Symmetry and Eigenvectors

Ada Chan and Chris D. Godsil [1]

Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1
http://bilby.uwaterloo.ca

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Abstract

We survey some of what can be deduced about automorphisms of a graph from information on its eigenvalues and eigenvectors. Two of the main tools are convex polytopes and a number of matrix algebras that can be associated to the adjacency matrix of a graph.

Eigenvectors and Eigenpolytopes

We study eigenvectors of the adjacency matrix of a graph, and how these interact with the graph's automorphisms. The treatment in the first four sections is based loosely on [10,16].

1. Eigenvalues and Automorphisms

Our aim in this section is to introduce the idea that the eigenvectors of the adjacency matrix of a graph are best viewed as a class of functions on the vertices of the graph. So we present some definitions and then use them to derive some classical results concerning simple eigenvalues of vertex-transitive graphs.

Let X be a graph with vertex set V. If i and j are vertices in X, we write $i \sim j$ to denote that i is adjacent to j. Let $F(V)$ denote the space of real functions on V. We denote the characteristic vector of the vertex i of X by e_i; the vectors e_i form the *standard basis* of $F(V)$. The *adjacency operator* $\mathcal{A}$ is the linear operator on $F(V)$ defined by

$$(\mathcal{A}f)(i) := \sum_{j \sim i} f(j).$$

The matrix that represents $\mathcal{A}$ relative to the standard basis is the usual adjacency matrix A of X.

The inner product of two functions f and g from $F(V)$ is

$$\sum_{i \in V} f(i)g(i);$$

the adjacency operator is self-adjoint relative to this inner product (equivalently, the adjacency matrix is symmetric). We denote the function on V that is constant and equal to 1 by $\mathbf{1}$. A function f is an eigenvector for $\mathcal{A}$ if it is not zero and there is a constant θ such that, for each vertex i in V

$$\theta f(i) = \sum_{j \sim i} f(j),$$

or $\mathcal{A}f = \theta f$. The constant θ is, of course, the eigenvalue of f. The graph X is regular if and only if $\mathbf{1}$ is an eigenvector for $\mathcal{A}$, with the corresponding eigenvalue equal to the valency of X.

Let f be an eigenvector for $\mathcal{A}$ and let i be the vertex of X such that $|f(i)| \geq |f(j)|$, for all $j \in V$. Then

$$|\theta||f(i)| = |\sum_{j \sim i} f(j)| \leq \sum_{j \sim i} |f(j)|,$$

and therefore,

$$|\theta| \leq \sum_{j \sim i} \frac{|f(j)|}{|f(i)|}.$$

If X is connected then equality holds if and only if $|f|$ is a constant function. If f itself is constant, then X is regular, otherwise X is regular and bipartite. In both cases, the function $|f|$ is an eigenvector. Moreover, by our choice of i, we can conclude that $|\theta|$ is less than or equal to the valency of i and, thus, the absolute value of all eigenvalues for $\mathcal{A}$ is bounded above by the maximum valency of X.

If X is a complete graph on the vertex set V of size n, then since it is regular, it has eigenvector $\mathbf{1}$ with eigenvalue $(n-1)$. Let $f \in F(V)$ such that $\sum_{i \in V} f(i) = 0$. Then for all $i \in V$,

$$\sum_{j \sim i} f(j) = -f(i),$$

and therefore -1 is an eigenvalue for X with $n-1$ linearly independent eigenvectors. If X is a circuit on n vertices and τ is the primitive n^{th} root of unity, then the function $f(i) = \tau^i$, for $i = 0 \dots n-1$ is an eigenvector for X with eigenvalue $(\tau + \tau^{-1})$.

We denote the automorphism group of X by $\mathrm{Aut}(X)$; if $i \in V$ and $a \in \mathrm{Aut}(X)$ then ia is the image of i under a. If $f \in F(V)$ and $a \in \mathrm{Aut}(X)$ then the composition $f \circ a$ is again a real function on X, with

$$(f \circ a)(i) = f(ia).$$

The map

$$f \mapsto f \circ a$$

is an invertible linear mapping of $F(V)$ onto itself. If we need to distinguish the above mapping from the automorphism a, we will denote it by $\hat{a}$. Our conventions imply that

$$\widehat{ab} f = \hat{a}\hat{b} f.$$

If $a \in \mathrm{Sym}(V)$ then

$$\hat{a} e_i = e_{ia^{-1}},$$

which is probably not what we would expect.

