Duality

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Outline

OUTLINE

1 FORMAL DUALITY AND DUALITY
- Theory
- An Application

2 NOMURA ALGEBRAS
- Type-II Matrices
- Spin Models

Title

Definition

Two association schemes are formally dual if the matrix of eigenvalues of one is the complex conjugate of the matrix dual eigenvalues of the other.

A Scheme on 512 Vertices

Let V be a vector space of dimension three over $GF(8)$. The 1-dimensional subspaces of V form a copy of $PG(2,8)$, which has 73 points. Let Ω be a hyperoval in $PG(2,8)$—10 points with no three collinear, 45 lines meet Ω in two points, 28 in zero.

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Construct a graph with vertex set V where two vectors are adjacent if and only if the 2-dimensional subspace they span lies in a parallel class corresponding to a point in Ω.

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Construct a graph with vertex set V where two vectors are adjacent if and only if the 2-dimensional subspace they span lies in a parallel class corresponding to a point in Ω.

This graph and its complement form an association scheme with two classes.

A Dual Scheme

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This scheme is dual to the first.

Duality Maps

Suppose $\mathcal{A}$ and $\mathcal{A}'$ are a formally dual pair of schemes, with

$$P_{\mathcal{A}} = \overline{Q}_{\mathcal{A}'}.$$

We define a map Θ from $\mathbb{C}[\mathcal{A}]$ to $\mathbb{C}[\mathcal{A}']$ by decreeing that

$$\Theta(A_i) = \sum_j p_i(j) A'_j.$$

Properties of Duality Maps

- $\Theta(A_i) = v\overline{E_i}$.

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- $\Theta(M \circ N) = \frac{1}{v}\Theta(M)\Theta(N)$.
- The matrix representing Θ relative to the bases provided by the Schur idempotents is P.

Self-dual Schemes

A scheme is formally self-dual if $P = \overline{Q}$. De Caen observed that if $\mathcal{A}$ and $\mathcal{A}'$ are formally dual, then

$$\mathcal{A} \otimes \mathcal{A}'$$

is formally self-dual.

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If $\mathcal{A}$ is formally self-dual with a duality map Θ and $\mathcal{B}$ is a subscheme of $\mathcal{A}$, then $\Theta(\mathcal{B})$ is formally dual to $\mathcal{B}$.

Outline

1. Formal Duality and Duality
   - Theory
   - An Application
2. Nomura Algebras
   - Type-II Matrices
   - Spin Models

$\mathbb{Z}_2^d$

Let V be the vector space $\mathbb{Z}_2^d$. We view V as a group of $2^d \times 2^d$ matrices A_u, for $u \in V$. Thus

$$A_u A_v = A_{u+v}$$

and so $A_u^2 = I$ and $A_0 = I$. These matrices form an association scheme with 2^d classes, its matrix of eigenvalues P satisfies $P^2 = vI$, and so it is formally selfdual.

If $C \subseteq V$, we use A_C to denote the sum

$$\sum_{u \in C} A_u.$$

Codes

Lemma

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$$\Theta\left(\sum_u x^{d-\mathrm{wt}(u)} y^{\mathrm{wt}(u)} A_u\right) = \sum_u (x+y)^{d-\mathrm{wt}(u)} (x-y)^{\mathrm{wt}(u)} A_u.$$

MacWilliams Theorem

Definition

If C is a linear code, its weight enumerator is the polynomial

$$W_C(x,y) := \sum_{u \in C} x^{n-\mathrm{wt}(u)} y^{\mathrm{wt}(u)}.$$

Theorem

If C is a binary linear code of length n, then

$$W_{C^\perp}(x,y) = \frac{1}{|C|} W_C(x+y, x-y).$$

A Proof of MacWilliams Theorem

Let $S = \sum_u x^{n-\mathrm{wt}(u)} y^{\mathrm{wt}(u)} A_u$. Then

$$\begin{aligned}
\mathrm{tr}(A_{C^\perp} S) &= 2^{-n}\,\mathrm{tr}(\Theta^2(A_{C^\perp} S)) \\
&= 2^{-n}\,\mathrm{sum}(\Theta(A_{C^\perp} S)) \\
&= 2^{-n}\,\mathrm{sum}(\Theta(A_{C^\perp}) \circ \theta(S)) \\
&= 2^{-n}|C^\perp|\,\mathrm{sum}(A_C \circ \Theta(S)) \\
&= |C|^{-1}\,\mathrm{sum}(A_C \circ \Theta(S)).
\end{aligned}$$

Outline

Definition of a Type-II Matrix

We use $M \circ N$ to denote the Schur product of matrices M and N, and we use $M^{(-)}$ to denote the Schur inverse of M.

Definition

A $v \times v$ complex matrix W is a *type-II matrix* if $WW^{(-)T} = vI$.

Examples of Type-II Matrices

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- Kronecker products of the above.

