}(\pi,\rho)) = dim(\mathcal{I}(\pi,\rho_{W_1})) + \cdots + dim(\mathcal{I}(\pi,\rho_{W_m}))$.

But by Schur's Lemma, $dim(\mathcal{I}(\pi,\rho_{W_i}))$ is either 0 or 1, depending on whether $\pi \nsim \rho_{W_i}$ or $\pi \sim \rho_{W_i}$, and the result follows. $\square$

Definition 3.55. *Let G be a finite group, let V, W be finite dimensional complex vector spaces, and let $\pi : G \to GL(V), \rho : G \to GL(W)$ be representations with π irreducible. We call $dim(\mathcal{I}(\pi,\rho))$ the* **multipicity of π in ρ**.

The following result shows that the multiplicities determine ρ up to similarity.

Corollary 3.56. *Let G be a finite group, let W be a finite dimensional complex vector space, let $\rho : G \to GL(W)$ be a representation and suppose that $\pi_i : G \to GL(V_i), i = 1, ..., k$ are inequivalent, finite dimensional, irreducible representations of G and represent all equivalence classes of finite dimensional irreducible representations, for which, $dim(\mathcal{I}(\pi,\rho)) \neq 0$. If $dim(\mathcal{I}(\pi_i,\rho)) = m_i, i = 1, ..., k$, then $\rho \sim \pi_1^{(m_1)} \oplus \cdots \oplus \pi_k^{(m_k)}$, where $\pi_i^{(m_i)} = \pi_i \oplus \cdots \oplus \pi_i, (m_i$ times).*

These results allow us to give a concrete formula for the dimension of the intertwining space for any two finite dimensional, complex representations.

Corollary 3.57. *Let G be a finite group, let $V_i, i = 1, ..., k$ be finite dimensional complex vector spaces, let $\pi_i : G \to GL(V_i), i = 1, ..., k$ be irreducible representations and let $\rho \sim \pi_1^{(m_1)} \oplus \cdots \oplus \pi_k^{(m_k)}, \gamma \sim \pi_1^{(n_1)} \oplus \cdots \oplus \pi_k^{(n_k)}$, then $dim(\mathcal{I}(\rho,\gamma)) = m_1n_1 + \cdots m_kn_k$.*

Proof. Since $dim(\mathcal{I}(\rho,\gamma))$ is invariant under similarities, it is enough to assume that $\rho = \pi_1^{(m_1)} \oplus \cdots \oplus \pi_k^{(m_k)}$ and $\gamma = \pi_1^{(n_1)} \oplus \cdots \oplus \pi_k^{(n_k)}$. Now a linear map T between a direct sum of vector space can be regarded as a block matrix $T = (T_{i,j})$ where each $T_{i,j}$ is a linear map from the j-th direct summand of the domain into the i-th direct summand of the range. Thus, any $T = (T_{i,j}) \in \mathcal{I}(\rho,\gamma)$ where each $T_{i,j} : V_{n_j} \to V_{n_i}$ satisfies $\pi_{n_i}(g)T_{i,j} = T_{i,j}\pi_{n_j}(g)$, for every $g \in G$.

Hence, by Shur's Lemma, when $n_j \neq n_i$, we will have that $T_{i,j} = 0$, while if $n_j = n_i$, then $T_{i,j} = \lambda_{i,j}I_{V_{n_j}}$, for some choice of scalars, $\lambda_{i,j} \in \mathbb{C}$.

From this it follows that as a vector space, $\mathcal{I}(\rho,\gamma)$ can be identified with the direct sum of rectangular matrices given by $M_{n_1,m_1} \oplus \cdots \oplus M_{n_k,m_k}$, which has dimension $m_1n_1 + \cdots m_kn_k$. $\square$

Corollary 3.58. *Let G be a finite group, let W be a finite dimensional complex vector space, let $\rho : G \to GL(W)$ be a representation and suppose that $\pi_i : G \to GL(V_i), i = 1, ..., k$ are inequivalent, finite dimensional, irreducible representations of G and represent all equivalence classes of finite dimensional irreducible representations, for which, $dim(\mathcal{I}(\pi,\rho)) \neq 0$. If $dim(\mathcal{I}(\pi_i,\rho)) = m_i, i = 1, ..., k$, then $dim(\rho(G)') = m_1^2 + \cdots + m_k^2$.*

Example 3.59. *Let G be a finite group, V be a finite dimensional vector space and let $\rho : G \to GL(V)$. If $dim(\rho(G)') = 5$, then what can be said about the decomposition of ρ into irreducible representations?*

Since there are only two ways to write 5 as a sum of squares, either $5 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2$, or $5 = 2^2 + 1^2$, we see that either ρ decomposes into 5 inequivalent irreducible representations each of multiplicity 1 or into 2 inequivalent irreducible representations, the first of multiplicity 2 and the second of multiplicity 1.

If, in addition, we knew that $dim(V) = 4$, then the first case would be impossible. Thus, we must have the second case, $\rho \sim \pi_1^{(2)} \oplus \pi_2$. If π_i is a representation on a d_i dimensional space, then we also have that $4 = 2d_1+d_2$, which forces that $d_1 = 1, d_2 = 2$.

Thus, we see that the above theorems give a lot of information for computing the multiplicities and dimensions of the irreducible subrepresentations of a given representation.

Problem 3.60. *Let $\pi : D_4 \to GL(2, \mathbb{C})$ be the two dimensional irreducible representation of D_4 given by regarding the square as embedded in the plane and let $\lambda : D_4 \to GL(8, \mathbb{C})$ be the left regular representation. Compute the multiplicity of π in λ.*

Problem 3.61. *Prove that the two-dimensional representation of $S_3 = D_3$ given by regarding the triangle as embedded in the plane is irreducible. Compute the multiplicity of this representation in the 6-dmensional, left regular representation of S_3.*

Problem 3.62. *Let G be a finite group, let V, W be finite dimensional complex vector spaces, and let $\pi : G \to GL(V), \rho : G \to GL(W)$ be representations. Prove that $dim(\mathcal{I}(\pi, \rho)) = dim(\mathcal{I}(\rho, \pi))$. Can you prove or disprove the same statement for real vector spaces?*

The Group Algebra

The vector space, $\mathbb{F}(G)$ is also naturally endowed with a product that makes it into an algebra over $\mathbb{F}$. This product is defined by

$$(\sum_{g\in G} \lambda_g e_g)(\sum_{h\in G} \mu_h e_h) = \sum_{g,h\in G} \lambda_g \mu_h e_{gh} = \sum_{k\in G} [\sum_{gh=k} \lambda_g \mu_h] e_k.$$

The basis vector corresponding to the identity element for G, is an identity for this algebra.

The group algebra has the property that every representation of G, $\pi : G \to GL(V)$, extends uniquely to a unital algebra homomorphism, $\tilde{\pi} : \mathbb{F}(G) \to \mathcal{L}(V)$, by setting $\tilde{\pi}(\sum_{g\in G} \lambda_g e_g) = \sum_{g\in G} \lambda_g \pi(g)$.

Conversely, given a unital algebra homomorphism, $\tilde{\pi} : \mathbb{F}(G) \to \mathcal{L}(V)$, then setting, $\pi(g) = \tilde{\pi}(e_g)$ defines a representation of G on V.

It is important to see what this product becomes when we regard $\mathbb{F}(G)$ as functions on G. Under this identification, $\sum_{g\in G} \lambda_g e_g$ is identified with the function, $f_1 : G \to \mathbb{F}$, with $f_1(g) = \lambda_g$. Similarly, $\sum_{h\in G} \mu_h e_h$ is identified with the function, f_2 satisfying, $f_2(h) = \mu_h$. Thus, the product of these two elements of $\mathbb{F}(G)$ is the function, denoted $f_1 * f_2$, satisfying,

$$(f_1 * f_2)(k) = \sum_{gh=k} \lambda_g \mu_h = \sum_{g\in G} \lambda_g \mu_{g^{-1}k} = \sum_{g\in G} f_1(g) f_2(g^{-1}k),$$

since $gh = k$ implies that $h = g^{-1}k$. Alternatively, solving for g instead, $g = kh^{-1}$, we see that,

$$(f_1 * f_2)(k) = \sum_{h\in G} f_1(kh^{-1}) f_2(h).$$

This product on functions is called the **convolution product.**

One final remark. Note that $dim(\mathbb{R}(D_4)) = card(D_4) = 8$. Thus, even though the map, $\pi : D_4 \to GL(2, \mathbb{R})$ is one-to-one, the map $\tilde{\pi} : \mathbb{R}(D_4) \to M_2(\mathbb{R})$ cannot be one-to-one, since $dim(M_2) = 4 < dim(\mathbb{R}(D_4))$.

4. Character Theory for Finite Groups

We will assume throughout this section that G is a finite group, unless specifically stated otherwise.

Definition 4.1. *Let $A = (a_{i,j}) \in M_n(\mathbb{F})$, then the* **trace of A** *is the quantity, $Tr(A) = \sum_{i=1}^n a_{i,i}$.*

The following summarizes the two most important properties of the trace.

Proposition 4.2. *The map, $Tr : M_n(\mathbb{F}) \to \mathbb{F}$ is linear and for $A = (a_{i,j}), B = (b_{i,j})$ in $M_n(\mathbb{F})$, we have $Tr(AB) = TR(BA)$.*

Proof. The linearity of Tr is clear. We have that $Tr(AB) = \sum_{i=1}^n \sum_{k=1}^n a_{i,k} b_{k,i} = \sum_{i=1}^n \sum_{k=1}^n b_{i,k} a_{k,i} = Tr(BA)$, after re-labeling the indices. □

Corollary 4.3. *Let $A \in M_n(\mathbb{F}), S \in GL(n, \mathbb{F})$, then $Tr(S^{-1}AS) = Tr(A)$. Consequently, $Tr(A)$ is the sum of the eigenvalues of A.*

Proof. We have that $Tr(S^{-1}AS) = Tr(SS^{-1}A) = Tr(A)$. Now, we may choose $S \in GL(n, \mathbb{C}))$ such that $S^{-1}AS$ is upper triangular with the eigenvalues of A for the diagonal entries. □

Remark 4.4. *If V is any n-dimensional space, then by choosing a basis for V we may identify $\mathcal{L}(V)$ with $M_n(\mathbb{F})$ and in this way define the trace of a linear map on V. If we choose a different basis for V, then the two matrix representations for a linear map that we obtain in this fashion will differ by conjugation by an invertible matrix. Thus, the value of the trace that one obtains in this way is independent of the particular basis and by the above corollary will always be equal to the sum of the eigenvalues of the linear transformation. Hence, there is a well-defined trace functional on $\mathcal{L}(V)$.*

Definition 4.5. *Let G be a group and let $\pi : G \to GL(V)$ be a representation of G on a finite dimensional vector space. Then the* **character of** π *is the function, $\chi_\pi : G \to \mathbb{F}$ defined by, $\chi_\pi(g) = Tr(\pi(g))$.*

Example 4.6. *Let G act on a finite set X and let π be the corresponding permutation representation on $\mathbb{F}(X)$. Then for any $g \in G, \chi_\pi(g) = card(\{x : g \cdot x = x\}) = \chi_\pi(g^{-1})$.*

Example 4.7. *Let G act on G by left multiplication and let λ be the left regular representation. Then for any $g \neq e, \chi_\lambda(g) = 0$, while $\chi_\lambda(e) = card(G)$.*

Example 4.8. *Let G be the group of affine transformations of G and let $\pi(\phi_{a,b}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, then $\chi_\pi(\phi_{a,b}) = a + 1$, while $\chi_\pi(\phi_{a,b}^{-1}) = a^{-1} + 1$.*

Example 4.9. *Suppose that $\rho : G \to GL(V)$ has been decomposed as $\rho \sim \pi_1^{(n_1)} \oplus \cdots \oplus \pi_k^{(n_k)}$, then as functions, $\chi_\rho = n_1\chi_{\pi_1} + \cdots + n_k\chi_{\pi_k}$. Thus, for finite groups we see that every character function is a linear combination with non-negative integer coefficients of the characters of irreducible representations!*

Example 4.10. *Let D_4 act on $\mathbb{R}^2$ as in Example 2.20, then $\chi_\pi(R) = \chi_\pi(F) = 0$, while $\chi_\pi(R^2) = -2$. When D_4 acts on the four vertices of the square, then for the induced permutation representation, ρ, we have that $\chi_\rho(R) = 0, \chi_\rho(F) = 2, \chi_\rho(R^2) = 0$. Thus, $\chi_\rho \neq 2\chi_\pi$ and so we know that $\rho \nsim \pi \oplus \pi$.*

We now look at some general properties of characters.

Proposition 4.11. *Let G be a finite group and let $\pi : G \to GL(V)$ be a representation where $dim(V) = n$ is finite. Then:*

(i) $\chi_\pi(e) = n$,
(ii) $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$,
(iii) $\chi_\pi(h^{-1}gh) = \chi_\pi(g)$,
(iv) *when $\pi \sim \rho$, then $\chi_\pi = \chi_\rho$.*

Proof. We have that $\chi_\pi(e) = Tr(I_V) = dim(V) = n$, and (i) follows.

Since G is a finite group, for any $g \in G$ there exists k such that $g^k = e$ and hence, $\pi(g)^k = I_V$. This fact forces all of the eigenvalues of $\pi(g)$ to lie on the unit circle. If $\{\lambda_1, \dots, \lambda_n\}$ denote the eigenvalues of $\pi(g)$, then the inverses of these numbers are the eigenvalues of $\pi(g^{-1})$ and hence, $\chi_\pi(g^{-1}) = \lambda_1^{-1} + \cdots + \lambda_n^{-1} = \bar{\lambda_1} + \cdots + \bar{\lambda_n} = \overline{\chi_\pi(g)}$, and (ii) follows.

Items (iii) and (iv), follow from the similarity invariance of the trace. □

Note that for the representation of the group of affine maps of $\mathbb{R}$, (ii) fails. This happens because the eigenvalues of these matrices need not lie on the unit circle.

4.1. **Averaging Over Groups.** Let G be a finite group and let $\pi_i : G \to GL(V_i), i = 1, 2$ be representations. Given $T \in \mathcal{L}(V_1, V_2)$, we set $E_G(T) = \frac{1}{|G|}\sum_{g \in G} \pi_2(g)T\pi_1(g^{-1})$, where $|G| = card(G)$. Note that $E_G(T) \in \mathcal{L}(V_1, V_2)$.

Definition 4.12. *We call $E_G(T)$ the* **expectation** *of T with respect to the two representations.*

The following results summarize the key facts about the expectation.

Proposition 4.13. *Let G be a finite group, let $\pi_i : G \to GL(V_i), i = 1, 2$ be representations on $\mathbb{C}$ vector spaces and let E_G be the correspondingexpectation. Then:*

(i) $E_G : \mathcal{L}(V_1, V_2) \to \mathcal{L}(V_1, V_2)$ *is a linear map,*
(ii) $E_G(T) \in \mathcal{I}(\pi_1, \pi_2)$,
(iii) $E_G^2 = E_G$,
(iv) *when $\pi_i, i = 1, 2$ are irreducible with $\pi_1 \nsim \pi_2$, then $E_G = 0$,*
(v) *when $\pi_1 = \pi_2$ is irreducible and $dim(V_1) = n$, then $E_G(T) = \frac{Tr(T)}{n} I_{V_1}$.*

Proof. The proof of (i) follows easily from the formula for E_G.

To see (ii), note that $\pi_2(h) = \frac{1}{|G|} \sum_{g \in G} \pi_2(hg) T \pi_1(g^{-1}) =$ $\frac{1}{|G|} \sum_{g_1 \in G} \pi_2(g_1) T \pi_1(g_1^{-1} h) = E_G(T) \pi_1(h)$, where $g_1 = hg$. Hence, $E_G(T) \in \mathcal{I}(\pi_1, \pi_2)$.

Note that if $T \in \mathcal{I}(\pi_1, \pi_2)$, then $E_G(T) = \frac{1}{|G|} \sum_{g \in G} \pi_2(g) T \pi_1(g^{-1}) = \sum_{g \in G} T \pi_1(g) \pi_1(g^{-1}) = T$. Thus, by (ii), the range of E_G is contained in $\mathcal{I}(\pi_1, \pi_2)$ and E_G leaves this space invariant. Hence, $E_G(E_G(T)) = E_G(T)$, and (iii) follows.

Result (iv) follows from (ii) and Schur's lemma, that $\mathcal{I}(\pi_1, \pi_2) = (0)$.

Finally, to see (v), note that $E_G(T) \in \mathcal{I}(\pi_1, \pi_1) = \pi_1(G)' = \{\lambda I_{V_1} : \lambda \in \mathbb{C}\}$. So let, $E_G(T) = \lambda I_{V_1}$ and compute, $n\lambda = Tr(E_G(T)) =$ $\frac{1}{|G|} \sum_{g \in G} Tr(\pi_1(g) T \pi_1(g^{-1})) = \frac{1}{|G|} \sum_{g \in G} Tr(T) = Tr(T)$, from which the result follows. $\square$

4.2. **The Schur Relations.** Let $\pi : G \to GL(n, \mathbb{F})$ and $\rho : G \to GL(m, \mathbb{F})$. Writing each matrix in terms of it's entires, $\pi(g) = (r_{i,j}(g)), \rho(g) = (s_{i,j}(g))$, we obtain functions, $r_{i,j} : G \to \mathbb{F}, s_{i,j} : G \to \mathbb{F}$. Now let $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m) = M_{m,n}(\mathbb{C})$ be an $m \times n$ matrix and let $E_G(T) = (t_{i,j}^G)$ denote its matrix entries.

We have that, $(t_{i,j}^G) = \frac{1}{|G|} \sum_{g \in G} \rho(g) T \pi(g^{-1})$, and so,

$$t_{i,j}^G = \frac{1}{|G|} \sum_{g \in G} \sum_{k,l} s_{i,k}(g) t_{k,l} r_{l,j}(g^{-1}).$$

Now suppose that $T = E_{k,l}$, the matrix that is 1 in the (k,l)-entry and 0 elsewhere, then we have that, $t_{i,j}^G = \sum_{g \in G} s_{i,k}(g) t_{l,j}(g^{-1})$.

If π, ρ are irreducible and inequivalent, then $E_G(T) = 0$, and so

(Schur 1) $$\frac{1}{|G|} \sum_{g \in G} s_{i,k}(g) r_{l,j}(g^{-1}) = 0 \text{ for all } i, j, k, l.$$

If on the other hand, $\pi = \rho$ is irreducible, then for $k \neq l, E_G(E_{k,l}) = 0$, while $E_G(E_{k,k}) = \frac{Tr(E_{k,k})}{n} I_n = 1/n I_n$. Thus, when $\pi = \rho$ is irreducible, then

(Schur 2) $$\frac{1}{|G|}\sum_{g\in G} r_{i,k}(g)r_{l,j}(g^{-1}) = \frac{\delta_{k,l}\delta_{i,j}}{n},$$

where $\delta_{i,j} = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$ denotes the *kronecker delta* function.

4.3. **Inner Products.** Given two vectors, $\sum \lambda_g e_g, \sum \mu_g e_g$ in $\mathbb{C}(G)$, the usual *inner product* of these vectors is given by

$$\langle \sum \lambda_g e_g, \sum \mu_g e_g \rangle = \sum \lambda_g \bar{\mu_g}.$$

If we regard these vectors as functions instead then this inner product becomes,

$$\langle f_1, f_2 \rangle = \sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

We wish to alter this definition slightly, thus we shall define,

$$(f_1|f_2) = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Note that, $|G|(f_1|f_2) = \langle f_1, f_2 \rangle$. We summarize the properties below.