Now we show that if $a \in \mathrm{Sym}(V)$ then $a \in \mathrm{Aut}(X)$ if and only if $\hat{a}$ and $\mathcal{A}$ commute. Equivalently, if $a \in \mathrm{Aut}(X)$ then $\hat{a}$ lies in the *centralizer* $C(\mathcal{A})$ of $\mathcal{A}$. Consider for any $f \in F(V)$ and $i \in V$,

$$(\mathcal{A}\hat{a})f(i) = \sum_{j \sim i} \hat{a} f(j) = \sum_{j \sim i} f(ja)$$

and

$$(\hat{a}\mathcal{A})f(i) = \mathcal{A}f(ia) = \sum_{l\sim ia} f(l).$$

Then $\mathcal{A}$ and $\hat{a}$ commute if and only if the right hand side of the above equations are equal. This occurs exactly when $i \sim j$ if and only if $ia \sim ja$ for all $i \in V$; that is $a \in \mathrm{Aut}(X)$.

If f is an eigenvector with eigenvalue θ and $a \in \mathrm{Aut}(X)$ then $f \circ a$ is again an eigenvector with eigenvalue θ; simply note that

$$\sum_{j\sim i}(f\circ a)(j) = \sum_{j\sim i} f(ja) = \sum_{\ell\sim ia} f(\ell) = \theta(f\circ a)(i).$$

Although we have not developed much theory yet, we can use what we have to derive some results concerning the eigenvalues of vertex-transitive graphs.

Assume X is a vertex-transitive graph with automorphism group G. Let θ be a simple eigenvalue of $\mathcal{A}$, and let f be a corresponding eigenvector. If $a \in G$ then $f \circ a$ is again an eigenvector with eigenvalue θ, as θ is simple it follows that

$$f \circ a = cf \tag{1.1}$$

for some constant c. Now there is a least positive integer r such that $a^r = 1$; from (1.1) we then find that $c^r = 1$. Hence c is an r-th root of unity. As both f and $f \circ a$ are real functions this implies that $c = \pm 1$.

1.1 Theorem (Petersdorf and Sachs [26]). *Let X be a vertex transitive graph on n vertices with valency k, and let θ be a simple eigenvalue of $\mathcal{A}$. Then $k - \theta$ is an even integer. If $\theta \neq k$ then $|V|$ is even.*

Proof. Let f be an eigenvector for θ. If $a \in \mathrm{Aut}(X)$ and $f \circ a = f$ then f is constant on the orbits of a. If $f \circ a = -f$ and S is an orbit of a then either f is zero on S, or $f(ia) = -f(i)$ for each i in S.

If f is constant then $\mathcal{A}f = kf$, i.e, $\theta = k$. So we may assume that f is not constant. As $\mathrm{Aut}(X)$ is vertex-transitive, it follows that there is a constant c such that $f(i) = \pm c$, for all vertices i in X. We may assume without loss that $c = 1$. Let i be a vertex of X and let ℓ_i be the number of neighbours j of i such that $f(j) = -f(i)$. Then

$$\theta f(i) = \sum_{j\sim i} f(j) = (k - 2\ell_i)f(i). \tag{1.2}$$

This proves our claim concerning θ.

Finally, as X is regular, the all ones vector $\mathbf{1}$ is an eigenvector for $\mathcal{A}$. As $\mathcal{A}$ is self-adjoint, eigenvectors with distinct eigenvalues are orthogonal. Hence

$$0 = \langle f, \mathbf{1} \rangle = \sum_{i \in V} f(i)$$

and thus we deduce that $|V|$ must be even. □

It is an interesting exercise to show that, if X is vertex transitive and has two simple eigenvalues, neither equal to the valency k, then $|V(X)|$ must be divisible by 4. It can also be shown that X has at least three vertices then it has at most $|V(X)|/2$ simple eigenvalues. This is another reasonable exercise. (Proofs of both these results are given in [16].)

A natural question is how many simple eigenvalues a vertex transitive graph can have.

