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Chapter 1

Incidence Structures

A reading course for projective geometry, based in large part on these notes. Use any source of help you can find. Collaboration is recommended.

1.1 Axioms

Read Section 1.1 on incidence structures.

1. Suppose $\mathcal{P}$ is a projective plane, possibly degenerate. Prove that the following are equivalent:
   (a) $\mathcal{P}$ contains a 4-arc.
   (b) The incidence graph of $\mathcal{P}$ is thick.

2. Prove that an incidence structure is a projective plane if and only if its incidence graph is bipartite with diameter three and girth six.

3. Determine the eigenvalues of the incidence graph of a (non-degenerate) projective plane.

4. Show that the set of points and lines fixed by a group of collineations of a projective plane is a projective plane, possibly degenerate. Determine the degenerate projective planes.

5. Let p be a point and ℓ a line in the projective plane $\mathcal{P}$ of order n. Let X be the subgraph of the incidence graph of $\mathcal{P}$ induced the points not on ℓ and the lines not on p. Show that X is an antipodal distance-regular cover of $K_{n.n}$. [The converse also holds.]

6. A *generalized quadrangle* is a partial linear space such that if p is a point and ℓ is a line not on q, there is a unique point on ℓ collinear with p. If thick, the incidence graph is semiregular (see e.g., G&R: AGT). Determine what the possibilities are if the incidence graph is not thick.

7. Prove that an incidence structure is a generalized quadrangle if and only if its incidence graph has diameter four and girth eight.

8. Let $\mathbb{E}$ be an extension field of $\mathbb{F}$ with degree three and assume $q = |\mathbb{F}|$. Then $\mathbb{E}$ is a 3-dimensional vector space over $\mathbb{F}$, and its 1- and 2-dimensional vector subspaces form a projective plane $\mathcal{P}$. Use the fact that the multiplicative group of a finite field is cyclic to show that there is a cyclic group of collineations of $\mathcal{P}$ of order q^2+q+1, acting regularly on the points of $\mathcal{P}$. [Hence we may assume that the incidence matrix of $\mathcal{P}$ is a circulant.]

Read Sections 4.4–5.4.

1.2 Collineations

A *collineation* of an incidence structure is an automorphism of its incidence graph that map each of the colour classes to itself. So it is a pair (P, B) of permutations such that P acts on the point and B acts on blocks and incidence is preserved. In matrix terms, if N is the incidence matrix of the structure, then (P, B) is a collineation if and only if $PNQ^T = N$.

Let $(\mathcal{P}, \mathcal{B})$ be an incidence structure with an incidence matrix N. Let ρ be a partition of $\mathcal{P}$ with characteristic matrix R. The i-th entry in the column of $R^T N$ corresponding to the block β is the number of points incident with β that lie in the i-th cell of ρ. Let ρ^* be the partition of $\mathcal{B}$, where two blocks lie in the same cell if and only if the corresponding columns of $R^T N$ are equal. We say ρ^* is the partition *induced* by ρ.

1.2.1 Theorem. *Let $(\mathcal{P}, \mathcal{B})$ be an incidence structure, let ρ be a partition of its points and let ρ^* be the induced partition of its blocks. If the rows of the incidence matrix of $(\mathcal{P}, \mathcal{B})$ are linearly independent, then $|\rho| \le |\rho^*|$.* □

1.2.2 Corollary. *Let $\mathcal{I}$ be an incidence structure and assume that the rows of its incidence matrix are linearly independent. If Γ is a group of* *collineations of $\mathcal{I}$, the number of orbits of Γ on blocks is at least as large as the number of orbits on points.* □

1. Let γ be a non-identity collineation of a projective plane of order n that fixes all points on some line ℓ. Show that either:

   (a) γ fixes exactly one point not on ℓ, and the order of γ divides $n-1$, or

   (b) γ does not fix any point off ℓ, and the order of γ divides n.

   A collineation as in (a) is known as a *homology*, in (b) we have an *elation*.

2. Characterize the structures that arise as the set of fixed points and fixed lines of a group of collineations of a generalized quadrangle.

Chapter 2

Coordinates

A projective space over a field $\mathbb{F}$ is the incidence structure with the 1-dimensional subspace of the vector space $V(d, \mathbb{F})$ as its points, and the 2-dimensional subspaces as its lines. We denote it by $PG(d-1, \mathbb{F})$. The *dimension* of the projective space is $d-1$, we refer to d as its *rank* (especially if we are talking to a matroid theorist).

An affine space over $\mathbb{F}$ has the vectors of the vector space $V(d, \mathbb{F})$ as points, and cosets of the 1-dimensional vector spaces as lines.