Proposition 4.14. *Let $f_1, f_2, f_3 \in \mathbb{C}(G)$ and let $\lambda \in \mathbb{C}$. Then:*

(i) $(f_1|f_1) \geq 0$ *and is equal to 0 if and only if* $f_1 = 0$,
(ii) $(f_1 + f_2|f_3) = (f_1|f_3) + (f_2|f_3)$,
(iii) $(f_1|f_2 + f_3) = (f_1|f_2) + (f_1|f_3)$,
(iv) $(\lambda f_1|f_2) = \lambda(f_1|f_2)$,
(v) $(f_1|\lambda f_2) = \overline{\lambda}(f_1|f_2)$,
(vi) $(f_1|f_2) = \overline{(f_2|f_1)}$.

Remark 4.15. *If V is a complex vector space, then any map, $(\cdot,\cdot) : V \times V \to \mathbb{C}$, satisfying (ii)–(v) is called a* **sesquilinear form.** *A sesquilinear form is called* **symmetric** *if it also satisfies (vi). A sesquilinear form is called* **non-negative** *if $(f_1|f_1) \geq 0$ and* **positive** *if it satisfies (i). A positive sesquilinear form is also called an it inner product. It is easy to show that any inner product automatically satisfies (vi). In fact, assuming (i)–(iv), implies (v) and (vi).*

Given a complex vector space with an inner product, vectors v, w are called **orthogonal** *or* **perpendicular** *if $(v|w) = 0$.*

Also, if one sets $||v|| = \sqrt{(v|v)}$, then this defines a **norm** *on V, and one has the* **Cauchy-Schwarz inequality,** $|(v|w)| \leq ||v|| \cdot ||w||$.

The following result shows the importance of the inner product on $\mathbb{C}(G)$.

Theorem 4.16. *Let $\pi_i : G \to GL(V_i), i = 1, 2$ be irreducible representations on finite dimensional vector spaces and let $\chi_i, i = 1, 2$ be the corresponding characters. Then*

$$(\chi_1|\chi_2) = \begin{cases} 1 & \pi_1 \sim \pi_2 \\ 0 & \pi_1 \nsim \pi_2 \end{cases}.$$

Proof. Let $\pi_1(g) = (s_{i,j}(g))$ and $\pi_2(g) = (r_{i,j}(g))$. Recall that $\overline{\chi(g)} = \chi(g^{-1})$. If $\pi_1 \nsim \pi_2$, then $(\chi_1|\chi_2) = \frac{1}{|G|}\sum_{g\in G}(\sum_i s_{i,i}(g))(\sum_j \overline{r_{j,j}(g)}) = \frac{1}{|G|}\sum_{g\in G}(\sum_i s_{i,i}(g))(\sum_j r_{j,j}(g^{-1})) = 0$, by Schur 1.

If $\pi_1 \sim \pi_2$, then $\chi_1(g) = \chi_2(g)$, and $dim(V_1) = dim(V_2) = n$, where n denotes their common dimension. Thus,

$$(\chi_1|\chi_2) = \frac{1}{|G|}\sum_{g\in G}(\sum_i r_{i,i}(g))(\sum_j r_{j,j}(g^{-1})) = \sum_{i,j=1}^{n} \frac{\delta_{i,j}\delta_{i,j}}{n} = 1,$$

by Schur 2. □

Corollary 4.17. *Let G be a finite group and let $\rho : G \to GL(V)$ be a representation and assume that $\rho \sim \pi_1^{(n_1)} \oplus \cdots \oplus \pi_k^{(n_k)}$, where $\pi_1, \ldots, \pi_k$ are inequivalent irreducible representations, then $(\chi_\rho|\chi_{\pi_i}) = n_i$, for all $i = 1, \ldots, k$, and $(\chi_\rho|\chi_\rho) = n_1^2 + \cdots + n_k^2$.*

Proof. We have that $\chi_\rho = n_1\chi_{\pi_1} + \cdots + n_k\chi_{\pi_k}$, and the result follows from the above theorem. □

Corollary 4.18. *Let G be a finite group and let $\rho : G \to GL(V)$ be a finite dimensional representation. Then ρ is irreducible if and only if $(\chi_\rho|\chi_\rho) = \frac{1}{|G|}\sum_{g\in G}|Tr(\rho(g))|^2 = 1$.*

Corollary 4.19. *Let G be a finite group and let $\rho_i : G \to GL(V_i), i = 1, 2$ be finite dimensional representations of G. Then $\rho_1 \sim \rho_2$ if and only if $\chi_{\rho_1} = \chi_{\rho_2}$.*

Proof. If they are similar, then we have that their characters agree. So assume that their characters are equal, let $\pi_1, \ldots, \pi_k$ be a set of inequivalent irreducible representations such that every irreducible representation that occurs as a subrepresentation of ρ_1 or ρ_2 is included in the set. Then we have that, $\rho_1 \sim \pi_1^{(n_1)} \oplus \cdots \oplus \pi_k^{(n_k)}$ and $\rho_2 \sim \pi_1^{(m_1)} \oplus \cdots \oplus \pi_k^{(m_k)}$ where we set n_i(respectively, m_i) equal to 0 if π_i is not a subrepresentation of ρ_1(respectively, ρ_2). Since the characters are equal, $n_i = (\chi_{\rho_1}|\chi_{\pi_i}) = (\chi_{\rho_2}|\chi_{\pi_i}) = m_i$, and hence, $\rho_1 \sim \rho_2$. □

4.4. **The Left Regular Representation.** Recall that the left regular representation, $\lambda : G \to GL(\mathbb{C}(G))$ is the induced permutation representation given by left multiplication on G. Thus, $\lambda(g)e_h = e_{gh}$, or in terms of functions, $(\lambda(g)f)(h) = f(g^{-1}h)$.

We will see that studying the left regular representation yields a great deal of information about the irreducible representations of G.

Theorem 4.20. *Let G be a finite group and let $\pi : G \to GL(n, \mathbb{C})$ be an irreducible representation of G. Then π is a subrepresentation of λ with multiplicity n.*

Proof. Note that $\chi_\lambda(g) = \begin{cases} |G|, & g = e \\ 0 & g \neq e \end{cases}$. Thus, $(\chi_\lambda|\chi_\pi) = \frac{1}{|G|}\sum_{g\in G}\chi_\lambda(g)\overline{\chi_\pi(g)} = \overline{\chi_\pi(e)} = n$. □

Theorem 4.21 (Sum of Squares)**.** *Let G be a finite group. Then there exists a finite number of finite dimensional irreducible representations. If these are on spaces of dimensions, $n_1, \ldots, n_k$, then*

$$n_1^2 + \cdots + n_k^2 = |G|.$$

Proof. Since every irreducible representation of G is a subrepresentation of λ there can be a most a finite number of inequivalent irreducible representations. Moreover, since each of these is of multiplicity, n_i, we have that $|G| = (\chi_\lambda|\chi_\lambda) = n_1^2 + \cdots + n_k^2$. □

Definition 4.22. *Let G be a group. Then the set of similarity equivalence classes of irreducible representations is called the* **spectrum of G** *and is denoted by $\widehat{G}$. A set $\{\pi_\alpha : \alpha \in A\}$ of irreducible representations of G is called a* **complete set of irreducible representations***, if it contains a representative of each equivalence class in $\widehat{G}$. That is, if every irreducible representation of G is equivalent to one of the representations in the set and for $\alpha \neq \beta$, we have $\pi_\alpha \nsim \pi_\beta$. We call the corresponding set of characters, $\{\chi_{\pi_\alpha} : \alpha \in A\}$ a* **complete set of characters of G.**

Note that for every group, we have a trivial one-dimensional representation, $\pi(g) = 1$, and hence in the decomposition of $|G|$ as a sum of squares, one of the terms is always, 1^2.

Example 4.23. *Find a complete set of irreducible representations of $S_3 = D_3$. We have seen that this group has a 2-dimensional irreducible representation and we always have the trivial 1-dimensional. Since $|S_3| = 6$ and $1^2 + 2^2 = 5$, we see that the only possiblity is that there is an additional 1-dimensional irreducible representation of S_3. Recall that for every permutation, we have the notion of the* **sign** *of the permutation, which is always ± 1 and that the sign of a product is the product of the signs. This defines the other representtion, by $\pi(g) = sign(g)$.*

Thus, the set of these three irreducible representations is a complete set of irreducible representations of S_3, and so $|\widehat{S_3}| = 3$.

	χ_0	χ_1	χ_2
e	1	1	2
R	1	1	-1
R^2	1	1	-1
F	1	-1	0
RF	1	-1	0
R^2F	1	-1	0

TABLE 1. **The Character Table for D_3**

Example 4.24. *Find a complete set of irreducible representations of D_4. We know that we have the trivial representation and at least one 2-dimensional irreducible. Since $|D_4| = 8$, we see that the only possible decomposition into a sum of squares is, $8 = 2^2+1^2+1^2+1^2+1^2$. Thus, we see that we altogether, we need four 1-dimensional representations and that $|\widehat{D_4}| = 5$.*

Using the relations, $F^2 = e, R^4 = e$ and $FR = R^3F$, we see that if $\pi : D_4 \to \mathbb{T}$, then $\pi(F) = \pm 1$. Using the last relation, we see that $\pi(F)\pi(R) = \pi(R)^3\pi(F)$ and since the range is abelian, $\pi(R) = \pi(R)^3$. Cancelling, yields, $\pi(R)^2 = 1$, or $\pi(R) = \pm 1$.

Thus, we have two potential values for $\pi(F)$ and two potential values for $\pi(R)$. Taking all possible choices for F and R, yields four possible 1-dimensional representations, which is exactly the number that we need!!

Thus, each of these four possibilities MUST yield a well-defined 1-dimensional representation of D_4 and these four one-dimensional representation together with the known 2-dimensional irreducible representation constitutes a complete set of irreducible representations for D_4.

The values of the characters of a complete set of characters are often summarized in a table called the **character table** of the group.

We record the character tables for the groups D_3 and D_4 below. Since $\chi(g) = \chi(h^{-1}gh)$ these are generally, only given for each conjugacy class. However, since we have yet to work out the conjugacy classes, we will just give the character for each group element. Note that the character of e always tells the dimension of the representation. We use the canonical generators, R and F.

In the table for D_3, the first column is the trivial character, the second column is the sign of a permutation and the third column is the character of the 2-dimensional irreducible representation that arises from embedding the triangle in the plane.

In the table for D_4, the first four columns are the characters of the four 1-dimensional representations that arise from sending R and F to ± 1 and the fifth column is the character of the 2-dimensional irreducible representation that arises from embedding the square in the plane.

	χ_0	χ_1	χ_2	χ_3	χ_4
e	1	1	1	1	2
R	1	1	-1	-1	0
R^2	1	1	1	1	-2
R^3	1	1	-1	-1	0
F	1	-1	1	-1	0
RF	1	-1	-1	1	0
R^2F	1	-1	1	-1	0
R^3F	1	-1	-1	1	0

TABLE 2. **The Character Table for** D_4

4.5. **The Space of Class Functions.** We get further information on the dimensions by studying the space of class functions.

Definition 4.25. *A function $f \in \mathbb{C}(G)$ is called a* **class function** *if it is constant on conjugacy classes, i.e., if $f(h^{-1}gh) = f(g)$ for every, $g, h \in G$. The set of class functions is denoted by $\mathbb{H}(G)$.*

Note that the set of class functions is a subspace of $\mathbb{C}(G)$ and that characters and linear combinations of characters are class functions.

Proposition 4.26. *The dimension of $\mathbb{H}(G)$ is equal to the number of conjugacy equivalence classes in G.*

Proof. Let $\{C_i : i \in I\}$ denote the conjugancy equivalence classes and define $f_i \in \mathbb{C}(G)$ by $f_i(g) = \begin{cases} 1 & \text{if } g \in C_i \\ 0 & \text{if } g \notin C_i \end{cases}$. Then clearly, the functions $\{f_i : i \in I\}$ are a basis for $\mathbb{H}(G)$. □

Proposition 4.27. *Let $\pi : G \to GL(V)$ be a representation of G and let $\tilde{\pi} : \mathbb{C}(G) \to \mathcal{L}(V)$, be the extension of π to the group algebra. If $f \in \mathbb{H}(G)$, then $\tilde{\pi}(f) \in \pi(G)'$. If π is also irreducible and $f \in \mathbb{H}(G)$, then $\tilde{\pi}(f) = \frac{(\chi_\pi|\bar{f})|G|}{n} I_V$, where $\bar{f}(g) = \overline{f(g)}$, and $n = dim(V)$.*

Proof. Note that $\pi(g^{-1})\tilde{\pi}(f)\pi(g) = \sum_{h\in G} \pi(g^{-1})f(h)\pi(h)\pi(g) = \sum_{h\in G} f(g^{-1}hg)\pi(g^{-1}hg) = \sum_{k\in G} f(k)\pi(k) = \tilde{\pi}(f)$, where $k = g^{-1}hg$ is a re-indexing of G. This calculation shows that $\tilde{\pi}(f)$ commutes with $\pi(g)$ and hence, $\tilde{\pi}(f) \in \pi(G)'$.

Now if π is irreducible, then $\pi(G)' = \{\alpha I_V : \alpha \in \mathbb{C}\}$ and hence, $\tilde{\pi}(f) = \alpha I_V$, for some scalar α. To compute α, we note that $n\alpha = Tr(\alpha I_V) = Tr(\tilde{\pi}(f)) = \sum_{h\in G} f(h)\chi_\pi(h) = |G|(\chi_\pi|\bar{f})$, and the result follows. □

Corollary 4.28. *Let π be irreducible and let $f \in \mathbb{H}(G)$. If $(\bar{f}|\chi_\pi) = 0$, then $\tilde{\pi}(f) = 0$.*

Theorem 4.29. *Let G be a finite group and let $\{\pi_i : i = 1, \ldots, m\}$ be a complete set of irreducible representations of G. Then $dim(\mathbb{H}(G)) = m$ and* *the corresponding complete set of characters,* $\{\chi_{\pi_i} : i = 1, \ldots, m\}$ *is an orthonormal basis for* $\mathbb{H}(G)$*.*

Proof. Note that the characters of these irreducible representations are orthogonal by Theorem 4.16. If the complete set of characters is not a basis for $\mathbb{H}(G)$, then there exists $f_1 \in \mathbb{H}(G)$, with $(f_1|\chi_{\pi_i}) = 0, i = 1, \ldots, m$. Set $f = \bar{f}_1$, then $f \in \mathbb{H}(G)$ and $(\chi_{\pi_i}|\bar{f}) = 0$, for all i and so, $\tilde{\pi_i}(f) = 0$ for all i. But since these are a complete set of irreducible representations of G, every representation decomposes as a direct sum of these representations and hence, $\tilde{\pi}(f) = 0$ for every finite dimensional representation of G.

Thus, in particular, $\tilde{\lambda}(f) = 0$, for the left regular representation. Let $e_e \in \mathbb{C}(G)$ be the basis vector corresponding to the identity of G. Then, $0 = \tilde{\lambda}(f)e_e = \sum_{g \in G} f(g)\lambda(g)e_e = \sum_{g \in G} f(g)e_g = f$. Thus, any function that is perpendicular to the complete set of character functions is 0, and hence they span $\mathbb{H}(G)$. $\square$

Corollary 4.30. *Let G be a finite group, then the cardinality of* $|\widehat{G}|$*, i.e., the cardinality of any complete set of irreducible representations of G is equal to the number of conjugacy equivalence classes in G.*

Proof. Recall that $dim(\mathbb{H}(G))$ is equal to the number of conjugacy equivalence classes. $\square$

Thus, we have a direct group thoeretic means, without referring to representations at all, to compute the cardinality of a complete set of irreducible representations. Or conversely, if we know the cardinality of such a set, then that gives us a means to use the representation theory to determine the number of conjugacy classes.

Thus, for example, we see that S_3 has 3 conjugacy classes and D_4 has 4 conjugacy classes.

Problem 4.31. *Compute the number of conjugacy classes in* D_5*. Use this result together with the sum of squares theorem to determine the number and dimensions of a complete set of irreducible representations.*

Problem 4.32. *Compute the number of conjugacy classes in* D_n*.*

Dimensions of Irreducible Representations

The above results can be used to give some sharper estimates on the dimensions of irreducible representations.

Proposition 4.33. *Let G be a finite abelian group. Then every irreducible representation is 1-dimensional and there are* $|G|$ *inequivalent irreducible representations.*

Proof. Since G is abelian, the conjugacy equivalence classes are singletons. Thus, the number of equivalency classes is $|G|$ and this is also the number of inequivalent irreducible representations. By the sum of squares result, every irreducible representation must be 1-dimensional. $\square$

Theorem 4.34. *Let G be a finite group and assume that, $A \subseteq G$ is an abelian subgroup. If $\pi : G \to GL(V)$ is an irreducible representation, then $dim(V) \le \frac{|G|}{|A|}$.*

Proof. Look at the restriction of π to $A, \pi_A : A \to GL(V)$. Since A is abelian, every irreducible representation of A is 1-dimensional. Hence, V will decompose into an internal direct sum of 1-dimensional subspaces that are all $\pi(A)$-invariant. Let $v_1 \in V$ be any non-zero vector that is invariant under $\pi(A)$. Let $\rho : A \to \mathbb{T}$ be a homomorphism, such that $\pi(a)v_1 = \rho(a)v_1$.

Let $m = \frac{|G|}{|A|}$. Then there are m left cosets in G/A. Choose, $g_1, \ldots, g_m \in G$, representatives from each coset and then G will be the disjoint union of the sets, g_iA. Let $W = span\{\pi(g_1)v_1, \ldots, \pi(g_m)v_1\}$. Given $g \in G$ and any i, there will exist j and $a \in A$ such that $gg_i = g_ja$. Hence, $\pi(g)[\pi(g_i)v_1] = \pi(g_j)\pi(a)v_1 = \pi(g_j)\rho_1(a)v_1 \in W$.

Therefore, $\pi(G)W \subseteq W$, but since V is irreducible, $W = V$. Hnece, $dim(V) = dim(W) \le m$. ☐

Example 4.35. *Since $|D_n| = 2n$ and $A = \{e, R, \ldots, R^{n-1}\}$ is an abelian subgroup, every irreducible of D_n is at most 2-dimensional.*

Recall that the bf center of G, Z(G) is the set of all elements in G that commute with every element of G. In particular, Z(G) is an abelian subgroup and so the dimension of every irreducible representation is at most, $\frac{|G|}{|Z(G)|}$, by the above result. The next result improves on this bound.

Theorem 4.36. *Let G be a finite group, $Z(G)$ the center of G and let $\pi : G \to GL(V)$ be an irreducible representation with $n = dim(V)$, then $n^2 \le \frac{|G|}{|Z(G)|}$.*

Proof. Let $h \in Z(G)$, then $\pi(h) \in \pi(G)'$. But since π is irreducible, this implies that $\pi(h) = \lambda I_V$, for some scalar, $\lambda \in \mathbb{T}$. Hence, $|\chi_\pi(h)| = n$, for every, $h \in Z(G)$.