Note that (1.2) implies that ℓ_i is independent of the vertex i. Let π be the partition of $V(X)$, where vertices i and j are in the same cell of π if $f(i) = f(j)$. Then π has exactly two cells, and the number of neighbours in a given cell of a vertex i of X is determined by the cell in which i lies. This means, in the terminology of Section 5, that π is an equitable partition of X.

2. Eigenspaces

If real-valued functions on $V(X)$ are interesting then it is not unreasonable to consider functions on $V(X)$ with values in the vector space $\mathbb{R}^m$, in particular those functions f from V to $\mathbb{R}^m$ such that, for some constant θ

$$\theta f(i) = \sum_{j \sim i} f(j). \tag{2.1}$$

By way of example, consider a regular dodecahedron in 3-space, with centre of mass at the origin. Let X be the 1-skeleton of this solid and let p be the map that associates each vertex of X with its image in $\mathbb{R}^3$. From the symmetry of the solid, we see that each vertex i there is a constant c_i such that

$$c_i p(i) = \sum_{j \sim i} p(j).$$

Since all vertices of the solid are equivalent under its symmetry group, c_i is independent of i and so (2.1) holds.

Our aim is to generalize this example. We use A to denote the adjacency matrix of X. Let θ be an eigenvalue of $\mathcal{A}$ with multiplicity m and let U_θ be a $|V| \times m$ matrix whose columns form an orthonormal basis for the eigenspace belonging to θ. As each column of U_θ is an eigenvector for $\mathcal{A}$ we have $\theta U_\theta = AU_\theta$. Hence, if $u_\theta(i)$ denotes the i-th row of U_θ, then (2.1) holds (with u_θ in place of f). We call the function u_θ from V to $\mathbb{R}^m$ the *representation* belonging to θ. As the columns of U_θ form an orthonormal set of vectors,

$$U_\theta^T U_\theta = I$$

and therefore, if we define $E_\theta := U_\theta U_\theta^T$ then $E_\theta = E_\theta^T$ and

$$E_\theta E_\tau = \begin{cases} E_\theta, & \text{if } \tau = \theta; \\ 0, & \text{otherwise.} \end{cases}$$

In fact E_θ is the matrix representing orthogonal projection onto the column space of U_θ. This implies that E_θ is an invariant of the eigenspace, and does not depend on the orthonormal basis we chose to form U_θ.

Next, if x and y are vectors in $\mathbb{R}^m$, for some m, then $\langle x, y\rangle$ denotes their inner product. If i and j are two vertices of X then

$$(E_\theta)_{i,j} = \langle u_\theta(i), u_\theta(j)\rangle.$$

This shows that the Euclidean distance between the points $u_\theta(i)$ and $u_\theta(j)$ is determined by i, j and θ.

A *convex polytope* is the convex hull of a finite set of points in $\mathbb{R}^m$. A *vertex* of a polytope is a point in it which is an endpoint of any closed line segment that contains it and is contained in the polytope. (Equivalently, a vertex is an extreme point.)

If S is a finite subset of $\mathbb{R}^m$ then the vertices of the convex hull C of S are a subset of S. Any linear mapping of $\mathbb{R}^m$ that fixes S as a set must fix the polytope generated by S, and a linear mapping fixes all points in C if and only if it fixes each vertex of C. The simplest example occurs when $m = 1$. If $|S| \geq 2$ then C has exactly two vertices and certainly any linear mapping that fixes these two vertices fixes all points in C.

The convex hull of the set of points $u_\theta(i)$, for all i in $V(X)$, is the *eigenpolytope* belonging to θ. Figure 1 illustrates the four eigenpolytopes of the cube. The corresponding matrices U_θ are listed in the following table. If h is an eigenvector for $\mathcal{A}$ with eigenvalue θ and S is the set of vertices on which h takes its maximum value then the convex hull of the points $u_\theta(i)$, for $i \in S$ is a face of the eigenpolytope. Conversely each face can be obtained from an eigenvector, since we can define the faces of a polytope as the set

