Periodic Graphs

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Outline

1. Periodicity & State Transfer
2. Some Results
3. Some Questions
Unitary Operators

Suppose $X$ is a graph with adjacency matrix $A$.

**Definition**

We define the operator $H_X(t)$ by

$$H_X(t) := \exp(iAt).$$
An Example

We have

$$H_{K_2}(t) = \begin{pmatrix} \cos(t) & i \sin(t) \\ i \sin(t) & \cos(t) \end{pmatrix}$$

Note that $H_X(t)$ is symmetric, because $A$ is, and unitary because

$$H_X(t)^* = \exp(-iAt) = H_X(t)^{-1}.$$
Probability Distributions

If $H$ is unitary, the Schur product

$$H \circ \overline{H}$$

is doubly stochastic. Hence each row determines a probability density. (It determines a continuous quantum walk.)
State Transfer

**Definition**

We say that perfect state transfer from the vertex $u$ to the vertex $v$ occurs at time $\tau$ if

$$|(H_X(\tau))_{u,v}| = 1.$$  

Example:

$$H_{K_2}(\pi/2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

thus we have perfect state transfer between the end vertices of $K_2$ at time $\pi/2$.  

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More Examples

Since

$$H_{X \Box Y}(t) = H_X(t) \otimes H_Y(t)$$

it follows that if perfect state transfer from $u$ to $v$ in $X$ occurs at time $\tau$, then we also have perfect state transfer from $(u, u)$ to $(v, v)$ in $X \Box X$ at time $\tau$.

So we get perfect state transfer between antipodal vertices in the $d$-cube $Q_d$ at time $\pi/2$. 
Squaring

If perfect state transfer from 1 to 2 occurs at time $\tau$, then

$$H_X(\tau) = \begin{pmatrix}
0 & \gamma & 0 & \ldots & 0 \\
? & 0 & ? & \ldots & ? \\
\vdots & \vdots & Q \\
? & 0
\end{pmatrix}$$

where $|ga| = 1$. Consequently $|(H_X(\tau))_{2,1}| = 1$ and

$$H_X(2\tau) = \begin{pmatrix}
\gamma^2 & 0 & 0 & \ldots & 0 \\
0 & \gamma^2 & 0 & \ldots & 0 \\
\vdots & \vdots & Q \\
0 & 0
\end{pmatrix}$$
Periodicity

**Definition**

We say that $X$ is periodic at the vertex $u$ with period $\tau$ if $|(H_X(\tau))_{u,u}| = 1$.

**Lemma**

*If perfect state transfer from $u$ to $v$ occurs at time $\tau$, then $X$ is periodic at $u$ and $v$.***
Spectral Decomposition

We have

$$A = \sum_{\theta} \theta E_{\theta}$$

where $\theta$ runs over the distinct eigenvalues of $A$ and the matrices $E_{\theta}$ represent orthogonal projection onto the eigenspaces of $A$. 

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Spectral Decomposition

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where \( \theta \) runs over the distinct eigenvalues of \( A \) and the matrices \( E_{\theta} \) represent orthogonal projection onto the eigenspaces of \( A \). Further if \( f \) is a function on the eigenvalues of \( A \), then

\[ f(A) = \sum_{\theta} f(\theta) E_{\theta} \]

and therefore

\[ H_X(t) = \sum_{\theta} \exp(i\theta t) E_{\theta}. \]
Integer Eigenvalues

**Lemma**

If the eigenvalues of $X$ are integers, it is periodic with period $2\pi$. 
Integer Eigenvalues

**Lemma**

*If the eigenvalues of $X$ are integers, it is periodic with period $2\pi$.***

In fact:

**Theorem**

*If $X$ is a connected regular graph, then $X$ is periodic if and only if its eigenvalues are integers.*
Vertex-Transitive Graphs

**Theorem**

If $X$ is vertex transitive and perfect state transfer occurs at time $\tau$, then

$$H_X(\tau) = \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots = \gamma T$$

where $|\gamma| = 1$ and $T$ lies in the center of $\text{Aut}(X)$. 
Antipodal Vertices

If $X$ has diameter $d$, we say that vertices at distance $d$ are antipodal. In all examples we have where perfect state transfer takes place, the vertices involved are antipodal.

- Is antipodality necessary?
- If $|V(X)| \geq 3$, can we get perfect state transfer between adjacent vertices?
Antipodal Vertices

If $X$ has diameter $d$, we say that vertices at distance $d$ are antipodal. In all examples we have where perfect state transfer takes place, the vertices involved are antipodal.

- If $|V(X)| \geq 3$, can we get perfect state transfer between adjacent vertices?
Efficiency

What is the minimum number of edges in a graph where perfect state transfer takes place between two vertices at distance $d$? (Beat $2^d$.)
Suppose $X$ is a Cayley graph for $\mathbb{Z}_2^d$ with connection set \{ $c_1, \ldots, c_m$ \} and set $s = c_1 + \cdots + c_m$. If $s \neq 0$, we get perfect state transfer from 0 to $s$ at time $\pi/2$. Can perfect state transfer occur if $s = 0$?
Mixing

We say that perfect mixing occurs at time $\tau$ if, for all vertices $u$ and $v$ in $X$,

$$|(H_X(\tau))_{u,v}| = \frac{1}{\sqrt{|V(X)|}}.$$

(For example $K_2$ of $Q_d$ at time $\pi/4$. What can we usefully say about graphs where perfect mixing occurs?)