A *hyperplane* in a partial linear space is a set of points H such if ℓ is a line that contains to points of H, then all points on ℓ lie in H. The hyperplanes in $PG(d, \mathbb{F})$ correspond to the subspace of $V(d+1, \mathbb{F})$ with codimension one.

Each invertible element of $\mathrm{Mat}_{d\times d}(\mathbb{F})$ determines a collineation of $PG(d-1, \mathbb{F})$; two invertible matrices determine the same collineation if one matrix is a non-zero scalar timrs the other. We define a *projective linear collineation* to be an element of $GL(d, \mathbb{F})$ modulo its centre, which consists of the non-zero scalar mappings. We denote this quotient by $PGL(d, \mathbb{F})$; it is the *projective linear group*.

There is a second class of collineations—any field automorphism gives a collineation. The group formed by all compositions of field automorphisms and projective linear maps is denoted $P\Gamma L(d, \mathbb{F})$. The fundamental theorem of projective geometry asserts that any collineation of a projective space over a field is the composition of a field automorphism with a linear map.

Each point in $PG(d-1, \mathbb{F})$ can be represented a non-zero vector in $V(d, \mathbb{F})$, two such vectors represent the same point if one vector is a non-zero scalar multiple of the other. If x and y are vectors, neither a scalar multiple of the other, then the non-zero vectors of the form

$$\lambda x + \mu y, \quad \lambda, \mu \in \mathbb{F}$$

represent the points of the line $x \vee y$.

If a is a non-zero vector in $V(d, \mathbb{F})$, then the kernel of the $1 \times d$ matrix a^T is a hyperplane. (More precisely, the 1-dimensional subspaces of $\ker(a^T)$ are the points of a hyperplane.)

A set S of m points can be represented by a $d \times m$ matrix M of $\mathbb{F}$ with distinct columns, with points given by the columns. The set of points represented does not change if we permute the columns of M, nor does it change if multiply columns by non-zero scalars. In other word, if Q of a permutation matrix with a non-zero diagonal matrix, then M and MQ determine the same points.

If H is a hyperplane with coordinate vector a^T, then the zero entries of $a^T M$ correspond to the points in S that lie in H. Thus we read of the sizes of the interesections $H \cap S$ from the elements of the row space of M. The row space of M is the *code* of M. The kernel of M is the *dual code*; since the columns of M are distinct, the minimum distance of the dual code is at least three. A linear code whose dual has minimum distance three is called a *projective code*. If $P \in GL(d, \mathbb{F})$, then M and PM have the same row space.

A Cayley graph for the additive group of a vector space is *linear* if its connection set is closed under multiplication by non-zero scalars. Thus the connection set of a linear Cayley graph corresponds to a set of points in $PG(d-1, \mathbb{F})$, and so each set of m points in a projective space $PG(d-1, \mathbb{F})$ corresponds to a linear Cayley graph with vertex set $\mathbb{F}^d$ and valency $m(|\mathbb{F}| - 1)$.

1. If $x, y \in V(d, \mathbb{F})$ and $x \neq y$, prove the points of the line $x \vee y$ are given by the vectors of the form $\lambda x + (1-\lambda)y$, for $\lambda \in \mathbb{F}$.

2. Assume $|\mathbb{F}| = q$ and let B be the vertex-edge incidence matrices of an orientation of a graph X on d vertices. If there is a vector a such that no entry of $a^T M$ is zero, show that X has a q-colouring.

3. Let S be a 4-arc in $PG(2, \mathbb{F})$. Prove that a projective linear mapping which fixes each point of S is the identity.

4. Let $\mathbb{F}$ be finite field of order q and suppose $q = p^k$ for some prime p. We use tr to denote the trace map from $\mathbb{F}$ to the field $\mathbb{Z}_p$. Let θ be a primitive p-th root of unitary. Let M be a $d \times m$ matrix over $\mathbb{F}$ and let $X = X(M)$ be the linear Cayley graph corresponding to M. Show that the map
$$x \mapsto \theta^{\mathrm{tr}(a^T x)}$$
is a character (on $\mathbb{F}^d$), and express its eigenvalue in terms of the Hamming weight of $a^T M$.

5. Show that $X(M)$ is strongly regular if and only the code of M has exactly two non-zero weights.

6. Let $V = V(d, q)$ and let U be a subspace with dimension e. Determine the number of 2-dimensional subspaces L such that $L \cap U = \langle 0 \rangle$.

7. Assume $V = V(d, q)$ and $d = 2e$. Let U_0 and U_1 be two subspaces of V with dimension e such that $U_0 \cap U_1 = \langle 0 \rangle$. Show that each point a of V not on U_0 or U_1 determines a collineation from U_0 to U_1.