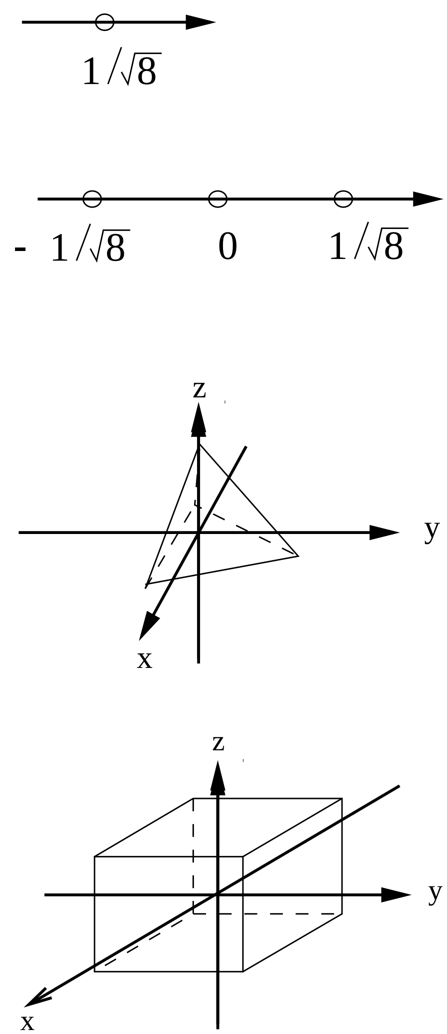

Figure 1: Eigenpolytopes of the Cube

of points on which a linear function takes its maximum value. Note that the eigenvector $-h$ determines a face disjoint from that gotten from h. Eigenpolytopes of distance-regular graphs are studied at some length in [14].

θ	m	U_θ
3	1	$\frac{1}{\sqrt{8}}\mathbf{1}$
-3	1	$\begin{pmatrix} \frac{-1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{-1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \\ \frac{-1}{\sqrt{8}} \\ \frac{-1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} \end{pmatrix}$
1	3	$\begin{pmatrix} \frac{-1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} \\ \frac{-1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{-1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} \\ \frac{-1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{-1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \end{pmatrix}$
-1	3	$\begin{pmatrix} \frac{1}{2} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{24}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{24}} \\ 0 & 0 & \sqrt{\frac{3}{8}} \\ \frac{-1}{2} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{24}} \\ \frac{-1}{2} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{24}} \\ 0 & 0 & \sqrt{\frac{3}{8}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{24}} \\ \frac{1}{2} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{24}} \end{pmatrix}$

3. Walks

A *walk* of length r in a graph X is a sequence $x_0, \dots, x_r$ of vertices from X, such that consecutive vertices are adjacent. A walk is *closed* if its first and last vertices are equal. In this section we show that the geometry of each eigenpolytope of X is determined by the number of walks of certain types in X. The number of walks $W_{i,j}(X,r)$ of length r from vertex i to vertex j in X is equal to

$$\langle e_i, \mathcal{A}^r e_j \rangle.$$

We will derive an alternative expression for this number. The *walk generating function* $W_{i,j}(X,t)$ is defined to be the formal sum

$$\sum_{r \geq 0} W_{i,j}(X,r) t^r.$$

The geometry of the eigenpolytopes of X is completely determined by these walk generating functions. Our main goal in this section is to demonstrate this.

If θ is an eigenvalue of $\mathcal{A}$, let E_θ denote orthogonal projection onto the associated eigenspace. For each eigenvalue θ of $\mathcal{A}$, we have

$$\mathcal{A} E_\theta = E_\theta \mathcal{A} = \theta E_\theta. \tag{3.1}$$

The *spectral decomposition* of $\mathcal{A}$ is equivalent to the identity

$$\mathcal{A} = \sum_\theta \theta E_\theta,$$

where we sum over all eigenvalues θ of $\mathcal{A}$. This is a standard result, of course, which can be found in many texts. (See [21], for example.) From (3.1) and the spectral decomposition we obtain, for any non-negative integer r:

$$\mathcal{A}^r = \sum_\theta \theta^r E_\theta.$$

Consequently

$$\langle e_i, \mathcal{A}^r e_j \rangle = \sum_\theta \theta^r \langle e_i, E_\theta e_j \rangle. \tag{3.2}$$

As $E_\theta = U_\theta U_\theta^T$, we obtain our next result.