Thus, $n^2|Z(G)| = \sum_{h \in Z(G)} |\chi_\pi(h)|^2 \le \sum_{g \in G} |\chi_\pi(g)|^2 = |G|(\chi_\pi|\chi_\pi) = |G|$, since π is irreducible. This inequality yields the result. ☐

Problem 4.37. *Let G be a finite group and let H be a subgroup. Let k be the maximum dimension of an irreducible representation of H. Prove that if $\pi : G \to GL(V)$ is an irreducible representation of G, then $dim(V) \le \frac{k|G|}{|H|}$.*

4.6. **The Isotypic Decomposition.** The group algebra can also be used to define a canonical decomposition of every representation in sums of irreducible representations.

Theorem 4.38. *Let G be a finite group, let $\{\pi_1, \ldots, \pi_m\}$ be a complete set of irreducible representations of G on spaces of dimensions, $\{n_1, \ldots, n_m\}$, with $\{\chi_1, \ldots, \chi_m\}$ the corresponding characters and let $\rho : G \to GL(V)$ be* *any representation with* $\tilde{\rho} : \mathbb{C}(G) \to \mathcal{L}(V)$ *the extension to the group algebra. If we set,* $P_i = \frac{n_i}{|G|}\tilde{\rho}(\bar{\chi}_i)$*, then*

(i) *each* $P_i \in \rho(G)'$ *and is a projection,*
(ii) $P_iP_j = 0$ *for* $i \neq j$*,*
(iii) $P_1 + \cdots + P_m = I_V$*,*
(iv) $V_i = P_i(V)$ *is* $\rho(G)$*-invariant,*
(v) *the subrepresentation,* $\rho_i = \rho_{|V_i}$ *is equivalent to* $\pi_i^{(k_i)}$ *where* $k_i = (\rho|\chi_{\pi_i})$*.*

Definition 4.39. *The decomposition of V into subrepresentations given by the above theorem is called the* **isotypic decomposition.**

Proof. We know that there exists an invertible S, so that $\rho(g) = S^{-1}\pi(g)S$, where $\pi = \pi_1^{(k_1)} \oplus \cdots \oplus \pi_m^{(k_m)}$, for some integers, k_i.

Note that $P_i = \frac{n_i}{|G|}S^{-1}\tilde{\pi}(\bar{\chi}_i)S$. By Proposition 4.27, we have that $\tilde{\pi}_j(\bar{\chi}_i) = \frac{|G|}{n_j}(\chi_j|\chi_i)I_{n_j} = 0$, when $i \neq j$ and $\tilde{\pi}_i(\bar{\chi}_i) = \frac{|G|}{n_i}I_{n_i}$.

Hence, $\frac{n_i}{|G|}\tilde{\pi}(\bar{\chi}_i) = 0 \oplus \cdots \oplus 0 \oplus I \oplus 0 \oplus \cdots \oplus 0$, which is the projection onto the subspace where $\pi_i^{(k_i)}$ acts. Since P_i is similar to this projection, the result follows. □

The following result explains the importance of this decomposition.

Corollary 4.40. *Let* ρ *and* V *be as above and let* $V = W_1 + \cdots + W_p$ *be any internal direct sum decomposition into irreducible subrepresntations. Let* U_i *be the sum of all of the* W_l*'s for which the subrepresentation is equivalent to* π_i*, then* $U_i = V_i$*.*

4.7. **Tensor Products of Representations and the Clebsch-Gordan Integers.** We assume that the reader has some familiarity with the concept of the tensor product of vector spaces and only review some of the key facts. Given two vector spaces, V and W, we let $V \otimes W$ denote their tensor product. Recall that if $\{v_1, \dots, v_n\}$ is a basis for V and $\{w_1, \dots, w_m\}$ is a basis for W, then $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes W$. In particular, $dim(V \otimes W) = dim(V) \cdot dim(W)$. If $A : V \to V$ and $B : W \to W$ are linear maps, then there is a unique, well-defined linear map, $A \otimes B : V \otimes W \to V \otimes W$ satisfying $(A \otimes B)(v \otimes w) = (Av) \otimes (Bw)$.

Note that each of the sets, $Z_i = \{v_i \otimes w : w \in W\}$ is a subspace of $V \otimes W$ of dimension m with basis $\{v_i \otimes w_j : 1 \leq j \leq m\}$ and that $V \otimes W = \sum_{i=1}^n Z_i$, is an internal direct sum decomposition. With respect to these bases and direct sum decomposition, if $A = (a_{i,j})$ and $B = (b_{k,l})$, then the matrix for $A \otimes B$ is given in block form by $(a_{i,j}B)$. If instead we let $X_j = \{v \otimes w_j : v \in V\}$, then this is a subspace of dimension n and $V \otimes W = \sum_{j=1}^m X_j$ is an internal direct sum decomposition and the block matrix of $A \otimes B$ for this decomposition is given by $(b_{k,l}A)$.

Alternatively, if one recalls that to write down a matrix for a linear transformation, one must choose, not just a basis, but an *ordered basis,* then one sees that the above block matrices are obtained from the ordered bases, $\{v_1 \otimes w_1, \ldots, v_1 \otimes w_m, v_2 \otimes w_1, \ldots, v_2 \otimes w_m, v_3 \otimes w_1, \ldots\ldots, v_n \otimes w_m\}$ and $\{v_1 \otimes w_1, \ldots, v_n \otimes w_1, v_1 \otimes w_2, \ldots, v_n \otimes w_2, v_1 \otimes w_3, \ldots\ldots, v_n \otimes w_m\}$, respectively.

From either of these representations, one sees that $Tr(A \otimes B) = Tr(A) \cdot Tr(B)$.

Proposition 4.41. *Let V and W be finite dimensional vector spaces, let G be a group, and let $\pi : G \to GL(V)$ and $\rho : G \to GL(W)$ be representations. Then setting $(\pi \otimes \rho)(g) = \pi(g) \otimes \rho(g)$ defines a representation $\pi \otimes \rho : G \to GL(V \otimes W)$, with character $\chi_{\pi\otimes\rho}(g) = \chi_\pi(g) \cdot \chi_\rho(g)$.*

Proof. First note that if $A \in GL(V)$ and $B \in GL(W)$, then $(A^{-1}) \otimes (B^{-1}) = (A \otimes B)^{-1}$, and so $A \otimes B \in GL(V \otimes W)$. This shows that $(\pi \otimes \rho)(g) \in GL(V \otimes W)$. Also, it is readily checked that $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ and hence, $(\pi(g) \otimes \rho(g))(\pi(h) \otimes \rho(h)) = \pi(gh) \otimes \rho(gh)$, so that the map $\pi \otimes \rho$ is a group homomorphism. Finally, the formula for the characters follows from the statement about traces. $\square$

Now let G be a finite group and let $\{\pi_1, \ldots, \pi_k\}$ be a complete set of irreducible representations of G. The representation $\pi_i \otimes \pi_j$, need not be irreducible but will always decompose, up to similarity, as a direct sum of the irreducible representations where the irreducible representation π_l will occur with some multiplicity, say $n_{i,j}^{(l)}$. This triply indexed set of integers are called the **Clebsch-Gordan integers.** Indeed, by our earlier results we have that,

$n_{i,j}^{(l)} = (\chi_{\pi_i \otimes \pi_j} | \chi_{\pi_l}) = (\chi_{\pi_i} \chi_{\pi_j} | \chi_{\pi_l})$.

4.8. **The Irreducible Representations of S_n.** In this section, we outline the theory of *Young tableaux* and their relation to the irreducible representations of S_n.

Given $g \in S_n$, it will be more convenient to use it's decomposition into *cycles.* Recall that if, say a permutation $g \in S_5$ is given by $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, then it would be represented in cycle notation as $g = (1, 2, 3)(4, 5)$.

We will need the following result from group theory, which is fairly easy to prove.

Proposition 4.42. *Let $g, h \in S_n$, then g and h are conjugate if and only if they have the same number of cycles of the same lengths.*

We do not prove the theorem, but only illustrate. Say, $g \in S_5$ is given by $g = (1, 2, 3)(4, 5)$ and $h \in S_5$ is arbitrary. Then one can see that $hgh^{-1} = (h(1), h(2), h(3))(h(4), h(5))$. Thus, every element conjugate to g has the same cycle structure. Conversely, if $k = (a_1, a_2, a_3)(a_4, a_5)$ has the same cycle structure, then setting $h(i) = a_i$, yields $hgh^{-1} = k$.

Thus, the number of conjugacy classes in S_n is determined by the number of possible cycle structures. If we list the cycles in a permutation from longest to shortest, then we see that the number of possible cycle structures and hence the number of conjugacy equivalence classes is given by the number of integer solutions to, $n = \alpha_1 + \ldots + \alpha_m$, with $\alpha_1 \geq \ldots \geq \alpha_m$. Thus, for $n = 3$ we have 3 solutions given by $3 = 3 = 2 + 1 = 1 + 1 + 1$, and hence 3 conjugacy classes in S_3. While for $n = 4$, we have 5 solutions, $4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1+1$, and hence 5 conjugacy equivalence classes in S_4.

Each such solution is called a **Young frame for S_n.** Thus, formally, a Young frame for S_n is just a set of numbers $\alpha_1 \geq \cdots \geq \alpha_m$, with $n = \alpha_1 + \cdots + \alpha_m$ and the number of Young frames for S_n is equal to the number of conjugacy equivalence classes of S_n, and hence, by our earlier results, is also equal to the number of irreducible representations of S_n.

Problem 4.43. *Compute the number of conjugacy equivalence classes in S_5.*

One way to picture all Young frames is by **Young schemes.** We illustrate the Young schemes for $n = 3, 4$, below.

INSERT PICTURE

Given a Young scheme, one forms a **Young tableaux**, by entering the numbers, 1–n, into each of the boxes in any order. Thus, to every Young scheme, there are $n!$ Young tableaux. A Young tableaux is called **standard** if the numbers in each row increase when we go from left to right and if the numbers in each column increase as we go from top to bottom.

For the Young scheme for $n = 5$ corresponding to $5 = 3 + 2$, we illustrate two different Young tableaux, $T_1 = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}, T_2 = \begin{matrix} 2 & 3 & 4 \\ 1 & 5 & \end{matrix}$. Note that T_1 is standard, while T_2 is not.

Given a tableau, T, we let R(T) denote the set of all possible permutations that only permute numbers appearing in the rows of the tableau. So, for T_1 a permutation that sent 1 to 3 would NOT belong to $R(T_1)$. Similarly, C(T) is defined to be the set of all permutations that only permute numbers belonging to the columns. So for example for T_1, every permutation in $C(T_1)$ would have to fix 4.

We list these permutations for T_1 in cycle notation.

$$R(T_1) = \{(1,2,4), (1,4,2), (1,2), (1,4), (2,4), (3,5), (1,2,4)(3,5), \\ (1,4,2)(3,5), (1,2)(3,5), (1,4)(3,5), (2,4)(3,5)\}$$

$$C(T_1) = \{(1,3), (2,5), (1,3)(2,5)\}$$

Given a tableau T, we define two elements of the group algebra, $\mathbb{C}(S_n)$,

$$P_T = \sum_{p \in R(T)} e_p,$$

and

$$Q_T = \sum_{q \in C(T)} (-1)^{sqn(q)} e_q.$$

Theorem 4.44 (Young-vonNeumann)**.** *Let $\lambda : S_n \to GL(\mathbb{C}(S_n))$ be the left regular representation and let T be a Young tableau. Then:*

(i) *$E_T = \tilde{\lambda}(P_T \cdot Q_T) \in \mathcal{L}(\mathbb{C}(S_n))$ is a projection,*

(ii) *$V_T = range(E_T)$ is $\lambda(S_n)$-invariant,*

(ii) *the restriction of λ to V_T is equivalent to $\pi^{(k)}$ for some irreducible representation π,*

(iii) *if T_1, T_2 are different tableau for the same Young frame, then the irreducible representations obtained as in (ii) are equivalent and their multiplicities are the same,*

(iv) *if T_1, T_2 are tableau for different Young frames, then the irreducible representations obtained as in (ii) are inequivalent.*

Thus, by choosing one tableau for each Young frame and proceeding as above one obtains a complete set of irreducible representations of S_n. The next theorem tells us how to compute the dimension of the irreducible representation corresponding to a particular young frame.

Theorem 4.45. *Let $\mathcal{F} = \{\alpha_1 \geq \ldots \geq \alpha_m\}$ be a Young frame for S_n. Then the dimension of the irreducible representation corresponding to $\mathcal{F}$ given by the Young-vonNeumann theorem is equal to the number of standard Young tableaux for $\mathcal{F}$, and is equal to*

$$n! \frac{\prod_{i<j}(\alpha_i - i - \alpha_j + j)}{\prod_i (\alpha_i - i + m)!}$$

5. Topological Groups

In this Chapter we cover the basic facts about topological groups. We assume that the reader is familar with all of the basic facts and definitions from topology.

Definition 5.1. *A* **topological group** *is a group G together with a topology on G that satisfies the following two properties:*

(i) *the map $p : G \times G \to G$ defined by $p(g, h) = gh$ is continuous when $G \times G$ is endowed with the product topology,*

(ii) *the map $inv : G \to G$ defined by $inv(g) = g^{-1}$ is continuous.*

We remark that to (i) is equivalent to the statement that, whenever $U \subseteq G$ is open, and $g_1 g_2 \in U$, then there exist open sets V_1, V_2 such that $g_1 \in V_1, g_2 \in V_2$ and $V_1 V_2 = \{h_1 h_2 : h_1 \in V_1, h_2 \in V_2\} \subseteq U$. Also (ii) is equivalent to showing that whenever $U \subseteq G$ is open, then $U^{-1} = \{g^{-1} : g \in U\}$ is open.

Example 5.2. *Following are some examples that are easily checked to be topological groups.*

1 *Let G be any group and endow G with the discrete topology.*
2 *Let G be any group and endow G with the indiscrete topology.*
3 $(\mathbb{R}, +)$ *with the usual topology is a topological group.*
4 $(\mathbb{R}^*, \cdot)$ *with the usual topology is a topological group.*
5 *Similarly,* $(\mathbb{C}, +)$ *and* $(\mathbb{C}^*, \cdot)$ *are topological groups.*
6 *Every subgroup of a topological group, endowed with the subspace topology, is a topological group.*

Here is an example that is not a topological group. Let $G = (\mathbb{Z}_2, +)$, and endow it with the topology where the only open sets are the empty set, $\{0\}$ and the whole group. Then $p^{-1}(\{0\}) = \{(0,0),(1,1)\}$ which is not open in the product topology.

We now look at some examples that are harder to show are topological groups. For these the following will be useful.

Proposition 5.3. *Let G be a group and assume that the topology on G comes from a metric, d. Then G is a topological group if and only if the following hold:*

(i) *for every $\epsilon > 0$, and $g_1, g_2 \in G$, there exists $\delta > 0$ such that if $d(g_1, h_1) < \delta$ and $d(g_2, h_2) < \delta$, then $d(g_1 g_2, h_1 h_2) < \epsilon$,*
(ii) *for every ϵ and $g \in G$ there exists δ so that whenever $d(g, h) < \delta$ then $d(g^{-1}, h^{-1}) < \epsilon$.*

Let $GL(n, \mathbb{R})$ be endowed with the Euclidean metric that it inherits by identifying every matrix with a vector in $\mathbb{R}^{n^2}$. Similarly, we endow $GL(n, \mathbb{C})$ by identifying every matrix with a vector in $\mathbb{R}^{2n^2}$.

Proposition 5.4. *$GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups.*

Proof. We only do the real case. To see that the product is continuous, fix matrices $A = (a_{i,j}), B = (b_{i,j})$ and $\epsilon > 0$. Using the continuity of the product on $\mathbb{R}$, we may choose for every $1 \le i, k, j \le n$ a number $\delta_{i,k,j}$ such that if $|a_{i,k} - c_{i,k}| < \delta_{i,k,j}$ and $|b_{k,j} - d_{k,j}| < \delta_{i,k,j}$ then $|a_{i,k} b_{k,j} - c_{i,k} d_{k,j}| < \epsilon/n^3$. Then we have that $|(AB - CD)_{(i,j)}| = |\sum_{k=1}^n a_{i,k} b_{k,j} - c_{i,k} d_{k,j}| < \epsilon/n^2$. Since each of the n^2 entries of the products are this close, we have that $d(AB, CD) < \epsilon$ and so the product is continuous.

To see the continuity of the inverse, first check that the map, $det : M_n \to \mathbb{R}$ is continuous and then use Cramer's formula for the inverse. □

Proposition 5.5. *Let G be a topological group, fix $g \in G$. Then the maps $L_g : G \to G$ and $R_g : G \to G$ defined by $L_g(h) = gh$ and $R_g(h) = hg$ are homeomorphisms. Consequently, $V \subseteq G$ is open(closed) if and only if gV is open(closed) if and only if Vg is open(closed).*

Proof. The map $\gamma_g : G \to G \times G$ defined by $\gamma_g(h) = (g, h)$ is clearly continuous. Hence, $L_g = p \circ \gamma_g$ is continuous. To see that L_g is a homeomorphism, note that $(L_g)^{-1} = L_{g^{-1}}$ is continuous.

The results for right multiplication follow similarly.

Finally, the last equivalences follow since, $L_h^{-1}(V) = h^{-1}V$ with $h = g^{-1}$ and similarly for right multiplication. □

Proposition 5.6. *Let G be a topological group, with $V \subseteq G$. Then V is open(closed) if and only if V^{-1} is open(closed).*

Proof. These statements follow by noticing that since $inv \circ inv$ is the identity, inv is a homeomorphism. □

Proposition 5.7. *Let G be a topological group and let $U \subseteq G$ be an open set with $e \in U$. Then there exists an open set V, with $e \in V$ such that $V = V^{-1}$ and $V \cdot V \subseteq U$.*

Proof. Since p is continuous, $p^{-1}(U)$ is open in $G \times G$ and $(e, e) \in p^{-1}(U)$. Hence, there exist open sets V_1, V_2 with $e \in V_1, e \in V_2$ such that $V_1 \cdot V_2 \subseteq U$.

By the above results, V_1^{-1}, V_2^{-1} are also open, hence, $V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1}$ is also open, $e \in V$, $V = V^{-1}$ and $V \cdot V \subseteq V_1 \cdot V_2 \subseteq U$. □

Recall that if a topological space satisfies the first separation axiom, then every singleton is a closed set, but this latter property is generally weaker and both of these properties are much weaker than being Hausdorff. This makes the following result somewhat surprising.

Proposition 5.8. *Let G be a topological group. Then G is Hausdorff if and only if $\{e\}$ is closed.*

Proof. If G is Hausdorff, then every singleton is closed. Conversely, assume that $\{e\}$ is closed, then $\{g\} = L_g(\{e\})$ is closed for every g.

We now show that $e \neq g$ can be separated by disjoint open sets. Since $\{g\}$ is closed there is an open set U with $e \in U$ and $g \notin U$. By an earlier result there exists $V = V^{-1}$ open with $e \in V$ and $V \cdot V \subseteq U$. Now $g \in gV$ and we claim that $V \cap gV$ is empty. Suppose that h is in the intersection, then $h = gh_1, h_1 \in V$. Hence, $g = hh_1^{-1} \in V \cdot V \subseteq U$, contradiction. Thus, we've shown that e and g can be separated by disjoint open sets.

