Chapter 33

Matrix Perturbation Theory

Let A and B be $n \times n$ Hermitian matrices. Our aim is to derive information about the eigenvalues of matrices of the form $A + tB$, for real t. The set of matrices

$$\{A + tB : t \in \mathbb{R}\}$$

is called a *matrix pencil*. If $\mathrm{rk}(B) = k$, we say that $A + B$ is a *rank-k update* of X.

We are interested in matrix pencils where A and B are $n \times n$ Hermitian and t is real, we call these *Hermitian pencils*. It is known [**?**] that, for a Hermitian pencil, there is an integer m and analytic functions $\theta_1(t), \dots, \theta_m(t)$ such that $\theta_i(t)$ is an eigenvalue of $A + tB$ (for each i). There are corresponding orthogonal projections $F_1(t), \dots, F_m(t)$; these are analytic functions of t and $F_r(t)$ is the projection onto the $\theta_r(t)$-eigenspace of $A + tB$.

33.1 Basics

The eigenvalues of a matrix A are continuous functions of the entries of A. If the entries of $A = A(t)$ are analytic functions of t, of are the eigenvalues A

For any matrices A and B, there is an open neighbourhood of zero such that

$$\mathrm{rk}(A + tB) \geq \mathrm{rk}(A).$$

The key is that if A is invertible, then $I + tA^{-1}B$ is invertible for small t, and so

$$A + tB = A(I + tA^{-1}B)$$

is invertible.

33.2 Rank-1 Updates

Let A be a Hermitian matrix with spectral decomposition $A = \sum_r \theta_r E_r$. The *eigenvalue support* of a vector x is the set

$$\{r : E_r x \neq 0\}.$$

Note that, since E_r is a projection, $E_r x = 0$ if and only if $x^* E_r x = 0$.

33.2.1 Theorem. *Let A be a Hermitian matrix and let x be a vector in $\mathbb{C}^n$. Then*

(a) The eigenvalues of $A + txx^T$ interlace the eigenvalues of A.

(b) The function $\theta_r(t)$ is constant if and only if $\theta_r(0)$ is not in the eigenvalue support of x.

33.3 Commutants

Let A be Hermitian with spectral decomposition $A = \sum_r \theta_r E_r$. Define a map Ψ_A on $\text{Mat}_{n\times n}(\mathbb{C})$ by

$$\Psi(M) := \sum_r E_r M E_r.$$

33.3.1 Theorem. *If A is Hermitian, Ψ is orthogonal projection on the commutant of A.*

Proof. As $E_r M E_r$ commutes with A, it is immediate that the image of Ψ lies in the commutant of A. If M commutes with A, it commutes with each idempotent E_r and accordingly

$$M = IMI = \sum_{r,s} E_r M E_s.$$

If $r \neq s$, then $E_r M E_s = M E_r E_s = 0$, and therefore the commutant of A is the imnage of Ψ.

It is also clear that $\Psi^2 = \Psi$, so Ψ is idempotent. Now if $M, N \in \text{Mat}_{n\times n}(\mathbb{C})$, then

$$\begin{aligned}
\langle N, \Psi(M)\rangle = \operatorname{tr} N^T \Psi(M) &= \sum_r \operatorname{tr}(N^T E_r M E_r) \\
&= \sum_r \operatorname{tr}(E_r N^T E_r M) \\
&= \langle \Psi(N), M\rangle
\end{aligned}$$

and so Ψ is self-adjoint.

If A is diagonal, then its Schur idempotents are diagonal 01-matrices. If the i-th eigenvalue of A is θ_i and has mutiplicity m_i (for $i = 1,\dots,k$), then the commutant of A consists of the block-diagonal matrices with k blocks, where the i-th block is $m_i \times m_i$. (Hence the dimension of the commutant is $\sum_i m_i^2$.) The orthogonal complement to the commutant consists of the matrices Schur-orthogonal to the block-diagonal matrix

$$J_{m_1} \oplus \cdots \oplus J_{m_k}.$$

If H commutes with A, we can express the eigenvalues of $A + tB$ in terms of the eigenvalues of A and B. To help with determining the eigenvalues of the pencil when A and B do not commute, we describe a more complicated way of getting at the eigenvalues in the commutative case.

Assume A has spectral decomposition

$$A = \sum_r \theta_r E_r$$

If B commutes with A, then each eigenspace of A is B-invariant and therefore has an orthogonal basis formed from eigenvectors of B. Let E be a spectral idempotent of A and assume its rank is m and that the corresponding eigenvalue is θ. There is an $n \times m$ matrix U such that $U^*U = I_m$ and $UU^* = E$; its column space is the eigenspace associated with E. The matrix that represents the restriction of B to $\operatorname{col}(U)$ is U^*BU and, if its eigenvalues are

$$\nu_1,\dots,\nu_r$$

with respective multiplicities

$$\mu_1,\dots,\mu_r,$$

the eigenvalues of the restriction of $A + tB$ to $\operatorname{col}(U)$ are

$$\theta + t\nu_1, \cdots, \theta + t\nu_r$$

with multiplicities as above. We will establish very similar expressions in the case where A and B need not commute.