3.1 Theorem. *For any two vertices i and j in the graph X and any non-negative integer r, we have*

$$W_{i,j}(X,r) = \sum_{\theta} \theta^r \langle u_\theta(i), u_\theta(j)\rangle,$$

where we sum over all eigenvalues θ of $\mathcal{A}$. □

3.2 Corollary. *The inner product $\langle u_\theta(i), u_\theta(j)\rangle$ is determined by θ and the sequence $W_{i,j}(X,r)$ for $r = 0, 1, \ldots$.*

Proof. ¿From (3.2), we have $\langle e_i, p(\mathcal{A})e_j\rangle = \sum_\theta p(\theta)\langle u_\theta(i), u_\theta(j)\rangle$, for any polynomial p. Let

$$p_\theta(x) := \prod_{\lambda\neq\theta} \frac{(x-\lambda)}{(\theta-\lambda)} = \begin{cases} 1, & \text{if } x = \theta; \\ 0, & \text{if } x \neq \theta,\ x \text{ an eigenvalue of } \mathcal{A}, \end{cases}$$

be a product over all eigenvalues of $\mathcal{A}$. Then $\langle u_\theta(i), u_\theta(j)\rangle = \langle e_i, p_\theta(\mathcal{A})e_j\rangle$ and the result follows. □

The significance of Corollary 3.2 is that it shows that the geometry of the set of points

$$\{u_\theta(i) : i \in V\}$$

is determined by θ and the walk generating functions $W_{i,j}(X,t)$ for all vertices i and j of X.

As one useful consequence we note that, if X is a vertex-transitive graph, the walk generating function $W_{i,i}(X,t)$ is independent of the vertex i and therefore the length of the vector $u_\theta(i)$ does not depend on i. Thus in this case all points $u_\theta(i)$ for i in V lie on a sphere with centre at the origin. The same conclusion will hold more generally for graphs X with the property that the walk generating functions $W_{i,i}(X,t)$ are independent of i. We will call such graphs *walk-regular*. They need not be vertex-transitive, but nonetheless have a number of properties in common with vertex-transitive graphs. For examples, the Petersdorf-Sachs theorem (Theorem 1.1) holds for walk-regular graphs. For if X is walk-regular and θ is a simple eigenvalue for X then $|u_\theta(i)|$ is independent of i, and the rest of the proof follows. (For further information, see [16,11].)

4. Automorphisms

It follows from the spectral decomposition of $\mathcal{A}$ that each projection E_θ is a polynomial in $\mathcal{A}$. Therefore each automorphism of X, viewed as a linear mapping, commutes with each projection E_θ. As $E_\theta = U_\theta U_\theta^T$, we thus have

$$\hat{g}U_\theta = \hat{g}U_\theta U_\theta^T U_\theta = U_\theta U_\theta^T \hat{g} U_\theta,$$

which implies that

$$u_\theta(ig) = u_\theta(i) U_\theta^T \hat{g} U_\theta.$$

Here $U_\theta^T \hat{g} U_\theta$ is an $m \times m$ matrix that is easily shown to be orthogonal. Let us denote it by g_θ. The mapping $g \mapsto g_\theta$ is thus a representation of the automorphism group into the group of $m \times m$ real orthogonal matrices; we obtain one such representation from each eigenspace of X.

An automorphism g lies in the kernel of the representation if and only if $\hat{g}U_\theta = U_\theta$, i.e., if and only if $\hat{g}$ fixes each column of U_θ. It follows that if g_θ is the identity for each eigenvalue θ then $\hat{g}$ fixes every eigenvector of $\mathcal{A}$, which implies that it is the identity mapping on $F(V)$.

Now consider the eigenpolytope of X associated to an eigenvalue θ of X, with multiplicity m.

If $m = 1$ then this is either a single point or a closed line segment in $\mathbb{R}^1$. In the first case the representation of $\mathrm{Aut}(X)$ afforded by the eigenspace is trivial (each automorphism is mapped to the identity). In the second case, the image of an automorphism either fixes each endpoint of the line segment, and hence induces the identity mapping, or it swaps the two endpoints and its square is the identity. Thus the representation of $\mathrm{Aut}(X)$ determines a homomorphism of $\mathrm{Aut}(X)$ onto the group of order 2, and therefore $\mathrm{Aut}(X)$ contains a subgroup of index 2. If each eigenvalue of X has multiplicity 1, it follows that $\mathrm{Aut}(X)$ must be an elementary abelian 2-group. (It is an interesting problem just how large $\mathrm{Aut}(X)$ can be in this case. Note that we saw in Section 1 that the eigenvalues of a vertex-transitive graph cannot all be simple, hence $|V| - 1$ is an upper bound.)