Now let $g_1 \neq g_2$ be any points in G. Then there exists U_1, U_2 disjoint, open with $e \in U_1$ and $g_1^{-1}g_2 \in U_2$. But then $g_1 \in g_1U_1$ and $g_2 \in g_1U_2$ and these are disjoint, open sets. □

Quotient Spaces and Quotient Groups

Let X be a topological space and let $\sim$ be an equivalence relation on X. There is a natural topology on the space of equivalence classes, $X/\sim$. To define this topology, let $q : X \to X/\sim$ be the quotient map and then we declare a set $U \subseteq X/\sim$ to be open if and only if $q^{-1}(U)$ is open in X.

An equivalence relation is called **closed,** if all of its equivalence classes are closed sets. It is fairly easy to see that if X is Hausdorff, but the equivalence relation is not closed, then $X/\sim$ will not be Hausdorff.

Finally, we recall that a map between topological spaces, $f : X \to Y$ is called **open** provided, for every open set U in X, $f(U)$ is open in Y.

Note that the above map q is open if and only if for every U open in X, $q^{-1}(q(U))$ is open in X.

Let G be a topological group and let H be a subgroup, and recall left coset equivalence. We write $G/H = G/\sim$ for the coset space.

Proposition 5.9. *Let G be a topological group and let H be a subgroup. Then the quotient map $q : G \to G/H$ is open.*

Proof. Let U be open G. Check that $q^{-1}(q(U)) = U \cdot H = \bigcup_{h \in H} Uh$, which expresses the set as a union of open sets and hence it is open. □

Proposition 5.10. *Let G be a Hausdorff, topological group and let H be a closed subgroup, then G/H is Hausdorff.*

Proof. Let $g_1H \neq g_2H$ be two points in G/H. Then $g_1^{-1}g_2 \notin H$ and hence there is an open set U with $g_1^{-1}g_2 \in U$ and $U \cap H$ empty.

Hence, $e \in g_1Ug_2^{-1}$ and we may choose an open set $V = V^{-1}$ with $e \in V$ and $V \cdot V \subseteq g_1Ug_2^{-1}$. Thus, $g_1^{-1}V \cdot Vg_2 \subseteq U$. Let $V_1 = g_1^{-1}V$ and $V_2 = Vg_2$. These sets are both open, $g_1 \in V_1^{-1}$ and $g_2 \in V_2$.

Hence, $g_1H \in q(V_1^{-1})$ and $g_2H \in q(V_2)$ and because q is an open map both of these sets are open in G/H.

It remains to show that they are disjoint, but this follows since, $q^{-1}(q(V_1^{-1})) \cap q^{-1}(q(V_2)) = V_1^{-1} \cdot H \cap V_2 \cdot H$ and any point in the intersection would satisfy $v_1^{-1}h_1 = v_2h_2$ which implies that $h_1h_2^{-1} = v_1v_2$. But the left-hand side of this equation is in H and the right-hand side is in U, which is a contradiction. □

Theorem 5.11. *Let G be a (Hausdorff) topological group, H a (closed) normal subgroup, then G/H is a(Hausdorff) topological group.*

Proof. Let U be open in G/H and let $O = q^{-1}(U)$. Suppose that, $g_1H, g_2H \in G/H$ with $g_1Hg_2H = g_1g_2H \in U$, then $g_1g_2 \in O$. Hence, there exist, V_1, V_2 open in G, such that $g_1 \in V_1, g_2 \in V_2$ and $V_1 \cdot V_2 \subseteq O$. Then $g_1H \in q(V_1), g_2 \in q(V_2)$ are open in G/H and $q(V_1)q(V_2) = \{v_1h_1v_2h_2 : v_i \in V_i, h_i \in H\} = \{v_1v_2h : v_i \in V_i, h \in H\} = q(V_1V_2) \subseteq q(O) = U$. Hence, the product is continuous in G/H.

The proof that the inverse is continuous is similar. □

Recall that $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group, $\mathbb{T}$. By the above results, both groups are topological groups, so it is natural to ask if the natural isomorphism is also a homeomorphism. The fact that it is will follow from the following results.

Homomorphisms and Isomorphisms of Topological Groups

Let G, K be two topological groups, we want to study **continuous homomorphisms.** We say that G and K are **topologically isomorphic** if there exists a map, $\pi : G \to K$ that is both a group isomorphism and a topological homeomorphism. Such a map is called a **topological isomorphism.**

Example 5.12. *Let $G = K = (\mathbb{R}, +)$ as groups, but let G have the discrete topology, and K the usual topology. Then the identity map is a continuous, algebraic isomorphism, but its inverse is not continuous, so that this map is not a topological isomorphism.*

Example 5.13. *Let G be any topological group and let $g \in G$, then the map $\pi(h) = ghg^{-1}$ is a topological isomorphism.*

Note that when K is Hausdorff, if $\pi : G \to K$ is any continuous homomorphism, then $ker(\pi) = \pi^{-1}(\{e\})$ is a closed, normal subgroup of K.

Proposition 5.14. *Let $\pi : G \to K$ be a homomorphism. If π is continuous at e, then π is continuous.*

Proof. Since π is continuous at e, given any open subset $U \subseteq K$ with $e_K \in U$, then $\pi^{-1}(U)$ is open in G. We show that π is continuous at an arbitrary $g \in G$.

Let $U \subseteq K$ be open with $\pi(g) \in K$. Then $\pi(e_G) = e_K \in \pi(g)^{-1}U$ which is open in K. Hence, by continuity at e, there exists $V \subseteq G$, open with $\pi(V) \subseteq \pi(g)^{-1}U$. But then gV is an open neighborhood of g, with $\pi(gV) = \pi(g)\pi(V) \subseteq \pi(g)\pi(g)^{-1}U = U$. Hence, π is continuous at g and since g was arbitrary, π is continuous on G. □

Proposition 5.15. *Let $\pi : G \to K$ be a continuous, homomorphism, with $H = ker(\pi)$. Then, $\tilde{\pi} : G/H \to K$ is a continuous, homomorphism.*

Proposition 5.16. *Let $\pi : G \to K$ be a continuous, onto homomorphism, with $H = ker(\pi)$. If π is an open map, then $\tilde{\pi} : G/H \to K$ is a topological isomorphism.*

Proof. All that remains to be shown is that $\tilde{\pi}^{-1}$ is continuous. But this will follow if we can show that $\tilde{\pi}$ is an open map. Note that U is open in G/H if and only if $V = q^{-1}(U)$ is open in G. Hence, if U is open in G/H, then $\tilde{\pi}(U) = \pi(V)$ is open in K. □

Example 5.17. *Consider $\pi : (\mathbb{R}, +) \to \mathbb{T}$ defined by $\pi(t) = e^{2\pi i t}$. This map is a continuous, homomorphism, onto with kernel $\mathbb{Z}$. It is easily checked that π carries open intervals to open arcs and hence is an open map. Thus, $\tilde{\pi} : \mathbb{R}/\mathbb{Z} \to \mathbb{T}$ is a topological isomorphism.*

Example 5.18. *Give the group, $G = \{\phi_{a,b} : a \neq 0, b \in \underset{\sim}{R}\}$ a topology by identifying it with a subset of $\mathbb{R} \times \mathbb{R}$. The homomorphism, $\pi : G \to GL(2, \mathbb{R})$ defined by $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is clearly a homeomorphism onto its range. But its range is a subgroup of the topological group $GL(2, \mathbb{R})$ and hence a topological group. Hence, G is a topological group with its given topology.*

Problem 5.19. *Let G_1, G_2 be topological groups and let $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ denote the product of the two groups with product,*

$(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$. *Prove that if* $G_1 \times G_2$ *is given the product topology, then it is a topological group.*

Problem 5.20. *Let* G *be a Hausdorff topological group,* $A, B \subseteq G$ *subsets. Prove that:*

(i) *A compact and B closed, implies that $A \cdot B$ is closed,*
(ii) *A,B compact, implies that $A \cdot B$ is compact,*
(iii) *A open, implies that $A \cdot B$ is open,*
(iv) *give an example to show that A closed and B closed does not guarantee that $A \cdot B$ is closed.*

Problem 5.21. *Let* G *be a Hausdorff topological group and let* $A, B \subseteq G$ *be connected subsets. Prove:*

(i) *$A \cdot B$ is connected,*
(ii) *A^{-1} is connected.*

Problem 5.22. *Let* G *be a Hausdorff topological group and let* C *denote the connected component of* e. *Prove:*

(i) *$g \cdot C$ is the connected component of g,*
(ii) *C is a normal subgroup of G,*
(iii) *C is obth open and closed in G.*

6. Representations of Topological Groups

Let V be a finite dimensional vector space over $\mathbb{R}$ with $dim(V) = n$. If we fix a basis, $v_1, \ldots, v_n$ for V then we have a group isomorphism, $\pi : GL(V) \to GL(n, \mathbb{R})$ by letting $\pi(T)$ be the matrix for $T \in GL(V)$ with respect to the ordered basis. Endowing $GL(V)$ with the topology of $GL(n, \mathbb{R})$ makes $GL(V)$ into a topological group. If we choose a different basis, $w_1, \ldots, w_n$ for V, we will get a different map $\rho : GL(V) \to GL(n, \mathbb{R})$ and we can also use ρ to endow $GL(V)$ with a topology, but it is fairly easy to see that the two topologies that one obtains in this fashion are really the same. This is because the maps π and ρ will only differ by conjugation by an element of $GL(n, \mathbb{R})$ and this map is a topological isomorphism of $GL(n, \mathbb{R})$. Thus, we define a unique topology making $GL(V)$ into a topological group that is topologically isomorphic to $GL(n, \mathbb{R})$.

In an analogous fashion, if V is a finite dimensional vector space over $\mathbb{C}$ then we may also endow $GL(V)$ with a unique topology.

Definition 6.1. *Let* G *be a topological group and let* V *be a finite dimensional vector space. By a* **continuous representation of G on V** *we mean a continuous, homomorphism* $\pi : G \to GL(V)$.

Note that if $\pi : G \to GL(V)$ and we pick a basis so that we have a matrix representation, $\pi(g) = (f_{i,j}(g))$ then π is continuous if and only if the functions $f_{i,j}$ are all continuous.

One Dimensional Representations of $\mathbb{R}$ and $\mathbb{T}$

We now take a careful look at the one dimensional representations of $\mathbb{R}$. First consider, homomorphisms $\pi : (\mathbb{R}, +) \to GL(\mathbb{R}) = (\mathbb{R}^*, \cdot)$. We have that $\pi(0) = 1, \pi(t+s) = \pi(t)\pi(s)$. One way to obtain such a map is to set $\pi(t) = e^{at}$ for some real a. We shall see that these are all of the continuous representations.

Example 6.2. *Here we exhibit some non-continuous representations of* $\mathbb{R}$. *First, note that* $\mathbb{R}$ *is a vector space over the field of rationals,* $\mathbb{Q}$. *Choose a basis(necessarily uncountable)* $\{r_\beta\}$ *for* $\mathbb{R}$ *over* $\mathbb{Q}$ *and also choose real numbers* $\{a_\beta\}$. *We may then define a homomorphism,* $\pi : \mathbb{R} \to \mathbb{R}^*$ *by setting,*

$$\pi(\sum_\beta q_\beta r_\beta) = \prod_\beta e^{a_\beta q_\beta}.$$

Note that if π *is any continuous representation of* $\mathbb{R}$, *then by the density of* $\mathbb{Q}$, *the map* π *is determined by its values on the rationals. But the above maps allow us to send a irrational real number to any value, independent of the values of rationals. In particular, by choosing one of the basis vectors to be* 1, *we could assign* $a = 0$ *for this "vector" and then the above map* π *would send every rational to* $e^0 = 1$ *but still send other reals to arbitrary values.*

Thus, we see that there are many discontinuous representations of $(\mathbb{R}, +)$.

Proposition 6.3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be any continuous function, such that* $f(x+y) = f(x)f(y)$ *and* $f(0) = 1$. *Then there exists,* $a \in \mathbb{R}$ *such that* $f(x) = e^{ax}$.

Proof. There is a unique number a such that $f(1) = e^a$. Then $f(n) = e^{an}$. Let $g(x) = f(x)e^{-ax}$, then $g(0) = 1, g(x+y) = g(x)g(y), g(n) = 1$ and g is continuous.

Hence, for any $m, g(n/m)^m = g(n) = 1$. Thus, $g(n/m) = 1$ for all rationals and by continuity for all $x \in \mathbb{R}$. Thus, $f(x) = e^{ax}$. $\square$

Corollary 6.4. *Let* $\pi : (\mathbb{R}, +) \to (\mathbb{R}^*, \cdot)$ *be a continuous homomorphism, then there exists a unique* $a \in \mathbb{R}$, *such that* $\pi(t) = e^{at}$.

We now turn our attention to the one dimensional complex representations of $(\mathbb{R}, +)$. Recall that if $\lambda = a + bi, a, b \in \mathbb{R}$ is any complex number then $e^\lambda = e^a(cos(b) + isin(b))$. Thus, for every $\lambda \in \mathbb{C}$, we have a continuous, homomorphism, $\pi : (\mathbb{R}, +) \to GL(\mathbb{C}) = (\mathbb{C}^*, \cdot)$, given by $\pi(t) = e^{\lambda t}$.

We shall prove that these are all of the continuous representations. To do this we will first consider continuous homomorphisms into $\mathbb{T}$.

Lemma 6.5. *Let* $\delta > 0$ *and let* $f : [-\delta, +\delta] \to \mathbb{R}$ *be a continuous function such that* $f(0) = 0$ *and* $f(t+s) = f(t) + f(s)$ *for all* $|t|, |s| \leq \delta/2$. *Then there exists a unique* $r \in \mathbb{R}$ *such that* $f(t) = rt$, *for all,* $-\delta \leq t \leq +\delta$.

Proof. Let $f(\delta) = a$, if f has the desired form, then $r = a/\delta$. Look at $g(t) = at/\delta$. We have that $f(0) = g(0), f(\delta) = g(\delta)$.

Since $a = f(\delta) = f(\delta/2 + \delta/2) = 2f(\delta/2)$, we have that $f(\delta/2) = a/2 = g(\delta/2)$. Inductively, we find, $f(\delta/2^n) = g(\delta/2^n) = a/2^n$. Now if k is an integer, $k < 2^n$, then $f(k\delta/2^n) = kf(\delta/2^n) = ka/2^n = g(k\delta/2^n)$.

Since the numbers of the form $k\delta/2^n$ are dense in $[0, \delta]$, by continuity, we have that $f(t) = g(t), 0 \leq t \leq \delta$. the case for negative t is similar. $\square$

Recall that $\mathbb{T}$ is topologically isomorphic to $\mathbb{R}/\mathbb{Z}$ with the quotient map given by $t \to e^{2\pi it}$. Moreover, since the quotient map is open, we have that $(-1/2, +1/2)$ is homeomorphic to $\mathbb{T}??\{-1\}$. Hence there exists a continuous function, $L : \mathbb{T}??\{-1\} \to (-1/2, +1/2)$ that is the inverse map, i.e., $e^{2\pi iL(z)} = z$ for z in this set. (For those familiar with complex, this is just a branch of the logarithm.)

Theorem 6.6. *Let $\rho : (\mathbb{R}, +) \to \mathbb{T}$ be a continuous homomorphism. Then there exists $b \in \mathbb{R}$, such that $\rho(t) = e^{ibt}$.*

Proof. Let $U = \rho^{-1}(\mathbb{T}\backslash\{-1\})$, so that this set is open and contains 0. Let $\delta > 0$ be chosen small enough that $[-\delta, +\delta] \subseteq U$, and so that the function $f : [-\delta, +\delta] \to \mathbb{R}$ defined by $f(t) = L(\rho(t))$ satisfies, $f([-\delta, +\delta]) \subseteq (-1/4, +1/4)$.

If $|t|, |s| \leq \delta/2$, then $f(t), f(s)$ and $f(t + s)$ are the unique numbers in $(-1/2, +1/2)$ satisfying $e^{2\pi if(t)} = \rho(t), e^{2\pi is} = \rho(s)$ and $e^{2\pi if(t+s)} = \rho(t+s)$. Hence, $e^{2\pi if(t+s)} = \rho(t + s) = \rho(t)\rho(s) = e^{2\pi if(t)}e^{2\pi if(s)} = e^{2\pi i(f(t)+f(s))}$. Since $f(t), f(s)$ are both in $(-1/4, +1/4), f(t) + f(s)$ is in $(-1/2, +1/2)$ so by the uniqueness in this interval, $f(t + s) = f(t) + f(s)$.

Hence, by the lemma, there exists $r \in \mathbb{R}$ such that $f(t) = rt$, for $|t| \leq \delta$. Thus, we have that $\rho(t) = e^{2\pi if(t)} = e^{2\pi irt} = e^{ibt}$, where $b = 2\pi r$ for any $|t| \leq \delta$.

Now given any $s \in \mathbb{R}$, there exists an integer n so that $|s/n| \leq \delta$ and so $\rho(s) = \rho(s/n)^n = (e^{ib(s/n)})^n = e^{ibs}$. $\square$

Theorem 6.7. *Let $\pi : (\mathbb{R}, +) \to \mathbb{C}^*$ be a continuous homomorphism. Then there exists $\lambda \in \mathbb{C}$ such that $\pi(t) = e^{\lambda t}$.*

Proof. Let $\pi_1 : (\mathbb{R}, +) \to \mathbb{R}^*$ be defined by $\pi_1(t) = |\pi(t)|$. Then π_1 is a continuous homomorphism and so there exists, $a \in \mathbb{R}$ such that $\pi_1(t) = e^{at}$.

Now let $\rho(t) = \pi(t)e^{at}$, then $\rho : (\mathbb{R}, +) \to \mathbb{T}$ and so there exists $b \in \mathbb{R}$ such that $\rho(t) = e^{ibt}$.

Hence, $\pi(t) = \pi_1(t)\rho(t) = e^{at}e^{ibt} = e^{(a+ib)t}$ and set $\lambda = a + ib$. $\square$

Corollary 6.8. *Let $\pi : (\mathbb{R}, +) \to \mathbb{C}^*$ be a bounded, continuous homomorphism, then $\pi(\mathbb{R}) \subseteq \mathbb{T}$ and there exists $b \in \mathbb{R}$ such that $\pi(t) = e^{ibt}$.*

Proof. Note that the fact that π is bounded forces $a = 0$ in the above theorem. $\square$

Theorem 6.9. *Let $\gamma : \mathbb{T} \to \mathbb{C}^*$ be a continuous homomorphism, then there exists an integer n, such that $\gamma(z) = z^n$.*

Proof. Let $\rho(t) = \gamma(e^{2\pi it})$, then ρ is a continuous hommomorphism of $\mathbb{R}$. Moreover, since $\mathbb{T}$ is compact, $\gamma(\mathbb{T})$ is compact and hence bounded. Thus, there exists, $b \in \mathbb{R}$ such that $\rho(t) = e^{ibt}$. But since $e^{ib} = \rho(1) = \gamma(1) = 1$ we see that $b = 2\pi n$ for some integer n. Now if $z = e^{2\pi i\theta} \in \mathbb{T}$, then $\gamma(z) = \rho(\theta) = e^{2\pi in\theta} = z^n$. □

Theorem 6.10. *Let $\pi : (\mathbb{R}^n, +) \to \mathbb{C}^*$ be a continuous homomorphism, then there exist, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, such that, $\pi((t_1, \dots, t_n)) = e^{\lambda_1 t_1 + \dots + \lambda_n t_n}$.*

Proof. Define $\pi_j : (\mathbb{R}, +) \to \mathbb{C}^*$ by setting, $\pi_j(t) = \pi((0, \dots, 0, t, 0, \dots, 0))$ where t occurs in the j-th component. Clearly, π_j is continuous and so there exists $\lambda_j \in \mathbb{C}$ such that $\pi_j(t) = e^{\lambda_j t}$. The proof is completed by noting that $\pi((t_1, \dots, t_n)) = \pi_1(t_1) \cdots \pi_n(t_n) = e^{\lambda_1 t_1 + \dots + \lambda_n t_n}$. □

Finite Dimensional Representations of $\mathbb{R}$ and $\mathbb{T}$

Let $A \in M_n(\mathbb{C})$, then the *exponential of A,* is given by the convergent power series,

$$e^A = I + A + \frac{A^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Proposition 6.11. *Let $A \in M_n$, then setting $\pi(t) = e^{At}$ defines a continuous, homomorphism $\pi : (\mathbb{R}, +) \to GL(n, \mathbb{C})$.*

Proof. Using the Cauchy product of power series, we have that, $e^{tA}e^{sA} = \sum_{n=0}^{\infty} \sum_{k+j=n} \frac{(tA)^k}{k!} \frac{(sA)^j}{j!} = \sum_{n=0}^{\infty} [\sum_{k+j=n} \frac{t^k s^j}{k!j!}] A^n = \sum_{n=0}^{\infty} [\sum_{k=0}^{n} \frac{t^k s^{n-k}}{k!(n-k)!}] A^n = \sum_{n=0}^{\infty} [\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k s^{n-k}] \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{(t+s)^n A^n}{n!} = e^{(t+s)A}$. Hence, $\pi(t)\pi(s) = \pi(t+s)$ and so the map is multiplicative and since $\pi(t)\pi(-t) = I$ the range is contained in $GL(n, \mathbb{C})$.