33.4 The General Case

We assume A has spectral decomposition $A = \sum_i \theta_i E_r$, and that θ_i has multiplicity m_i. Assume $E_r = U_r U_r^*$, as before. Let B_0 be the orthogonal projection of B onto the commutant of A and set $B_1 = B - B_0$. Then

$$B_0 = \sum_i E_i B E_i.$$

and

$$E_i B_1 E_i = 0$$

for all i. There is a constant $\epsilon > 0$ such that the eigenspaces of $A + tB_0$ are the same for all t in the open interval $(0, \epsilon)$, and hence in this interval the projections onto the eigenspaces of $A + tB_0$ are independent of t. Hence if F represents projection onto one of these eigenspaces, then $FB_1F = 0$.

We now appeal to Theorem 7.9.1 of Lancaseer "Theory of Matrices", which tells us that if $\sum_i E_i B E_i = 0$, then the linear terms in the series expansions of the eigenvalues of $A + tB$ are zero. Equivalently, the linear terms depend only on the eigenvalues of $\sum_i E_i B E_i$.

The Inertia Bound for $L(K_5)$

For any graph X the size of a coclique is bounded (above) by the minimum of the number of non-negative eigenvalues and the number of non-positive eigenvalues of any symmetric matrix S such that $S_{i,j} \neq 0$ only if $ij \in E(X)$. This is the *inertia bound*, which is due to Dragos Cvetkovic. Following Elzinga, we prove that the inertia bound for $L(K_5)$ is tight.

We take the vertex set of K_5 to be $\{0, 1, 2, 3, 4\}$. There is a partition of the vertices of $L(K_5)$ into three cliques. The first clique consists of the vertices that contain 0. The remaining six vertices split into the K_3 formed by the vertices that contain 1, and a second K_3 formed by the vertices that do not contain 1. These three cliques form a spanning subgraph of $L(K_5)$ with 12 edges; let C denote its adjacency matrix. Let A be the adjacency matrix of $L(K_5)$.

Now compute the eigenvalues of the matrix

$$C + t(A - C).$$

In sage we can do this with the magic words

```
import scipy
from scipy import linalg

lk5 = graphs.CompleteGraph(5).line_graph()
A = lk5.am()
c0 = lk5.subgraph([vx for vx in lk5.vertices() if 0 in vx])
c1 = lk5.subgraph([vx for vx in lk5.vertices() if 0 not in vx and 1 in vx])
c2 = lk5.subgraph([vx for vx in lk5.vertices() if 0 not in vx and 1 not in vx])
clqs = c0.disjoint_union(c1.disjoint_union(c2))
C = clqs.am()

def evals(t):
return vector(linalg.eigh( C +t*(A-C), eigvals_only = True))

# sage: evals(-0.9)
# (-2.8, -2.8, -1.31661554144, -1.01151297144, -0.1, -0.1, -0.1, -0.1, 4.11661554144, 4.21151297144)
```

If $C + t(A - C))$ is not invertible, $-1/t$ is an eigenvalue of $C^{-1}(A - C)$; we calculate these by

```
vector(linalg.eig( C^(-1)*(A-C), right=False))
```

$$(1.61803398875, -0.5 + 0.866025403784 * I, -0.5 - 0.866025403784 * I, -0.61803398875, 1.0, -2.0, -2.0, 1.0, 1.0, 1.0)$$

Here the real eigenvalues determine the values where the inertia of $C + t(A - C)$ changes. They are:

$$t = -1, \; -0.618, \; 0.5, \; 1.618$$

Now we find the eigenvalues of $C + t(A - C))$ for t in $-1.5, -1.4, \ldots, 1.9, 2.0$, and we see that if *t* is one of -0.9, -0.8, -0.7 then $C + t * (A - C)$ has only two non-negative eigenvalues. Therefore the inertia bound is tight for $L(K_5)$. [Each row of the table is the value of t followed by the 10 eigenvalues.]