If $m = 2$ then the eigenpolytope is either a closed line segment or a convex polygon. In the first case our previous analysis suffices, in the second we obtain a representation of $\mathrm{Aut}(X)$ as a subgroup of a dihedral group. If each eigenvalue of X has multiplicity at most 2 then $\mathrm{Aut}(X)$ is therefore a subgroup of a direct product of dihedral groups.

Babai, Grigoryev and Mount [1] have proved that if the multiplicity of the eigenvalues are bounded by some constant, then $\mathrm{Aut}(X)$ can be computed in polynomial time.

Equitable Partitions

Let $C_1, \ldots, C_m$ be the orbits of a group of automorphisms of X. Any two vertices in C_i must have the same number of neighbours in C_j, for all $1 \le i,\ j \le m$. Equitable partitions are partitions of $V(X)$ with this regularity property, that are not necessarily orbit partitions. Our treatment in this and the next section follows [15] and [11: Chapter 5].

5. Basics

If π is a partition of a set, let $|\pi|$ denote the number of cells of π. A partition is *discrete* if its cells are all singletons.

Let X be a graph with vertex set V and let π be a partition of V with cells $C_1, \ldots, C_m$. We say that π is *equitable* if, for each i and j, the number of neighbours in C_j of a given vertex u in C_i is determined by i and j, and is independent of the choice of u in C_i. If G is a subgroup of $\mathrm{Aut}(X)$, then the orbits of G form a partition π of V with the properties that the subgraph induced by a cell of π is regular and the subgraph formed by the edges joining any two cells of π is bipartite and semiregular. Hence, the orbits of an automorphism group of X form an equitable partition. However, the converse is not true, Figure 2 shows an example of an equitable partition that is not an orbit partition.

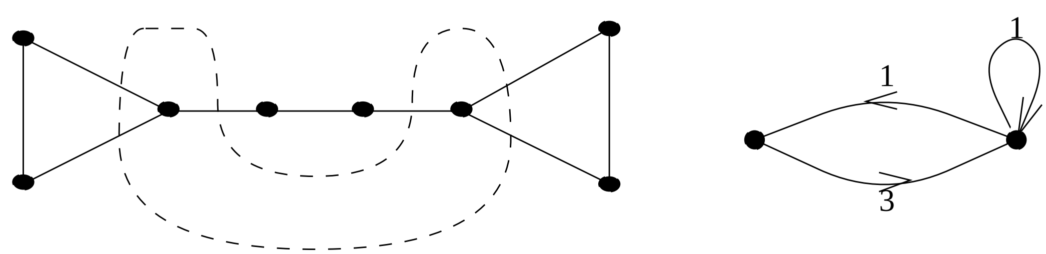

Figure 2: An equitable partition that is not an orbit partition

The *quotient* X/π of X relative to the equitable partition π is the directed graph with the cells $C_1, \ldots, C_m$ of π as its vertices and $c_{i,j}$ arcs from cell i to cell j, where $c_{i,j}$ is the number of vertices in C_j adjacent to a given vertex in C_i. The adjacency matrix of X/π is the $m \times m$ matrix with ij-entry equal to $c_{i,j}$.

Figure 3 lists four equitable partitions of the Petersen graph with their quotients.

If π is a partition of V, let $F(V, \pi)$ denote the space of real functions on $V(X)$ that are constant on the cells of π.