Nomura Algebras

Definition

If W is a Schur invertible matrix, we define vectors $W_{i/j}$ by

$$W_{i/j} = (We_i) \circ (We_j)^{(-)}.$$

The Nomura algebra $\mathcal{N}_W$ of W consists of the matrices for which all the vectors $W_{i/j}$ are eigenvectors.

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- $J \in \mathcal{N}_W$ if and only if W is a type-II matrix.
- $\mathcal{N}_W$ is a matrix algebra.

A Transform

Definition

Suppose W is a type-II matrix of order $v \times v$. If $M \in \mathcal{N}_W$, let $\Theta_W(M)$ be the $v \times v$ matrix such that

$$MW_{i/j} = (\Theta_W(M))_{i,j}\, W_{i/j}.$$

Thus if $M, N \in \mathcal{N}_W$, then

$$\Theta(MN) = \Theta(M) \circ \Theta(N).$$

We also see that $\Theta(I) = J$ and $\Theta(J) = vI$.

Duality

Theorem

If W is a type-II matrix and $M, N \in \mathcal{N}_W$, then $\Theta_W(M) \in \mathcal{N}_{W^T}$ and $\Theta_W(M \circ N) = \frac{1}{v}\Theta_W(M)\Theta_W(N)$.

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Corollary

If W is a type-II matrix, then $\mathcal{N}_W$ is the Bose-Mesner algebra of an association scheme, and $\mathcal{N}_{W^T}$ is the Bose-Mesner algebra of a scheme dual to $\mathcal{N}_W$.

Outline

Some Action

If $A, B \in \mathrm{Mat}_{v\times v}(\mathbb{C})$, we define linear maps X_A and Δ_B by

$$X_A(M) := AM, \quad \Delta_B(M) := B \circ M.$$

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If $\mathcal{A}$ is an association scheme then the algebra generated by the matrices

$$X_M,\ \Delta_M,\ M \in \mathbb{C}[\mathcal{A}]$$

is essentially the Terwilliger algebra of $\mathcal{A}$.

Another View of the Nomura Algebra

Theorem

If W is a type-II matrix then the following are equivalent:

- $R \in \mathcal{N}_W$.
- *There is a matrix S such that* $X_R \Delta_{W^{(-)T}} X_W = \Delta_{W^{(-)T}} X_W \Delta_S$.
- $X_R \Delta_{W^{(-)T}} X_W = \Delta_{W^{(-)T}} X_W \Delta_S$ *and* $S = \Theta_W(R)$.

Duality as an Endomorphism of $\mathrm{Mat}_{v\times v}(\mathbb{C})$

Corollary

If W is type-II and $R \in \mathcal{N}_W$, then

$$\Theta_W(R) = W^{-1}(W \circ (R(W^{(-)} \circ (WJ)))).$$

Spin Models

Definition

A type-II matrix W is a spin model if $W \in \mathcal{N}_W$.

If W is a spin model, then $\mathcal{N}_W = \mathcal{N}_{W^T}$.

Lemma

If W is a spin model then

$$X_W \Delta_{W(-)} X_W = \Delta_{W(-)} X_W \Delta_{W(-)}.$$

EXAMPLES OF SPIN MODELS

CYCLIC Let θ be a primitive $2v$-th root of unity, and define the $v \times v$ matrix W by

$$W_{i,j} := \theta^(i - j)^2.$$

Then W is a spin model.

Examples of Spin Models

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Jaeger Let A be the adjacency matrix of the Higman-Sims graph. There are scalars s and t such that

$$I + sA + t(J - I - A)$$

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Nomura The Bose-Mesner algebra of a distance-regular double cover of $K_{n,n}$ with index two contains a spin model.

The Braid Group

The braid group on three strands is generated by elements σ and τ subject to the relation

$$\sigma\tau\sigma = \tau\sigma\tau.$$

It follows that

$$(\sigma\tau\sigma)^2 = (\sigma\tau)^3$$

and therefore $(\sigma\tau\sigma)^2$ is central.

Spin and Braids

If W is a spin model then the assignments

$$\sigma \mapsto X_W, \quad \tau \mapsto \Delta_{W(-)}$$

give a representation of the braid group B_3 (and its trace gives rise to a link invariant).

Conjugation and Duality

Theorem (Godsil and Chan)

If W is a spin model, then

$$X_{W^T}\Delta_{W^{(-)}}X_W = \Delta_{W^{(-)}}X_W\Delta_{W^{(-)T}}.$$

If Λ denotes either side of this identity and $R \in \mathcal{N}_W$, then

$$\Lambda^{-1}X_R\Lambda = \Delta_{\Theta_W(R)}.$$

Conjugation by Λ^2 is the transpose map, and Λ^4 is central.

REMARKS AND PROBLEMS

- Type-II matrices occur widely in combinatorics: symmetric designs, equiangular lines, strongly regular graphs.
- We have no examples of a type-II matrix W where $\mathcal{N}_W$ is interesting and W is not a spin model.