The continuity of π follows from facts about power series. □

The converse of the above theorem is also true, but is a bit beyond our means.

Theorem 6.12. *(M.Stone) Let $\pi : (\mathbb{R}, +) \to GL(n, \mathbb{C})$ be a continuous, homomorphism. Then $lim_{t\to 0} \frac{\pi(t) - I}{t}$ exists and if we denote this limit by A, then $\pi(t) = e^{tA}$.*

Thus, to understand these representations we need to understand the behavior of matrix exponentials. This is often covered in undergraudate differential courses and we review the key facts. If $A = S^{-1}JS$ where J is the Jordan canonical form of A, then $e^{tA} = S^{-1}e^{tJ}S$ so it is enough to understand the exponential of a Jordan cell.

If A is diagonalizable, i.e., if each Jordan block for A is 1×1, so that $J = Diag(\lambda_1, \dots, \lambda_n)$, then $e^{tJ} = Diag(e^{t\lambda_1}, \dots, e^{t\lambda_n})$.

If for example $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is a 2×2 Jordan cell, then $e^{tJ} = \begin{pmatrix} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix}$.

Proposition 6.13. *Let $A \in M_n$. Then the set $\{e^{tA} : t \in \mathbb{R}\}$ is bounded if and only if A is similar to a diagonal matrix and all the eigenvalues of A are purely imaginary.*

Proof. If any of the Jordan blocks in the Jordan form of A is larger than 1×1, then e^{tJ} will contain a coefficient of the form $te^{t\lambda}$ which is unbounded as a function of t, for every λ.

Hence, each of the Jordan cells is 1×1, which means that A is similar to a diagonal matrix, $J = Diag(\lambda_1, \dots, \lambda_n)$ where the $\lambda_i's$ are the eigenvalues of A. But in order for $e^{t\lambda}$ to be bounded, λ must be purely imaginary. □

Theorem 6.14. *Let $\pi : \mathbb{T} \to GL(n, \mathbb{C})$ be a continuous, homomorphism, then there exist integers, $k_1, \dots, k_n$ and an invertible matrix, S, such that $\pi(z) = S^{-1}Diag(z^{k_1}, \dots, z^{k_n})S$.*

Proof. Look at $\rho : \mathbb{R} \to GL(n, \mathbb{C})$ defined by $\rho(t) = \pi(e^{it})$. Since $\mathbb{T}$ is compact the image of π is a bounded set and hence ρ is bounded. Hence by the above, $\rho(t) = S^{-1}Diag(e^{ib_1t}, \dots, e^{ib_nt})S$.

Arguing as in the 1-dimensional case, each $b_j = 2\pi k_j$ for an integer k_j. Thus, if $z = e^{i\theta}$, then $\pi(z) = \rho(\theta) = S^{-1}Diag(e^{2\pi ik_1\theta}, \dots, e^{2\pi ik_n\theta})S = S^{-1}Diag(z^{k_1}, \dots, z^{k_n})S$. □

Problem 6.15. *Let G_1, G_2 be topological groups and let $G_1 \times G_2$ be the product group. Prove that $\pi : G_1 \times G_2 \to GL(n, \mathbb{C})$ is a continuous homomorphism if and only if there exist continuous homomorphisms, $\pi_i : G_i \to GL(n, \mathbb{C}), i = 1, 2$ with $\pi_1(g_1)\pi_2(g_2) = \pi_2(g_2)\pi_1(g_1)$ for all $g_1 \in G_1, g_2 \in G_2$ such that $\pi((g_1, g_2)) = \pi_1(g_1)\pi_2(g_2)$.*

Problem 6.16. *Let $\mathbb{T}^m$ denote the group that is the product of m copies of the circle group(often called the m-torus). Given $z = (z_1, \dots, z_m) \in \mathbb{T}^m$ and $K = (k_1, \dots, k_m)$ an m-tuple of integers, we set $z^K = z_1^{k_1} \cdots z_m^{k_m}$. Prove that if $\pi : \mathbb{T}^m \to GL(n, \mathbb{C})$ is a continuous homomorphism, then there exists an invertible S and m-tuples of integers, $K_1, \dots, K_n$, such that $\pi(z) = S^{-1}Diag(z^{K_1}, \dots, z^{K_n})S$*

Problem 6.17. *Prove that every continuous, homomorphism from $\mathbb{T}$ into $\mathbb{R}^*$ is constant.*

Problem 6.18. *Find analogues of the above results for continuous, homomorphisms of $\mathbb{C}^*$ into $\mathbb{R}^*, \mathbb{T}$ and $\mathbb{C}^*$.*

7. Compact and Amenable Groups

In this section we will show that many results from the representation theory of finite groups can be extended to compact and amenable groups.

Recall that a(Hausdorff) topological space, X, is called *compact*, if every open cover of X has a finite subcover. A space, X, is called *locally compact*, if for every $x \in X$, there is an open set, U, with $x \in U$ such that the closure of U is compact. For example, $\mathbb{R}$ is locally compact, but not compact, while the circle group $\mathbb{T}$ is compact. If G is a discrete group, then it is compact if and only if it is a finite set, while every discrete group is locally compact. In this chapter we study the properties and representations of compact and amenable groups.

First, we take a look at a couple of important examples of such groups.

Definition 7.1. *The* **unitary group,** $\mathcal{U}(n)$, *is the subgroup of* $GL(n,\mathbb{C})$, *defined by,*

$$\mathcal{U}(n) = \{A \in GL(n,\mathbb{C}) : A^*A = I\}.$$

The **special unitary group,** $\mathcal{SU}(n)$ *is the subgroup of* $\mathcal{U}(n)$, *defined by*

$$\mathcal{SU}(n) = \{A \in \mathcal{U}(n) : det(A) = 1\}.$$

The **orthogonal group,** $\mathcal{O}(n)$, *is the subgroup of* $GL(n,\mathbb{R})$ *defined by,*

$$\mathcal{O}(n) = \{A \in GL(n,\mathbb{R}) : A^tA = I\}.$$

The **special orthogonal group,** $\mathcal{SO}(n)$ *is the subgroup of* $\mathcal{O}(n)$, *defined by,*

$$\mathcal{SO}(n) = \{A \in \mathcal{O}(n) : det(A) = 1\}.$$

It is easily checked that a matrix $A \in \mathcal{U}(n)(respectively, \mathcal{O}(n))$ if and only if the columns are an orthonormal basis for $\mathbb{C}^n(respectively, \mathbb{R}^n)$. Thus, both of these subgroups are compact, because they are closed and bounded subsets of the corresponding spaces of matrices.

Since $det : \mathcal{U}(n) \to \mathbb{C}^*$ is a continuous, homomorphism, and $\mathcal{SU}(n)$ is the kernel of this homomorphism it is a closed, normal subgroup of $\mathcal{U}(n)$. Similarly, $O(n)$ is a closed, normal subgroup of $\mathcal{O}(n)$.

Let's first take a closer look at $\mathcal{O}(2)$ as a group and topological space. If for any angle θ, we set $R(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$, then $R(\theta) \in \mathcal{SO}(2)$ and it is the matrix of the linear map given by counterclockwise rotation through the angle θ. The map $\pi : \mathbb{T} \to \mathcal{SO}(2)$ defined by $\pi(e^{i\theta}) = R(\theta)$ is a continuous, homomorphism. Also, given any matrix in $\mathcal{O}(2)$, if its first column is the unit vector, $\begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix}$, then since its second column must be perpindicular to this vector, we see that the second column is either $\pm \begin{pmatrix} -sin(\theta) \\ cos(\theta) \end{pmatrix}$. If we let $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then we see that $\mathcal{O}(2) = \{R(\theta) : 0 \le \theta < 2\pi\} \cup \{R(\theta)F : 0 \le \theta < 2\pi\}$ and so topologically, $\mathcal{O}(2)$ is the union of two circles. Since, $det(R(\theta)F) = -1$, we see that the image of the circle group is $\mathcal{SO}(2)$ and , thus, this map defines a continuous, isomorphism between $\mathbb{T}$ and $\mathcal{SO}(2)$. It follows that $\mathcal{O}(2)$ is a semidirect product of $\mathbb{T}$ by $\mathbb{Z}_2$. Also, note that $FR(\theta)F^{-1} = R(-\theta)$.

Now let's look at $\mathcal{U}(2)$. Given any $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$, if we let $R(a,b) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, then $R \in \mathcal{SU}(2)$. Moreover, if we multiply any two such matrices together, then we get another such matrix!!(Check this yourself.) Thus, we get a closed(and hence compact) subgroup of $\mathcal{SU}(2)$.

What is a little surprising, is that the set of such pairs, $(a, b) \in \mathbb{C}^2 = \mathbb{R}^4$ corresponds to the unit sphere in $\mathbb{R}^4$, which is 3-dimensional. Hence, in this manner, we see the 3-sphere, S_3 can be made into a compact topological group.

Now in a manner similar to the above, we see that if $\begin{pmatrix} a \\ b \end{pmatrix}$ is the first column of any matrix in $\mathcal{U}(2)$ then the second column must have the form $\begin{pmatrix} -e^{i\theta}\bar{b} \\ e^{i\theta}\bar{a} \end{pmatrix}$. Thus, if we set $F(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, then these matrices generate another representation of $\mathbb{T}$ and every matrix in $\mathcal{U}(2)$ is uniquely of the form $R(a, b)F(\theta)$.

Thus, as above, we see that $\mathcal{SU}(2) = \{R(a, b) : |a|^2 + |b|^2 = 1\}$ and topologically, $\mathcal{SU}(2)$ is homeomorphic to S_3. Also, topologically, $\mathcal{U}(2)$ is homeomorphic to $S_3 \times \mathbb{T}$. We have that $F(\theta)R(a, b)F(\theta)^{-1} = R(a, e^{i\theta}b)$ and that $\mathcal{U}(2)$ is isomorphic to the semidirect product of S_3 by $\mathbb{T}$.

Given a topological space, X, we let $C(X)$ denote the set of continuous, functions from X to $\mathbb{C}$. It is easy to see that $C(X)$ is a vector space over $\mathbb{C}$, that is, sums and scalar multiplies of continuous functions are again continuous. Given $f \in C(X)$, the support of f, denoted *supp*(f) is the closure of the set, $\{x \in X : f(x) \neq 0\}$. A function is said to have *compact support* provided that supp(f) is compact. We let $C_c(X) \subseteq C(X)$ denote the set of continuous functions with compact support. A function f is called *bounded* provided that there is a constant,M, such that $|f(x)| \leq M$ and we let $C_b(X)$ denote the set of continuous, bounded functions on X. Note that every function with compact support is bounded, so $C_c(X) \subseteq C_b(X) \subseteq C(X)$ and when X is compact, $C_c(X) = C_b(X) = C(X)$.

Proposition 7.2. *The sets $C_c(X), C_b(X)$ are vector subspaces of $C(X)$.*

Proof. If $f_1, f_2 \in C_c(X)$, then *supp*$(f_1 + f_2) \subseteq$ *supp*$(f_1) \cup$ *supp*(f_2). Since the union of two compact sets is again compact and closed subsets of compact sets are compact, we have that *supp*$(f_1 + f_2)$ is compact and hence, $f_1 + f_2 \in C_c(X)$.

If $f \in C_c(X)$ and $\lambda \in \mathbb{C}$, then *supp*$(\lambda f) =$ *supp*(f) when $\lambda \neq 0$ and when $\lambda = 0$, *supp*(λf) is the empty set(which is closed). Thus, $C_c(X)$ is a vector subspace.

The proof for $C_b(X)$ is similar. □

If G is a topological group, $f \in C(G)$ and $g \in G$ then we define functions $g \cdot f$ and $f \cdot g$ by setting $(g \cdot f)(h) = f(g^{-1}h)$ and $(f \cdot g)(h) = f(hg^{-1})$. The reason for using the inverses in these definitions is for associativity, that is, with the above definitions, we have that $g_1 \cdot (g_2 \cdot f) = (g_1 g_2) \cdot f, (f \cdot g_1) \cdot g_2 = f \cdot (g_1 g_2)$. Thus, these definitions yield actions of G on C(G), as soon as we have shown that these new functions are actually continuous. Also, we set $\tilde{f}(h) = f(h^{-1})$.

Proposition 7.3. *Let $f \in C(G)$, then $g \cdot f, f \cdot g$ and $\tilde{f}$ are in $C(G)$. Moreover, if $f \in C_c(X)(respectively, C_b(X))$, then $g \cdot f, f \cdot g$ and $\tilde{f}$ are all in $C_c(G)(respectively, C_b(X))$.*

Proof. Recall that $L_{g^{-1}} : G \to G$ defined by $L_{g^{-1}}(h) = g^{-1}h$ is continuous and hence $g \cdot f = f \circ L_{g^{-1}}$ is continuous. Also $supp(g \cdot f) = \{h : (g \cdot f)(h) \neq 0\}^- = \{h : f(g^{-1}h) \neq 0\}^- = g \cdot supp(f)$ and so if the support of f is compact, then so is the support of $g \cdot f$.

The proofs for $f \cdot g$ and $\tilde{f}$ are similar and use the continuity of $R_{g^{-1}}$ and inv.

The proofs for $C_b(X)$ are similar. □

Definition 7.4. *A linear map from a complex vector space into $\mathbb{C}$ is called a* **linear functional.** *When G is a topological group, a linear functional, L whose domain is either, $C_c(G), C_b(G)$ or $C(G)$ is called* **left invariant** *provided that $L(f) = L(g \cdot f)$ for every f and $g \in G$. Similarly, L is called* **right invariant** *if $L(f \cdot g) = L(f)$ and* **invariant** *if it is both left and right invariant. Finally, L is called* **positive** *if $L(f) \geq 0$ whenever $f \geq 0$, i.e, whenever $f(h) \geq 0$, for every $h \in G$. If L is both positive and invariant then it is called an* **invariant mean.** *When there exists a non-zero invariant mean on $C_b(G)$, the group G is called* **amenable.**

Example 7.5. *If G is finite, then $C(G) = \mathbb{C}(G)$ and we obtain an invariant mean on C(G), by setting,*

$$L(f) = \frac{1}{|G|} \sum_{h \in G} f(h).$$

Example 7.6. *For $G = (\mathbb{R}, +)$ and $f \in C_c(\mathbb{R})$, note that $\int_{-\infty}^{+\infty} f(t)dt$ is well-defined and finite since $supp(f) \subseteq [a, b]$ and so the integration is really only over the interval [a,b]. Also if $s \in \mathbb{R}$, then $(s \cdot f)(t) = (f \cdot s)(t) = f(t-s)$ and we have $\int_{-\infty}^{+\infty} f(t-s)dt = \int_{-\infty}^{+\infty} f(t')dt'$ by doing the substitution $t' = t - s, dt' = dt$.*

Thus, if we let $L(f)$ denote this integral, then it is easy to see that L is an invariant mean on $C_c(\mathbb{R})$.

Example 7.7. *The group $(\mathbb{R}, +)$ is actually amenable, that is, there exists a non-zero invariant mean on $C_b(\mathbb{R})$, not just on $C_0(\mathbb{R})$. The proof requires a bit of functional analysis and we sketch the main ideas for the interested student. For each natural number N, define $L_N : C_b(\mathbb{R}) \to \mathbb{C}$, by $L_N(f) = \frac{1}{2N} \int_{-N}^{+N} f(t)dt$. This defines a sequence of bounded linear functionals in the unit ball of the dual space $C_b(\mathbb{R})^*$ of $C_b(\mathbb{R})$. By the Banach-Alaoglu theorem the unit ball of the dual space is compact in the weak*-topology. Hence, this set of functionals has a limit point, L, since $L_N(1) = 1$ for all N, we have* *that $L(1) = 1$, and hence, $L \neq 0$. Fix, $s \in \mathbb{R}$ and f and note that,*

$$|L_N(f) - L_N(s \cdot f)| = \frac{1}{2N} \Big| \int_{-N}^{+N} f(t) - f(t-s) dt \Big|$$
$$= \frac{1}{2N} \Big| \int_{-N-s}^{-N} f(t)dt + \int_{N-s}^{N} f(t)dt \Big| \leq \frac{\|f\|_\infty |s|}{N} \to 0,$$

as $N \to +\infty$. This implies that the limit point L, satisfies $L(f) = L(s \cdot f)$, and hence, is invariant. If $f \geq 0$, then $L_N(f) \geq 0$, for all N and hence, $L(f) \geq 0$. Thus, L is an invariant mean on $(\mathbb{R}, +)$. Note that when $f \in C_0(\mathbb{R})$, we have that $L_N(f) \to 0$, and it will follow that, $L(f) = 0$. This shows that the limit L cannot be given as an integration, even though it is a functional obtained as a limit of integrals.

Example 7.8. *Let $G = \mathbb{T}$, and for $f \in C(\mathbb{T})$ set*

$$L(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt.$$

If $z = e^{is} \in \mathbb{T}$, then $(z \cdot f)(e^{it}) = (f \cdot z)(e^{it}) = f(e^{i(t-s)})$ and again a simple substitution shows that $L(z \cdot f) = L(f)$. Thus, it readily follows that L is an invariant mean on $C(\mathbb{T})$.