-1.5	-4.0	-4.0	-3.18330013267	-1.67423461417	0.5	0.5	0.5	0.5	5.18330013267	5.67423461417
-1.4	-3.8	-3.8	-2.82045915678	-1.61140997322	0.4	0.4	0.4	0.4	5.02045915678	5.41140997322
-1.3	-3.6	-3.6	-2.45786823163	-1.54962684489	0.3	0.3	0.3	0.3	4.85786823163	5.14962684489
-1.2	-3.4	-3.4	-2.09558536927	-1.48904374382	0.2	0.2	0.2	0.2	4.69558536927	4.88904374382
-1.1	-3.2	-3.2	-1.73368792320	-1.42985148151	0.1	0.1	0.1	0.1	4.5336879232	4.62985148151
-1.0	-3.0	-3.0	-1.37228132327	-1.37228132327	0.0	0.0	0.0	0.0	4.37228132327	4.37228132327
-0.9	-2.8	-2.8	-1.31661554144	-1.01151297144	-0.1	-0.1	-0.1	-0.1	4.11661554144	4.21151297144
-0.8	-2.6	-2.6	-1.2632011236	-0.651595203261	-0.2	-0.2	-0.2	-0.2	3.8632011236	4.05159520326
-0.7	-2.4	-2.4	-1.21246761636	-0.3	-0.3	-0.3	-0.3	-0.292844953646	3.61246761636	3.89284495365
-0.6	-2.2	-2.2	-1.16495033058	-0.4	-0.4	-0.4	-0.4	0.0642440249314	3.36495033058	3.73575597507
-0.5	-2.0	-2.0	-1.12132034356	-0.5	-0.5	-0.5	-0.5	0.418861169916	3.12132034356	3.58113883008
-0.4	-1.8	-1.8	-1.08242276016	-0.6	-0.6	-0.6	-0.6	0.769586530435	2.88242276016	3.43041346957
-0.3	-1.6	-1.6	-1.04932420089	-0.7	-0.7	-0.7	-0.7	1.11372195088	2.64932420089	3.28627804912
-0.2	-1.4	-1.4	-1.02336879396	-0.8	-0.8	-0.8	-0.8	1.44559962547	2.42336879396	3.15440037453
-0.1	-0.1	-1.2	-1.00623784042	-0.9	-0.9	-0.9	-0.9	1.75192593016	2.20623784042	3.04807406984
0.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	2.0	2.0	3.0
0.1	-1.1	-1.1	-1.1	-1.1	-1.00712472795	-0.8	-0.8	1.80712472795	2.13095842402	3.06904157598

0.2	-1.2	-1.2	-1.2	-1.2	-1.03041346957	-0.6	-0.6	1.63041346957	2.12554373535	3.27445626465
0.3	-1.3	-1.3	-1.3	-1.3	-1.07279220614	-0.4	-0.4	1.47279220614	2.03842268941	3.56157731059
0.4	-1.4	-1.4	-1.4	-1.4	-1.13693168769	-0.2	-0.2	1.33693168769	1.91511421982	3.88488578018
0.5	-1.5	-1.5	-1.5	-1.5	-1.22474487139	0	0	1.22474487139	1.77525512861	4.22474487139
0.6	-1.6	-1.6	-1.6	-1.6	-1.33693168769	0.2	0.2	1.13693168769	1.62690801373	4.57309198627
0.7	-1.7	-1.7	-1.7	-1.7	-1.47279220614	0.4	0.4	1.07279220614	1.47373234984	4.92626765016
0.8	-1.8	-1.8	-1.8	-1.8	-1.63041346957	0.6	0.6	1.03041346957	1.31757723984	5.28242276016
0.9	-1.9	-1.9	-1.9	-1.9	-1.80712472795	0.8	0.8	1.00712472795	1.15946434976	5.64053565024
1.0	-2.0	-2.0	-2.0	-2.0	-2.0	1.0	1.0	1.0	1.0	6.0
1.1	-2.20623784042	-2.1	-2.1	-2.1	-2.1	0.839565251632	1.00623784042	1.2	1.2	6.36043474837
1.2	-2.42336879396	-2.2	-2.2	-2.2	-2.2	0.678411014052	1.02336879396	1.4	1.4	6.72158898595
1.3	-2.64932420089	-2.3	-2.3	-2.3	-2.3	0.516708968124	1.04932420089	1.6	1.6	7.08329103188
1.4	-2.88242276016	-2.4	-2.4	-2.4	-2.4	0.354580419753	1.08242276016	1.8	1.8	7.44541958025
1.5	-3.12132034356	-2.5	-2.5	-2.5	-2.5	0.192113447068	1.12132034356	2.0	2.0	7.80788655293
1.6	-3.36495033058	-2.6	-2.6	-2.6	-2.6	0.0293735125905	1.16495033058	2.2	2.2	8.17062648741
1.7	-3.61246761636	-2.7	-2.7	-2.7	-2.7	-0.133589736004	1.21246761636	2.4	2.4	8.533589736
1.8	-3.8632011236	-2.8	-2.8	-2.8	-2.8	-0.296737973824	1.2632011236	2.6	2.6	8.89673797382
1.9	-4.11661554144	-2.9	-2.9	-2.9	-2.9	-0.460041152089	1.31661554144	2.8	2.8	9.26004115209
2.0	-4.37228132327	-3.0	-3.0	-3.0	-3.0	-0.62347538298	1.37228132327	3.0	3.0	9.62347538298