Example 7.9. *Let $G = (\mathbb{R}^+, \cdot) \subseteq (\mathbb{R}^*, \cdot)$. For $f \in C_c(\mathbb{R}^+)$ we set $L(f) = \int_0^\infty f(t) \frac{dt}{t}$. If $s \in \mathbb{R}^+$ then $L(s \cdot f) = \int_0^\infty f(s^{-1}t) \frac{dt}{t} = \int_0^\infty f(t') \frac{dt'}{t'}$, where $t' = s^{-1}t$. It is easily checked that L is linear and, thus, L is an invariant mean.*

Example 7.10. *One can also define an invariant mean on $C_b(\mathbb{R}^*)$, in a method analogous to the one used for $(\mathbb{R}, +)$. One sets $L_N(f) = \frac{1}{2ln(N)} \int_{1/N}^{N} \frac{f(t)}{t} dt$, and takes a weak*-limit point of this sequence of bounded, linear functionals.*

Definition 7.11. *An invariant mean L is called* **normalized***, provided its domain contains $C_b(G)$ and $L(1) = 1$, where 1 denotes the function that is constantly equal to 1.*

Thus, the above examples show that every finite group, $(\mathbb{R}, +), (\mathbb{R}^*, \cdot)$ and $\mathbb{T}$ have normalized,invariant means. Note that the normalized invariant mean on $C_b(\mathbb{R})$, does not arise from any sort of integration, even though it was a limit of integrals. Similarly, $\mathbb{Z}$ is also known to be an amenable group, the invariant mean can be obtained as a limit of finite sums, but also is not given as a sum.

In fact, it is known that every abelian group is amenable. We will see below that one consequence of Haar's Theorem, presented below, is that every compact group is amenable. Some familiar groups that are not amenable, are the free groups on two or more generators. There are some "good" characterizations of amenable groups, but there is also a great deal that is not known. One conjecture is that a group is not amenable if and only if it contains a subgroup isomorphic to the free group on two generators.

The proof of the following theorem uses results from measure theory and functional analysis and so is beyond our current background.

Theorem 7.12. *(Haar) Let G be a compact topological group, then there is a unique normalized, left invariant mean $M : C(G) \to \mathbb{C}$.*

Corollary 7.13. *(Haar) Let G be a compact topological group, then the unique normalized left invariant mean, M, is right invariant. Thus there is a unique normalized, invariant mean on G and so every compact group is amenable. In addition, for any $f \in C(G)$, we have that $M(\tilde{f}) = M(f)$.*

Proof. Fix $g \in G$ and define, $M_g : C(G) \to \mathbb{C}$, by setting $M_g(f) = M(f \cdot g)$. We will show that M_g is a normalized, left invariant mean and hence, $M(f) = M_g(f) = M(f \cdot g)$, so that M is right invariant.

It is clear that M_g is normalized and linear. Also, $(g_1 \cdot f) \cdot g_2 = g_1 \cdot (f \cdot g_2)$, hence $M_g(g_1 \cdot f) = M((g_1 \cdot f) \cdot g) = M(g_1 \cdot (f \cdot g)) = M(f \cdot g) = M_g(f)$. Thus, M_g is left invariant. Finally, if $f(h) \geq 0$ for all h, then $f \cdot g(h) \geq 0$ for all h and hence, $f \geq 0$ implies that $M_g(f) = M(f \cdot g) \geq 0$, and so M_g is a normalized, left invariant mean on G and hence, $M = M_g$.

Next, define $\tilde{M}(f) = M(\tilde{f})$. Similar arguments show that $\tilde{M}$ is a normalized, left invariant mean on G and hence, $M(f) = \tilde{M}(f) = M(\tilde{f})$. □

Definition 7.14. *Let G be a compact, topological group. The unique, normalized, invariant mean on G is called the* **Haar mean** *on G.*

Thus, the first and third examples are the unique Haar means.

We well see that normalized, invariant means on compact groups allow us to prove many of the results for finite groups that required "averaging" also hold for compact groups.

Problem 7.15. *Find the unique invariant mean on $\mathbb{T}^m$.*

Problem 7.16. *Show that the unique normalized, invariant mean on $\mathcal{O}(2)$ is given by its identification with two circles and then taking the average of the two integrals over the circles.*

Measure Theoretic Interpretation

For those of you familiar with concepts from measure theory, we give a measure theoretic interpretation of Haar's theorem. Given a compact topological space, X, the σ-algebra generated by the open subsets of X is called the **Borel sets** and denoted $\mathcal{B}$.

One version of the Riesz Representation Theorem, says that for every, positive, linear functional, $L : C(X) \to \mathbb{C}$ there exist a unique positive measure, μ, on $\mathcal{B}$, such that $L(f) = \int_X f(x)d\mu(x)$, and conversely every such measure gives rise to such a positive linear functional. Note that $L(1) = \mu(X)$.

Moreover, L is invariant, if and only if μ has the property that, $\mu(g \cdot B) = \mu(B \cdot g) = \mu(B)$, for every Borel set, B and every $g \in G$. Such measures are called **invariant measures** or **Haar measures.**

Thus, Haar's theorem is equivalent to the fact that given a compact group G, there is a unique invariant measure μ with $\mu(G) = 1$. This measure is called, the **normalized, Haar measure on G.**

Thus, we see that Lebesgue measure on $\mathbb{R}$ is a Haar measure on $(\mathbb{R}, +)$, but we cannot normalize this measure, since $m(\mathbb{R}) = +\infty$ and arc length measure on $\mathbb{T}$ is the unique, normalized Haar measure on $\mathbb{T}$.

For the fourth example, $(\mathbb{R}^+, \cdot)$, we see that the Haar measure satisfies, $\mu([a,b]) = \int_a^b 1\frac{dt}{t} = ln(b) - ln(a) = ln(b/a)$ and again this measure cannot be normalized since $\mu(\mathbb{R}^+) = +\infty$.

By the above problem, the unique normalized Haar measure on $\mathcal{O}(2)$ is the average of the arc length measures on its two circles.

Since $(\mathbb{R}, +)$ is an amenable group, there is a normalized mean on $C_b(\mathbb{R})$, but this mean is not given by a measure on $\mathbb{R}$.

Matrix-Valued Functions

Let $\pi : G \to GL(n, \mathbb{C})$ be a continuous, bounded homomorphism. If we write, $\pi(h) = (t_{i,j}(h))$ in terms of its coordinate functions, then the continuity of π is equivalent to the continuity of $t_{i,j}$ for all i and j, and the fact that π is bounded guarantees that $t_{i,j} \in C_b(G)$. We shall often need to consider expressions of the form $\pi(h)A\pi(h^{-1})$ where $A \in M_n$. A moments reflection shows that the above expression is a matrix of continuous functions. Explicitly, if we let $A = (a_{i,j})$, then $\pi(h)A\pi(h^{-1}) = (f_{i,j}(h))$ where, $f_{i,j}(h) = \sum_{k,l=0}^n t_{i,k}(h)a_{k.l}t_{l,j}(h^{-1})$. Thus, as functions, we have that $f_{i,j} = \sum_{k,l=0}^n t_{i,k}a_{k,l}\tilde{t}_{l,j}$.

We will often think of a matrix of continuous functions, $(f_{i,j}(h))$ from G to $\mathbb{C}$, as a continuous, function $F = (f_{i,j})$ from G to M_n.

Now given $F = (f_{i,j})$, we may apply an invariant mean, M, to each function to obtain a scalar-matrix, $(M(f_{i,j}))$ we shall denote this scalar matrix as $M^{(n)}(F)$. When F is rectangular, say $m \times n$, the same ideas apply and we denote the scalar matrix obtained by $M^{(m,n)}(F)$. An example helps to solidfy this idea.

Example 7.17. *Let* $\pi : \mathbb{T} \to GL(2, \mathbb{C})$ *be defined by* $\pi(z) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}$, *let* $\rho : \mathbb{T} \to GL(3, \mathbb{C})$ *be defined by* $\rho(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & z^3 & 0 \\ 0 & 0 & z^2 \end{pmatrix}$, *let* $A = (a_{i,j})$ *be a* 2×3 *matrix of scalars, let* $F(z) = \pi(z)A\rho(z^{-1})$ *and compute* $M^{(2,3)}(F)$, *where M is the unique Haar mean on the circle group.*

We see that $F(z) = \begin{pmatrix} za_{1,1}z^{-1} & za_{1,2}z^{-3} & za_{1,3}z^{-2} \\ z^2a_{2,1}z^{-1} & z^2a_{2,2}z^{-3} & z^2a_{2,3}z^{-2} \end{pmatrix}$. Since, $M(z^n) = \int_0^1 e^{2\pi int}dt = 0$, for $n \neq 0$, we have that $M^{(2,3)}(F) = \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \end{pmatrix}$.

Lemma 7.18. *Let $B = (b_{i,j}) \in M_n$ and let $F = (f_{i,j})$ be an $n \times n$ matrix of functions, then $M^{(n)}(BF) = BM^{(n)}(F)$ and $M^{(n)}(FB) = M^{(n)}(F)B$.*

Proof. The (i,j)-th entry of BF is $\sum_{k=0}^{n} b_{i,k} f_{k,j}$ and so the (i,j)-th entry of $M^{(n)}(BF)$ is $M(\sum_{k=0}^{n} b_{i,k} f_{k,j}) = \sum_{k=0}^{n} b_{i,k} M(f_{k,j})$ which is the (i,j)-th entry of the product $BM^{(n)}(F)$.

The proof for multiplication on the right is identical. □

Note that in the above result, we don't need the matrices to all be square, just of compatible sizes so that we can do the matrix multiplications.

Proposition 7.19. *Let G be an amenable group, with normalized, invariant mean, M, let $\pi : G \to GL(n, \mathbb{C}), \rho : G \to GL(m, \mathbb{C})$ be continuous, bounded, homomorphisms and let A be a $n \times m$ matrix of scalars. If we set $F(h) = \pi(h)A\rho(h^{-1})$, then the scalar matrix $M^{(n,m)}(F) \in \mathcal{I}(\pi, \rho)$.*

Definition 7.20. *We shall denote the above scalar matrix by A_G.*

Proof. Fix $g \in G$. Then by the above lemma, $\pi(g)M^{(n,m)}(F) = M^{(n,m)}(\pi(g)F)$. But $\pi(g)F(h) = \pi(gh)A\rho(h^{-1}) = \pi(gh)A\rho((gh)^{-1})\rho(g) = F(gh)\rho(g)$.

Note that if $F = (f_{i,j})$, then $F(gh) = (f_{i,j}(gh)) = ((g \cdot f_{i,j})(h))$. Thus, $\pi(g)F = (g \cdot f_{i,j})\rho(g)$.

Finally, we have that, $\pi(g)M^{(n,m)}(F) = M^{(n,m)}(\pi(g)F) = M^{(n,m)}((g \cdot f_{i,j})\rho(g)) = (M(g \cdot f_{i,j}) \cdot \rho(g) = (M(f_{i,j})) \cdot \rho(g) = M^{(n,m)}(F) \cdot \rho(g)$. □

Note Example 6.15 actually computed, A_G. The following result gives an instance where one can definitely determine A_G.

Proposition 7.21. *Let G be an amenable group with a normalized, invariant mean, M, let $\pi : G \to GL(n, \mathbb{C})$ be a continuous, bounded, irreducible representation of G and let $A \in M_n$. Then $A_G = \frac{Tr(A)}{n} I_n$.*

Proof. Since π is irreducible, $\pi(G)' = \{\lambda I_n : \lambda \in \mathbb{C}\}$. Hence, $A_G = \lambda I_n$ for some λ.

Note that for any $h \in G, Tr(F(h)) = Tr(\pi(h)A\pi(h^{-1})) = Tr(A)$. Hence, $n\lambda = Tr(A_G) = Tr(M^{(n)}(F)) = M(Tr(F)) = M(Tr(A)) = Tr(A)$, and we have that $\lambda = \frac{Tr(A)}{n}$, as was to be shown. □

Theorem 7.22. *Let G be an amenable group with normalized, invariant mean, M, let $\pi : G \to GL(V)$ be a continuous, bounded, representation of G on a finite dimensional vector space, V, and let $W \subseteq V$ be a $\pi(G)$-invariant subspace. Then there exists a projection onto W that is $\pi(G)$-invariant.*

Proof. We identify $V = \mathbb{C}^n$ and $GL(V) = GL(n, \mathbb{C})$. By choosing any complementary subspace for W, we may define a $n \times n$ projection matrix, P from $V = \mathbb{C}^n$ onto W.

Let $F(h) = \pi(h)P\pi(h^{-1})$, and let $P_G = M^{(n)}(F)$. Then by the above lemma, $\pi(g)P_G = P_G\pi(g)$ for any $g \in G$. We will show that P_G is a projection onto W.

For any vector $w \in W, P_G w = M^{(n)}(F)w = M^{(n,1)}(F \cdot w)$, where we have applied the lemma to the $n \times 1$ matrix-valued function $F(h)w$. But, since $w \in W$ and this space is invariant, $F(h)w = \pi(h)P\pi(h^{-1})w = \pi(h)\pi(h^{-1})w = w$. Thus, $M^{(n,1)}(Fw) = w$ and we have that $P_g w = w$ for all $w \in W$.

Hence, for any $v \in V, P_G(Pv) = Pv$, and so, $P_G P = P$. Now, $P_G^2 = P_G M^{(n)}(F) = M^{(n)}(P_G F)$, but $P_G F(h) = P_G \pi(h)P\pi(h^{-1}) = \pi(h)P_G P\pi(h^{-1}) = F(h)$. Thus, $M^{(n)}(P_G F) = M^{(n)}(F) = P_G$, and we have that $P_G^2 = P_G$.

Thus, P_G is a projection, is invariant and fixes W. We just need to see that it doesn't project onto a larger space. Now, $P\pi(h)P = \pi(h)P$ since W is invariant, and hence, $PF(h) = P\pi(h)P\pi(h^{-1}) = F(h)$. Thus, $PP_G = PM^{(n)}(F) = M^{(n)}(PF) = M^{(n)}(F) = P_G$. Thus, if $P_G v = v$, then $v = P_G v = P(P_G v) \in W$.

This shows that the range of P_G is contained in W and so it must be a projection onto W. □

Corollary 7.23. *Let G be amenable, $\pi : G \to GL(V)$ be a continuous, bounded representation with V finite dimensional. If $W \subseteq V$ is a $\pi(G)$-invariant subspace, then W is $\pi(G)$-complemented.*

Proof. Let P_G be a $\pi(G)$-invariant projection onto W. Then $I - P_G$ is a projection onto a $\pi(G)$-invariant complementary subspace. □

Corollary 7.24. *Let G be an amenable group, and let $\pi : G \to GL(V)$ be a continuous, bounded representation with V finite dimensional. Then there exists $\pi(G)$-invariant subspaces, $W_1, \dots, W_k$ of V, such that $W_i \cap W_j = (0)$ for $i \neq j, V = W_1 + \cdots + W_k$ and the restriction of the π to each W_i is irreducible.*

Proof. If π is not irreducible, then there exists a $\pi(G)$-invariant subspace, W with a $\pi(G)$-complement. Clearly, the restriction of a continuous, bounded representation to a subspace is still bounded and continuous. So unless W is irreudcible, we may repeat. □

Recall that over $\mathbb{C}$ a representation is irreducible if and only if $\pi(G)' = \{\lambda I : \lambda \in \mathbb{C}\}$.

Theorem 7.25. *(Schur's Lemma) Let G be an amenable group, and let $\pi_i : G \to GL(V_i)$ be continuous, bounded irreducible representations, with V_i finite dimensional, complex vector spaces for $i = 1, 2$. Then either:*

(1) *π_1 and π_2 are inequivalent and $dim(\mathcal{I}(\pi_1, \pi_2)) = 0$ or,*
(2) *π_1 and π_2 are equivalent and $dim(\mathcal{I}(\pi_1, \pi_2)) = 1$.*

Proof. Let $A \in \mathcal{L}(V_1, V_2)$ and look at $A_G \in \mathcal{I}(\pi_1, \pi_2)$ and proceed exactly as in the proof of Schur's lemma for finite groups. □

Theorem 7.26. *Let G be an amenable group and let $\pi : G \to GL(V), \rho : G \to GL(W)$ be bounded, continuous representations on finite dimensional spaces with π irreducible. If $W = W_1 + \cdots + W_k$ is a direct sum decomposition of W into invariant subspaces such that each subrepresentation $\rho_i = \rho|_{W_i}$ is* *irreducible for each i, then $dim(\mathcal{I}(\pi,\rho)) = \#\{i : \rho_i \sim \pi\}$, and consequently, this number is independent of the particular decomposition of ρ into irreducible subrepresentations.*

Proof. The proof is the same as for Theorem 2.23. Given $T \in \mathcal{L}(V,W)$ write $Tv = (T_1v, \dots, T_kv)$ where $T_i \in \mathcal{L}(V,W_i)$. □

8. Character Theory for Amenable Groups

Many of our earlier results on characters for finite groups also hold for amenable groups.

Given $\pi : G \to GL(V)$ be a continuous, bounded representation, with V finite dimensional, the **character of** π, χ_π is the continuous function, $\chi_\pi(g) = Tr(\pi(g))$. Note that if $\pi : G \to GL(n,\mathbb{C})$ and we write $\pi(g) = (r_{i,j}(g))$, then $\chi_\pi = \sum_{i=1}^n r_{i,i}$ as functions.

The **Ugly Identities** play a key role again. Let $\pi : G \to GL(n,\mathbb{C}), \rho : G \to GL(m,\mathbb{C})$ be continuous, bounded, irreducible, representations with $\pi(g) = (r_{i,j}(g)), \rho(g) = (t_{i,j}(g))$, then when π and ρ are not equivalent,

$$M(r_{i,k}\widetilde{t_{l,j}}) = 0,$$

and, when $\pi = \rho$,

$$M(r_{i,k}\widetilde{r_{l,j}}) = \frac{\delta_{i,j}\delta_{k,l}}{n}.$$

To prove these identities, we consider the $n \times m$ matrix $E_{k,l}$ and let $F(h) = \pi(h)E_{k,l}\rho(h^{-1}) = (r_{i,k}(h)t_{l,j}(h^{-1}))$. When π and ρ are inequivalent, then $M^{(n,m)}(F) = 0$.

If they are equivalent, then n=m and $M^{(n)}(F) = \lambda I_n$, for some $\lambda \in \mathbb{C}$. But, $n\lambda = Tr(M^{(n)}(F) = M(Tr(F)) = M(Tr(E_{k,l})) = \delta_{k,l}$, since $Tr(F(h)) = Tr(\pi(h)E_{k,l}\pi(h^{-1})) = Tr(E_{k,l}) = \delta_{k,l}$. Thus, $\lambda = \delta_{k,l}/n$ and examining the entries of $F(h)$ yields the result.

Inner Products

Given $f_1, f_2 \in C_b(G)$ we define an inner product by setting $(f_1|f_2) = M(f_1\bar{f_2})$. It is easily checked that it has the following properties.

Proposition 8.1. *Let G be an amenable group with an invariant mean M, and let $f_1, f_2, f_3 \in C_b(G)$ and let $\lambda \in \mathbb{C}$, then:*

- $(f_1|f_1) \geq 0$,
- $(f_1 + f_2|f_3) = (f_1|f_3) + (f_2|f_3), (f_1|f_2 + f_3) = (f_1|f_2) + (f_1|f_3)$,
- $(\lambda f_1|f_2) = \lambda(f_1|f_2), (f_1|\lambda f_2) = \bar{\lambda}(f_1|f_2)$,
- $(f_1|f_2) = \overline{(f_2|f_1)}$.

For a general amenable group, there can exist non-zero functions for which $(f|f) = 0$. In fact, $\mathbb{Z}$ is amenable and $(f|f) = 0$ for every $f \in C_c(\mathbb{Z})$. However, for compact groups we can say more.

Proposition 8.2. *Let G be a compact group and let M be the Haar mean. If $f \in C(G)$ and $(f|f) = 0$, then $f = 0$.*

Proof. Let $f \neq 0$, and say that $f(h_0) \neq 0$. then for $\delta = |f(h_0)|/2$, there exists an open set $U, h_0 \in U$ such that $|f(h)| > \delta$ for every $h \in U$.

The collection of sets, $\{gU : g \in G\}$ is an open cover of G and hence, there exist $g_1, \ldots, g_n \in G$ such that $G \subseteq g_1U \cup \cdots \cup g_nU$.

Let $f_i(h) = f(g_i^{-1}h) = g_i \cdot f$, so that each f_i is continuous and $M(|f_i|^2) = M(g_i \cdot |f|^2) = M(|f|^2) = (f|f)$.

For $h \in g_iU, h = g_iu, u \in U$ and so we have, $|f_i(h)| = |f(g_i^{-1}h)| = |f(u)| > \delta$. Thus, if we let $p(h) = |f_1(h)|^2 + \cdots + |f_n(h)|^2$, then $p(h) > \delta^2$ for every $h \in G$. Hence, by the positivity property, $M(p) \geq M(\delta^2 1) = \delta^2$, but $M(p) = nM(|f|^2)$. □

Lemma 8.3. *Let G be an amenable group, $\pi : G \to GL(n, \mathbb{C})$ be a continuous, bounded homomorphism, then for each $g \in G$ the eigenvalues of $\pi(g)$ are on the unit circle.*

Proof. Fix S so that $\rho(g) = S\pi(g)S^{-1} = (t_{i,j}(g))$ is upper triangular. Then each diagonal entry is an eigenvalue of $\pi(g)$. But since π is bounded, so is ρ and hence $t_{i.i}(g)^n = t_{i,i}(g^n)$ is bounded, which forces $t_{i,i}(g)$ to be on the unit circle. □

Theorem 8.4. *Let G be an amenable group, let $\pi_i : G \to GL(V_i), i = 1, 2$ be bounded, continuous, irreducible representations with $V_i, i = 1, 2$ finite dimensional, complex vector spaces. Then:*

- *when π_1 and π_2 are inequivalent, $(\chi_{\pi_1}|\chi_{\pi_2}) = 0$,*
- *when π_1 and π_2 are equivalent, $(\chi_{\pi_1}|\chi_{\pi_2}) = 1$.*

Proof. Let $\pi_1(g) = (r_{i,j}(g)), \pi_2(g) = (t_{i,j}(g))$, then $(\chi_{\pi_1}|\chi_{\pi_2}) = \sum_{i,j} M(r_{i,i}t_{j,j}) = 0$ by the first ugly identity, when the representations are inequivalent.

Note that for any bounded representation, since $\chi(g)$ is the sum of the eigenvalues of $\pi(g)$ and these are all of modulus one, we have that $\bar{\chi(g)} = \bar{\lambda_1} + \cdots + \bar{\lambda_n} = \lambda_1^{-1} + \cdots + \lambda_n^{-1} = \chi(g^{-1})$, where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $\pi(g)$.

If π_1 and π_2 are equivalent, then $\chi_{\pi_1}(g) = \chi_{\pi_2}(g)$ and hence, $(\chi_{\pi_1}|\chi_{\pi_2}) = (\chi_{\pi_1}|\chi_{\pi_1}) = \sum_{i,j=1}^n M(r_{i,i}\bar{r_{j,j}}) = \sum_{i,j=1}^n M(r_{i,i}\tilde{r_{j,j}}) = 1$, by the second ugly identity. □

Corollary 8.5. *Let G be an amenable group, let $\rho : G \to GL(V), \pi : G \to GL(W)$ be continuous, bounded representations on finite dimensional, complex vector spaces with π irreducible, then $dim(\mathcal{I}(\pi, \rho)) = (\chi_\rho|\chi_\pi)$.*

Proof. Write $V = V_1 + \cdots + V_k$ as a direct sum of irreducible subrepresentations, $\rho_1, \ldots, \rho_k$, then $\chi_\rho = \chi_{\rho_1} + \cdots + \chi_{\rho_k}$ and hence, $(\chi_\rho|\chi_\pi) = \sum_i(\chi_{\rho_i}|\chi_\pi) = \#\{i : \rho_i \sim \pi\} = dim(\mathcal{I}(\rho, \pi))$. □

Corollary 8.6. *Let G be an amenable group, let $\rho_i : G \to GL(V_i), i = 1, 2$ be continuous, bounded representations, with $V_i, i = 1, 2$ finite dimensional,complex vector spaces, then ρ_1 and ρ_2 are equivalent if and only if $\chi_{\rho_1} = \chi_{\rho_2}$.*

Corollary 8.7. *Let G be an amenable group, $\rho : G \to GL(V)$ be a continuous, bounded representation, with V a finite dimensional, complex vector space and assume that ρ is equivalent to $\pi_1^{(n_1)} \oplus \cdots \oplus \pi_k^{(n_k)}$, where $\pi_1, \ldots, \pi_k$ are irreducible, then $(\chi_\rho|\chi_\rho) = \sum_{i=1}^k n_i^2$.*

We will need the following result later.

Recall that a (complex) matrix is called *unitary* if $U^* = U^{-1}$ and a (real) matrix is *orthogonal* if $U^t = U^{-1}$. A representation, $\pi : G \to GL(n, \mathbb{C})$ is called a *unitary representation* if $\pi(g)$ is a unitary matrix for every $g \in G$ and $\pi : G \to GL(n, \mathbb{R})$ is called a *orthogonal representation* if $\pi(g)$ is a orthogonal matrix for every $g \in G$.

Finally, recall that given a given a (finite) dimensional vector space with an *inner product*, via the *Gramm-Schmidt orthogonalization process*, one can choose a basis that is also an *orthonormal set*, i.e., $(v_i|v_j) = 0$, when $i \neq j$, and $(v_i|v_i) = 1$, for all$i$.

Theorem 8.8. *Let G be an amenable group, and let $\pi : G \to GL(n, \mathbb{C})$ be a bounded, continuous homomorphism, then there exists an invertible matrix S, such that, $\rho(g) = S^{-1}\pi(g)S$ is a unitary representation. In the case that, $\pi : G \to GL(n, \mathbb{R})$, is a bounded, continuous homomorphism, there exists an invertible S, such that $\rho(g) = S^{-1}\pi(g)S$ is a orthogonal representation.*

Proof. We only prove the complex case.

For each pair of vectors $x, y \in \mathbb{C}^n$ define $f_{x,y} \in C_b(G)$ by setting $f_{x,y}(h) =< \pi(h)x, \pi(h)y >$, where ¡,¿ denotes the usual inner product on $\mathbb{C}^n$.

Note that these functions satisfy, $f_{x,x} \geq 0, f_{x_1+x_2,y} = f_{x_1,y} + f_{x_2,y}, f_{x,y_1+y_2} = f_{x,y_1} + f_{x,y_2}, f_{\lambda x,y} = \lambda f_{x,y}, f_{x,\lambda y} = \bar{\lambda} f_{x,y}$ for all vectors, x, x_1, x_2, y, y_1, y_2 and scalars λ.

We define, $[x, y] = M(f_{x,y})$ and claim that this is a new inner product on $\mathbb{C}^n$. To see this note that, $[x_1 + x_2, y] = M(f_{x_1+x_2,y}) = M(f_{x_1,y} + f_{x_2,y}) = [x_1, y] + [x_2, y]$. The other properties of an inner product follow similarly.

Now, for any fixed $g \in G, f_{\pi(g)x,\pi(g)y}(h) =< \pi(h)\pi(g)x, \pi(h)\pi(g)y >= f_{x,y}(hg) = (f_{x,y} \cdot g)(h)$. Hence, $[\pi(g)x, \pi(g)y] = M(f_{x,y} \cdot g) = M(f_{x,y}) = [x, y]$.

This last equation shows that $\pi(g)$ is a unitary matrix in the [,] inner product. To see this note that if $\{v_i\}$ is an orthonormal basis for $\mathbb{C}^n$ in this new inner product, then $[\pi(g)v_i, \pi(g)v_j] = [v_i, v_j]$ and so, $\{\pi(g)v_i\}$ is also an orthonormal set.

Now let $\{e_i\}$ be the usual orthonormal basis for $\mathbb{C}^n$ in the usual inner product, ¡,¿ and let $S \in GL(n, \mathbb{C})$ be defined by $Se_i = v_i, S^{-1}v_i = e_i$ be the matrix for this change of basis.

Given, $x = \sum_i \alpha_i e_i, y = \sum_j \beta_j e_j$, we have that $[Sx, Sy] = [\sum_i \alpha_i v_i, \sum_j \beta_j v_j] = \sum_{i,j} \alpha_i \bar{\beta}_j [v_i, v_j] = \sum_i \alpha_i \bar{\beta}_i =< x, y >$. Similarly, $< S^{-1}u, S^{-1}w >= [u, w]$.

Now let $\rho(g) = S^{-1}\pi(g)S$, we claim that ρ is unitary on $\mathbb{C}^n$ with the usual inner product. To see this, compute, $< \rho(g)e_i, \rho(g)e_j >=< S^{-1}\pi(g)v_i, S^{-1}\pi(g)v_j >=$

$[\pi(g)v_i, \pi(g)v_j] = [v_i, v_j] =< e_i, e_j >$. Thus, $\rho(g)$ carries an orthonormal set to an orthonormal set, and hence is unitary. $\square$

9. Tensor Products

We assume that the reader has seen the tensor product of vector spaces before and only review their key properties.

If V,W are vector spaces, then one can form a new vector space called their *tensor product,* $V \otimes W$. The tensor product is the linear span of vectors of the form, $\{v \otimes w : v \in V, w \in W\}$ that are called, *elementary tensors.* Elementary tensors obey the following rules:

- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$,
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$,
- $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$,

where $v, v_1, v_2 \in V, w, w_1, w_2 \in W$ and λ is a scalar.

Moreover, if $\{v_i : i \in I\}$ is a basis for V and $\{w_j : j \in J\}$ is a basis for W, then $\{v_i \otimes w_j : i \in I, j \in J\}$ is a basis for for $V \otimes W$. Consequently, $dim(V \otimes W) = dim(V)dim(W)$.

Finally, if $A \in \mathcal{L}(V)$ and $B \in \mathcal{L}(W)$ are linear maps, then there is a map denoted, $A \otimes B \in \mathcal{L}(V \otimes W)$ such that $A \otimes B(v \otimes w) = (Av) \otimes (Bw)$.

Now, if $C \in \mathcal{L}(V), D \in \mathcal{L}(W)$, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, and it follows that $(A \otimes I_W)(I_V \otimes B) = A \otimes B = (I_V \otimes B)(A \otimes I_W)$.

When V and W are finite dimensional vector spaces, then $V \otimes W$ is also finite dimensional and hence $A \otimes B$ can be represented by a matrix, as soon as we choose an ordered basis for $V \otimes W$. If $\{v_1, \dots, v_n\}$ is a basis for V and $A = (a_{i,j})$ is the $n \times n$ matrix for A, $\{w_1, \dots, w_m\}$ is a basis for W and $B = (b_{i,j})$ is the matrix for B, then $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes W$ and as soon as we order it, we may represent $A \otimes B$ by a matrix. When we order the basis for $V \otimes W$ by, $\{v_1 \otimes w_1, v_1 \otimes w_2, \dots, v_1 \otimes w_m, v_2 \otimes w_1, \dots, v_2 \otimes w_m, \dots, v_n \otimes w_1, \dots, v_n \otimes w_m\}$, then the $nm \times nm$ matrix for $A \otimes B$ is given in block form by,

$$\begin{pmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & & \vdots \\ a_{n,1}B & \dots a_{n,n}B & \end{pmatrix}.$$

The matrix obtained from $(a_{i,j}), (b_{i,j})$ in this fashion is called the *Kronecker tensor* of A and B. When we order the basis for $V \otimes W$ by, $\{v_1 \otimes w_1, \dots, v_n \otimes w_1, v_1 \otimes w_2, \dots, v_n \otimes w_2, \dots, v_1 \otimes w_m, \dots w_m\}$, then the block matrix for $A \otimes B$ in this basis is,

$$\begin{pmatrix} b_{1,1}A & \dots & b_{1,m}A \\ \vdots & & \vdots \\ b_{m,1}A & \dots & b_{m,m}A \end{pmatrix},$$

which is the Kronecker tensor of the matrices $(b_{i,j})$ and $(a_{i,j})$(with the order reversed).

Example 9.1. *Let* $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, *then the Kronecker tensor is*

$$A \otimes B = \begin{pmatrix} B & 2B \\ 0 & -B \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 0 & -2 & 0 \\ 1 & 0 & -1 & 2 & 0 & -2 \\ 0 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

while the Kronecker tensor,

$$B \otimes A = \begin{pmatrix} A & 2A & A \\ 0 & -A & 0 \\ A & 0 & -A \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 4 & 1 & 2 \\ 0 & -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 9.2. *Let G be a group and let $\pi : G \to GL(V)$ and $\rho : G \to GL(W)$ be homomorphisms, then setting $\gamma(g) = \pi(g) \otimes \rho(g)$, defines a homomorphism, $\gamma : G \to GL(V \otimes W)$. If G is a topological group, V and W are finite dimensional, and π, ρ are continuous, (bounded), homomorphisms, then γ is a continuous,(bounded), homomorphism.*

Proof. We have that $\gamma(gh) = \pi(gh) \otimes \rho(gh) = (\pi(g)\pi(h)) \otimes (\rho(g)\rho(h)) = (\pi(g) \otimes \rho(g))(\pi(h) \otimes \rho(h)) = \gamma(g)\gamma(h)$, and so γ is a homomorphism.

If V and W are finite dimensional, then we may choose bases so that $\pi(g) = (r_{i,j}(g)), \rho(g) = (t_{i,j}(g))$ and the fact that these maps are continuous (bounded) is equivalent to all of the functions $r_{i,j}, t_{i,j}$ being continuous (bounded) functions. But the matrix for γ, is of the form, $\gamma(g) = (r_{i,j}(g)t_{k,l}(g))$ and since the products of (bounded) continuous are again (bounded) and continuous, the proof is complete. □

We set $\pi \otimes \rho = \gamma$.

Note that when we consider the characters of these representations, then we have that,

$$\chi_{\pi\otimes\rho}(g) = \chi_\pi(g)\chi_\rho(g).$$

Thus, products of characters are again characters! Hence, the linear span of the character functions is an algebra of functions on G.

Problem 9.3. *Let π denote the 2-dimensional irreducible representation of D_4. Compute the decomposition of the 4-dimensional representation, $\pi \otimes \pi$ into a sum of irreducible representations. Let ρ denote the 4-dimensional representation of D_4 obtained as permutations on the 4 vertices of the square. Compute the decomposition of $\rho\otimes\rho$ into a sum of irreducible representations.*

There is one other way to obtain a new representation from an old representation that plays a role.

Let $\pi : G \to GL(n, \mathbb{C})$, say, $\pi(g) = (r_{i,j}(g))$, be a homomorphism and define $\bar{\pi} : G \to GL(n, \mathbb{C})$ by $\bar{\pi}(g) = (\bar{r_{i,j}})$. It is easily checked that $\bar{\pi}$ is also a homomorphism and that if G is a topological group and π is a continuous(bounded) homomorphism, then $\bar{\pi}$ is also a continuous(bounded) homomorphism.

Note that $\chi_{\bar{\pi}} = \overline{\chi_\pi}$, so that the complex conjugate of a character is also a character.

Problem 9.4. *Prove that if π is irreducible, then $\bar{\pi}$ is irreducible.*

10. The Peter-Weyl Theory for Compact Groups

In this section, we discuss a collection of results for compact groups attributed to Peter and Weyl. This collection of theorems form the basis for what is often referred to as "abstract harmonic analysis on groups". The key fact is that these theorems, generalize classical Fourier theory, which can be seen as the application of the Peter-Weyl theorems to the group $\mathbb{T}$.

Theorem 10.1 (Peter-Weyl 1)**.** *Let G be a compact group, let $g \neq h$ be elements of G. Then there exists n and a continuous, irreducible homomorphism, $\pi : G \to GL(n, \mathbb{C})$ such that $\pi(g) \neq \pi(h)$.*

Thus, there are enough continuous, irreducible representations on finite dimensional to separate the elements of G.

A proof of this theorem is beyond us at this stage, since the proof uses the theory of compact operators on on Hilbert space. However, we will be able to prove most of the rest of the Peter-Weyl theory using this theorem as the basis.

Given $\pi : G \to GL(n, \mathbb{C})$, write $\pi(g) = (r_{i,j}(g))$, the the functions, $r_{i,j}$ are called the *coefficients* of π.

Theorem 10.2 (Peter-Weyl 2)**.** *Let G be a compact group, then every continuous, complex-valued function on G, is a uniform limit of finite linear combinations of coefficients of finite dimensional, irreducible representations of G.*

Proof. We shall use the Stone-Weierstrass theorem, which says that if a set of continuous functions on a compact space X is an algebra that separates points, contains the function that is constantly equal to one, and is closed under the taking of conjugates, then it is uniformly dense in C(X). To this end we let $\mathcal{A} \subseteq C(G)$ denote the set of functions that are linear combinations of coefficients of irreducible representations.

Clearly, $\mathcal{A}$ is a vector space.

Now by the first Peter-Weyl theorem, given two points there is $\pi = (r_{i,j})$ such that $\pi(g) \neq \pi(h)$ and so for some coefficient, we must have that $r_{i,j}(g) \neq r_{i,j}(h)$, and thus $\mathcal{A}$ separates points.

Since the function that is constantly equal to 1, is a one-dimensional irreducible representation, it belongs to $\mathcal{A}$.

Now if $f_1 = r_{i,j}$ is a coefficient of the irreducible, π and $f_2 = t_{k,l}$ is a coefficient of the irreducible, representation, ρ, then $f_1 f_2$ is a coefficient of the representation $\pi \otimes \rho$. Now this later representation is not necessarily irreducible, but it can be decomposed into a direct sum of irreducible representations and this decomposition expresses $f_1 f_2$ as a sum of coefficients of the irreducible representations appearing in the decomposition of $\pi \otimes \rho$. Thus, $\mathcal{A}$ is an algebra of functions that separates points.

Finally, if f is a coefficient of π, then $\bar{f}$ is a coefficient of $\bar{\pi}$ and one can argue as in the last paragraph that this is a sum of coefficients of irreducible representation, or use the last problem. Thus, $\mathcal{A}$ is closed under the taking of conjugates. $\square$

Example 10.3. *Consider $G = \mathbb{T}$. We have shown that every irreducible of $\mathbb{T}$ is one-dimensional and they are all of the form, $\pi_n(z) = z^n$. Thus, by Peter-Weyl 2, we have that every continuous function on $\mathbb{T}$, can be uniformly approximated by a function of the form, $\sum_{n=-N}^{+N} a_n z^n$, where a_n are scalars.*

Example 10.4. *More generally, for $\mathbb{T}^k$, we have that every irreducible representation is of the form $\pi_N(z) = z^N$ where we are using multi-index notation, $z = (z_1, \ldots, z_k), N = (n_1, \ldots, n_k)$ and $z^N = z_1^{n_1} \cdots z_k^{n_k}$. Thus, by Peter-Weyl 2, every continuous function on $\mathbb{T}^k$ can be approximated by a function that is a finite sum of the form $\sum a_N z^N$, where a_N are scalars.*

There is another characterization of $\mathcal{A}$ that is often useful. Note that if $\pi : G \to GL(n, \mathbb{C})$ is a continuous, representation with $\pi(g) = (t_{i,j}(g))$, the coefficient functions and $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ are vectors in $\mathbb{C}^n$, then the function, $f(g) = \langle \pi(g)x, y \rangle = \sum a_i \bar{b_j} t_{i,j}(g)$ is in $\mathcal{A}$.

Conversely, if f is any function in $\mathcal{A}$, then f is a sum of coefficients of continuous representations and we claim that by considering the representation that is the direct sum of these representations and choosing vectors x and y, appropriately, we can write $f(g) = \langle \pi(g)x, y \rangle$.

We illustrate how to do this when $f = at_{1,2} + br_{3,4}$ where $\pi = (t_{i,j}), \rho = (r_{k,l})$. We have that $f(g) = \langle \pi(g)(ae_1), e_2 \rangle + \langle \rho(g)(be_3), e_4 \rangle = \langle (\pi(g) \oplus \rho(g))x, y \rangle$, where $x = (ae_1) \oplus (be_3), y = e_2 \oplus e_4$.

Thus, we have shown that $\mathcal{A} = \{\langle \pi(g)x, y \rangle : \pi \text{continuous}, x, y \text{vectors}\}$.

The Space $L^2(G)$

For these results, we will need some basic results about *Hilbert spaces.*

Let G be a compact group and let m be the unique, normalized Haar measure on G. The there are two ways that we can define the space, $L^2(G)$, for those familiar with measure theory,

$$L^2(G, m) = \{[f] : f measurable, \int_G |f|^2 dm < +\infty\},$$

where $[f] = [g]$ if and only if $f = g, a.e.m.$

For those unfamiliar, recall that for $f_1, f_2 \in C(G)$, setting $(f_1|f_2) = M(f_1\bar{f}_2) = \int_G f_1\bar{f}_2 dm$, defines an inner product on C(G) and so we have a natural metric on C(G), $d(f_1, f_2) = (f_1 - f_2|f_1 - f_2)^{1/2}$. The set C(G) is not complete in this metric and so it has a completion as a metric space and that is $L^2(G)$.

In either case, it is important to know that $L^2(G, m)$ is a complete, inner product space, i.e., a Hilbert space with inner product, $(f_1|f_2) = \int_G f_1\bar{f}_2 dm$, metric, $d(f_1, f_2) = (\int_G |f_1 - f_2|^2 dm)^{1/2}$ and $C(G) \subseteq L^2(G)$ is a dense subset.

Recall that if we are given a complete inner product space, V, then a collection of vectors $\{e_i : i \in I\}$ is called *orthonormal* provided, $(e_i|e_j) = 0$ when $i \neq j$ and $\|e_i\|^2 = (e_i|e_i) = 1$. An orthonormal set is called *complete* provided $v \in V$ and $(v|e_i) = 0$ for all $i \in I$, implies that $v = 0$. A complete orthonormal set is also called an *orthonormal basis* for V.

One of the key facts about orthonormal bases is Parseval's theorem, which is just a natural extension of Pythagoras' theorem.

Theorem 10.5 (Parseval)**.** *Let V be a Hilbert space and assume that $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for V. Then for every $v \in V$, we have:*

- $\|v\|^2 = \sum_{n=1}^{+\infty} |(v|e_n)|^2$,
- $v = \sum_{n=1}^{+\infty}(v|e_n)e_n$, *in norm.*

The second statement means that, $lim_{N\to+\infty}\|v - \sum_{n=1}^{N}(v|e_n)e_n\| = 0$.

Definition 10.6. *Given a compact group, G, we let $\widehat{G}$ denote the set of equivalence classes of finite dimensional, continuous, irreducible representations of G. The set $\widehat{G}$ is called the* **dual of G.** *A collection $\{\pi_\alpha : \alpha \in A\}$ of finite dimensional, continuous irreducible representations of G is called* **complete**, *provided that $\pi_\alpha \nsim \pi_\beta$ for $\alpha \neq \beta$, and $\widehat{G} = \{[\pi_\alpha] : \alpha \in A\}$, where $[\pi]$ denotes the equivalence class of a representation π.*

By an earlier, result, every continuous bounded representation of a compact group is equivalent to a unitary representation, so we may choose a complete set consisting of unitary representations.

Example 10.7. *For $G = \mathbb{T}$, we have that $\widehat{\mathbb{T}} = \{[\pi_n] : n \in \mathbb{Z}\}$, where $\pi_n(z) = z^n$. Thus, we may identify, $\widehat{\mathbb{T}}$ with $\mathbb{Z}$. Moreover, since π_n takes values on the unit circle, it is a unitary representation.*

Example 10.8. *More generally, for $\mathbb{T}^k$, we have that $\widehat{\mathbb{T}^k} = \{[\pi_N] : N \in \mathbb{Z}^k\}$, where $\pi_N(z) = z^N$, in multi-index notation. Thus, we may identify, $\widehat{\mathbb{T}^k}$ with $\mathbb{Z}^k$. Again this is a complete set of unitary representations.*

We can now state the third Peter-Weyl result.

Theorem 10.9 (Peter-Weyl 3)**.** *Let G be a compact group, and let $\pi_\alpha : G \to GL(n_\alpha, \mathbb{C}), \alpha \in A$ be a complete set of unitary representations for G, with $\pi_\alpha(g) = (t_{i,j}^{(\alpha)}(g))$ the coefficients of π_α. Then the set $\{\sqrt{n_\alpha}t_{i,j}^{(\alpha)} : 1 \leq i, j \leq n_\alpha, \alpha \in A\}$ is an orthonormal basis for $L^2(G)$.*

Before proving this theorem, we tie it into classical Fourier theory. By the above theorem, $\pi_n(z) = z^n$ is an orthonormal basis for $L^2(\mathbb{T}, m)$ where, recall that arc-length measure is the Haar measure for the $\mathbb{T}$. By identifying $\mathbb{T}$ with the interval, $[0, 2\pi)$ via the map, $t \to e^{it}$, we identify $L^2(\mathbb{T}, m)$ with $L^2([0, 2\pi), \mu)$ where $\mu([0, 2\pi)) = 1$, so that μ is $\frac{1}{2\pi}$ times Lebesgue measure. Thus, we see that the functions, $f_n(t) = \pi_n(e^{it}) = e^{int} = cos(nt) + isin(nt)$ are a complete orthonormal set with respect to this measure.

Similar results hold for $\mathbb{T}^k$ and $[0, 2\pi)^k$.

Proof. Since each π_α is unitary, $(t^{(\alpha)}_{i,j}(g^{-1})) = \pi_\alpha(g^{-1}) = \pi_\alpha(g)^* = (\overline{t^{(\alpha)}_{j,i}(g)})$. Thus, we have that, $\overline{t^{(\alpha)}_{j,i}} = \widetilde{t^{(\alpha)}_{i,j}}$.

Now by the ugly idenities, it follows that, $(t^{(\alpha)}_{i,j} | t^{(\beta)}_{k.l}) = M(t^{(\alpha)}_{i,j} \widetilde{t^{(\beta)}_{l,k}}) = \begin{cases} 0, \alpha \neq \beta \\ \frac{\delta_{i,k}\delta_{j,l}}{n_\alpha}, \alpha = \beta \end{cases}$, which proves that the set of functions is orthonormal.

We set, $e^{(\alpha)}_{i,j} = \sqrt{n_\alpha} t^{(\alpha)}_{i,j}$.

It remains to show that if $f \in L^2(G, m)$ and $(f | e^{(\alpha)}_{i,j}) = 0$ for all, i, j, α, then f=0.

Note that if $\{f_n\}$ is a sequence in $L^2(G, m)$ such that $\|f - f_n\| \to 0$, then $(f_n | \tilde{f}) \to (f | \tilde{f})$ for any $\tilde{f} \in L^2(G, m)$. By the second property of $L^2(G, m)$, we may choose such a sequence of functions with $f_n \in C(G)$. However, by Peter-Weyl 2, we may choose functions, $h_n \in \mathcal{A}$ such that, $|f_n(g) - h_n(g)| < 1/n$ for every $g \in G$. Hence, $\|f_n - h_n\|^2 = (f_n - h_n | f_n - h_n) = \int_G |f_n(g) - h_n(g)|^2 dm(g) \leq \int_G \frac{1}{n^2} dm = \frac{1}{n^2}$. Hence, $\|f_n - h_n\| < 1/n$ and so, $\|f - h_n\| \to 0$.

Note that since each h_n is a linear combination of coefficients, $(f | h_n) = 0$. But, $\|h_n\|^2 = (h_n | h_n) = (h_n | h_n) - (f | h_n) = (h_n - f | h_n) \leq \|f - h_n\| \cdot \|h_n\|$. Therefore, $\|h_n\| \leq \|f - h_n\| \to 0$, and so $\|h_n\| \to 0$.

Finally, $\|f\|^2 = (f | f) = lim_n (h_n | f) \leq lim_n \|h_n\| \cdot \|f\| = 0$ and it follows that $\|f\| = 0$ and so f=0. □

There are many other theorems that relate properties of $L^2(G, m)$ to properties of G. The following one is typical, but is beyond our current methods.

Theorem 10.10. *Let G be a compact group. Then the following are equivalent:*

(i) $\hat{G}$ *is countable,*
(ii) $L^2(G, m)$ *is a separable, Hilbert space,*
(iii) $L^2(G, m)$ *has a countable orthonormal basis,*
(iv) G *is a metrizable topological space.*

Those of you familiar with Hilbert spaces, should recognize that the equivalence of (ii) and (iii) is true in general. Also, that (i) implies (iii) follows from the Peter-Weyl 3. The fact that (iii) implies (i) follows from the fact that any two complete orthonormal sets in a Hilbert space, must have the same cardinality. So the real depth in this theorem is the equivalence of (i)-(iii) with (iv).

We close this section with a result that is useful for finding a complete set of unitary representations of a compact group.

Theorem 10.11. *Let G be a compact group and suppose that $\pi : G \to GL(n, \mathbb{C})$ is a continuous, unitary representation of G with the property that for any $g \neq h$, we have $\pi(g) \neq \pi(h)$, i.e., that π separates points. Let $\pi^{(m)} : G \to GL(n^m, \mathbb{C})$ denote the continuous unitary representation of G that is obtained by tensoring π with itself m times. Then every continuous, finite dimensional, irreducible representation of G is equivalent to a subrepresentation of $\pi^{(m)} \otimes \bar{\pi}^{(k)}$, for some integers m and k, where $\bar{\pi}$ denotes the complex conjugate of the representation π.*

Proof. Let $S = \{\pi_\alpha : \alpha \in A\}$ be a set that contains one representation from each equivalence class of irreducible, representations that can be obtained as subrepresentations of the set of representations, $\pi^{(m) \otimes \bar{\pi}^{(k)}}$.

Assume that $\pi_\alpha : G \to GL(n_\alpha, \mathbb{C})$ and that $\pi_\alpha = (t_{i,j}^{(\alpha)})$, so that the functions, $\{\sqrt{n_\alpha} t_{i,j}^{(\alpha)} : \alpha \in A\}$ are an orthonormal set in $L^2(G, m)$. We will show that these are a complete orthonormal set.

Note that linear combinations of these functions will contain all of the functions that occur as coefficients in any of the representations, $\pi^{(m)} \otimes \bar{\pi}^{(k)}$. But if $\pi = (t_{i,j})$, then using the Kronecker tensor, we see that the coefficients of $\pi^{(2)}$ are all products of any two of the functions, $t_{i,j} t_{k,l}$. Similarly, the coefficients of $\pi^{(m)}$ are all products of m coefficient functions. Thus, the coefficients of $\pi^{(m)} \otimes \bar{\pi}^{(k)}$ contain all products of m coefficient functions and of k complex conjugates of coefficient functions.

Thus, the linear combinations of the above set of orthonormal functions is the subalgebra $\mathcal{B}$ of $C(G)$, generated by the coefficients of π and their complex conjugates. Since π separates points, this subalgebra separates points and the complex conjugates of functions in $\mathcal{B}$ are again in $\mathcal{B}$. Finally, the constant function belongs to $\mathcal{B}$, since it is equal to $|det(\pi(g))|^2$, which belongs to $\mathcal{B}$. Hence, by the generalized Stone-Weierstrass theorem, $\mathcal{B}$ is uniformly dense in $C(G)$.

From this it follows that if a continuous function is orthogonal to the above orthonormal set, then that function is 0. But since the continuous functions are dense in $L^2(G, m)$, this set of functions is a complete orthonormal set.

Now suppose that S was not a complete set of irreducible representations. Then there would exist $\rho = (r_{i,j})$ an irreducible, unitary representation, that was not equivalent to any representation in S. But this would imply that the functions, $r_{i,j}$ are orthogonal to the functions obtained from S, a contradiction. Hence, S contains every a representative from every irreducible, finite dimensional, continuous representation of G. $\square$

Thus, for example, for the groups, $\mathcal{U}(n)$ and $\mathcal{SU}(n)$ we see that their identity representations as subsets of $GL(n,\mathbb{C})$, separate points. Hence every irreducible representation of these groups can be found by tensoring the identity representation and its complex conjugate representation with itself finitely many times and restricting to subspaces that are irreducible.

Class Functions

Let G be a compact group, then for any $g \in G$, the conjugacy class of g, $C_g = \{h^{-1}gh : h \in G\}$ is a compact subset of G, since it is the image of the compact set G under the continuous function, $f(h) = h^{-1}gh$.

Recall *conjugacy equivalence*, $g_1 \sim g_2$ if and only if $C_{g_1} = C_{g_2}$ which is also if and only if $g_2 \in C_{g_1}$. Since each equivalence class is closed, we get that the space of cosets, $\widetilde{G} = G/\sim$, is aalso a compact (Hausdorff) space. We let $q : G \to \widetilde{G}$ denote the quotient map.

A function $f : G \to \mathbb{C}$ is called a **class function** if it is constant on equivalence classes, i.e., if $f(h^{-1}gh) = f(g)$. We let $C_{inv}(G) \subseteq C(G)$ denote the set of continuous, class functions(in this setting inv stands for "invariant"). From general topology, we know that $f \in C_{inv}(G)$ if and only if there exists, $\tilde{f} \in C(\widetilde{G})$ such that $f = \tilde{f} \circ q$.

Proposition 10.12. *Let G be a compact topological group, then $C_{inv}(G)$ is a uniformly closed subalgebra of C(G).*

Proof. Easy! □

When G was a finite group, $C_{inv}(G)$ was our space, $H \subseteq \mathbb{C}(G)$.

Theorem 10.13 (Peter-Weyl 4)**.** *Let G be a compact group.*

(i) *Finite linear combinations of characters of continuous, finite dimensional, irreducible representations are uniformly dense in $C_{inv}(G)$,*
(ii) *If $f \in C_{inv}(G)$ and $(f|\chi_\pi) = 0$ for all such characters, then f=0.*

Proof. Given $f \in C_{inv}(G)$ and $\epsilon > 0$, there exists, $f_1 \in \mathcal{A}$, such that $|f(g) - f_1(g)| < \epsilon$ for every $g \in G$. By the remarks following Peter-Weyl 2, there exists a continuous, homomorphism, $\pi : G \to GL(n,\mathbb{C})$ and vectors, $x, y \in \mathbb{C}^n$, such that $f_1(g) = \langle \pi(g)x, y\rangle$.

Fix $g \in G$ and let $h \in G$ be arbitrary. Since $f(h^{-1}gh) = f(g)$, we have that $|f(g) - \langle \pi(h^{-1}gh)x, y\rangle| < \epsilon$. Regarding these as functions of h and integrating with respect to Haar measure, we obtain $|f(g) - \int_G \langle \pi(h^{-1}gh)x, y\rangle dm(h)| < \epsilon$. We now prove that the function represented by the integral is a sum of characteristic functions, which will complete the proof of (i).

Note that, $\int_G \langle \pi(h^{-1}gh)x, y\rangle = \langle [\int_G \pi(h^{-1})\pi(g)\pi(h)dm(h)]x, y\rangle$.

Recall that if π is irreducible and A is any matrix, then $\int_G \pi(h^{-1})A\pi(h)dm(h) = \frac{Tr(A)}{n} I_n$. So, if π was irreducible, then $\int_G \langle \pi(h^{-1})\pi(g)\pi(h)x, y\rangle dm(h) = \frac{\chi_\pi(g)}{n}\langle x, y\rangle$. For a general, π, write $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ with each π_i irreducible and on a space of dimension n_i, and decompose $x = x_1 \oplus \cdots \oplus x_k$

and $y = y_1 \oplus \cdots \oplus y_k$. Then we have that,

$$\int_G \langle \pi(h^{-1})\pi(g)\pi(h)x, y\rangle dm(h) = \sum_{i=1}^{k} \int_G \langle \pi_i(h^{-1})\pi_i(g)\pi_i(h)x_i, y_i\rangle dm(h) = \sum_{i=1}^{k} \frac{\chi_{\pi_i}(g)}{n_i}\langle x_i, y_i\rangle,$$

which completes the proof of (i).

To prove (ii), given $f \in C_{inv}(G)$, we can find functions, f_n that are finite linear combinations of characters such that $|f(g) - f_n(g)| < 1/n$ for all n and all $g \in G$. This implies that $\|f - f_n\|_2^2 = (f - f_n|f - f_n) = \int_G |f(h)-f_n(h)|^2 dm(h) < 1/n^2$. But since, $(f|\chi_\pi) = 0$, for all π, we have that $(f|f_n) = 0$, and hence, $0 = \lim_{n\to\infty}(f - f_n|f - f_n) = \lim_{n\to\infty}(f|f) + (f_n|f_n)$ and it follows that $\int_G |f(h)|^2 dm(h) = \|f\|_2^2 = 0$, But since f is continuous, by an earlier result, this implies that $f = 0$. $\square$

Definition 10.14. *Let G be a compact group, then $L^2_{inv}(G,m) = \{f \in L^2(G,m) : f(h^{-1}gh) = f(g), a.e.m\}$ is called the space of* **square integrable class functions**.

The following useful characterization of $L^2_{inv}(G)$ is sometimes used as the definition.

Proposition 10.15. *Let G be a compact group, then $L^2_{inv}(G)$ is the closure of $C_{inv}(G)$ in $L^2(G)$.*

Proof. Since $C(G) \subseteq L^2(G)$ is a dense subset, given any $f \in L^2_{inv}(G)$ there is a sequence of functions, $f_n \in C(G)$ such that, $\|f - f_n\| < 1/n$. Set $\tilde{f}_n(g) = \int_G f_n(h^{-1}gh)dm(h)$, then $\tilde{f}_n \in C_{inv}(G)$ and $\|f - \tilde{f}_n\| < 1/n$. $\square$

Corollary 10.16. *Let G be a compact group and let $\{\pi_\alpha : \alpha \in A\}$ be a complete set of finite dimensional irreducible representations for G. Then $\{\chi_{\pi_\alpha} : \alpha \in A\}$ is an orthonormal basis for $L^2_{inv}(G,m)$.*

Proof. Assume that $f \in L^2_{inv}(G)$ is perpendicular to all such characters, then we must show that $\|f\|_2 = 0$. By the above proposition, there exists $f_n \in C_{inv}(G)$ such that $\|f - f_n\|_2 \to 0$. But since each f_n can be approximated in sup norm by a linear combination of characters, we can assume that each f_n is actually a linear combination of characters. But then $(f|f_n) = 0$ and the fact that $\|f\| = 0$ follows as in earlier arguments. $\square$