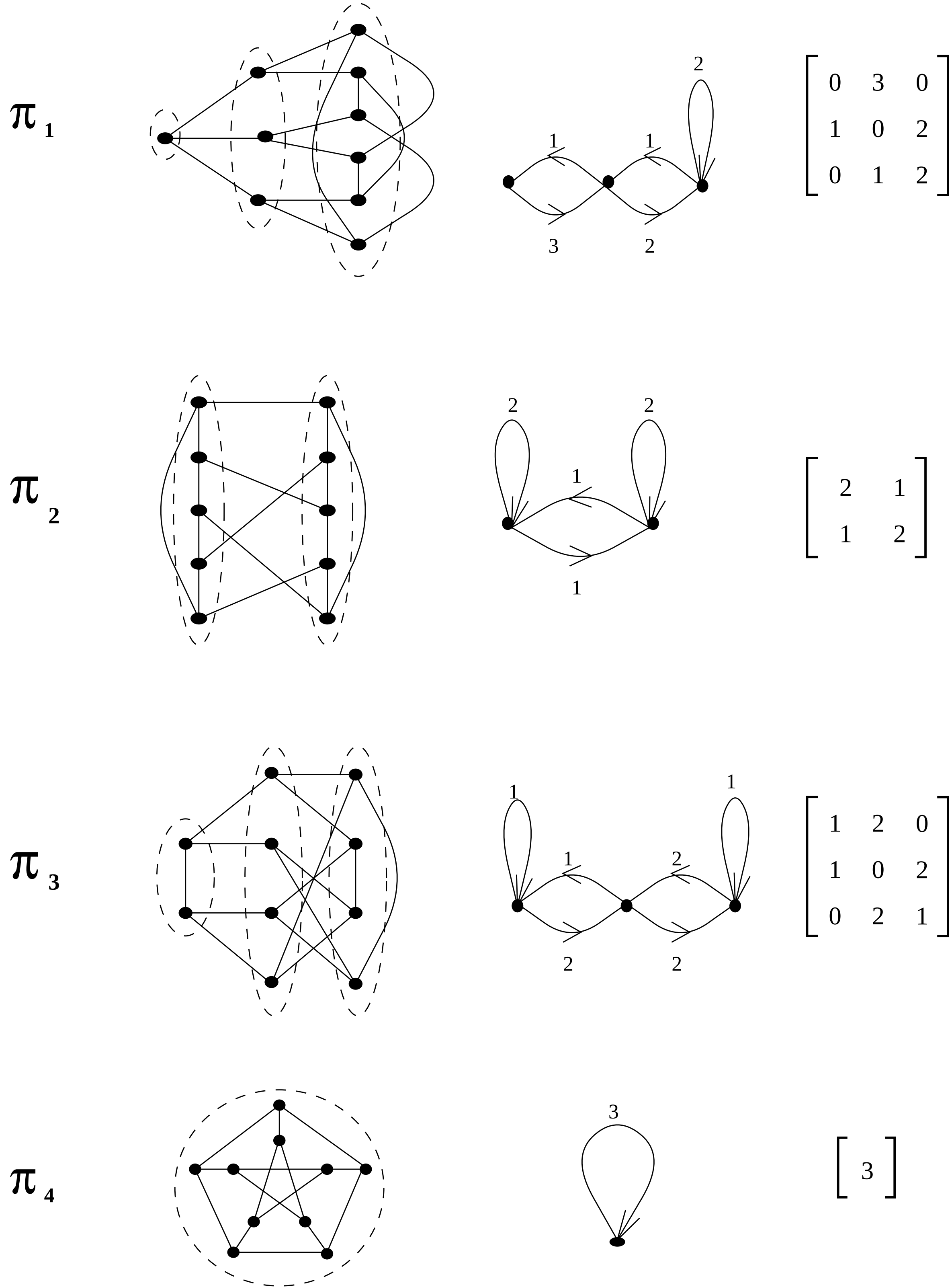

Figure 3: Equitable partitions of Petersen graph

5.1 Lemma. *A subspace U of $F(V)$ equals to $F(V,\pi)$ for some partition π of V if and only if U is a subalgebra of $F(V)$ which contains the constant functions.*

Proof. If π a partition of V then $F(V,\pi)$ is a subspace of $F(V)$ that contains all the constant functions; it is also closed under multiplication and is thus a subalgebra of $F(V)$.

Conversely, suppose U is a subalgebra of $F(V)$ which contains the constant functions and with basis $\{x_1,\dots,x_m\}$. For each vector x_i, let π_i be the partition of V such that vertices u,v belong to the same cell if and only if $x_i(u)=x_i(v)$. Let π be the partition of V whose cells are the non-empty pairwise intersections of the cells of $\pi_1,\dots\pi_m$. Let

$$c = bx_1 + b^2x_2 + \dots + b^m x_m,$$

where the real number b is chosen large enough that c takes distinct values on vertices in distinct cells of π. Suppose $v_1,\dots,v_r$ are vertices in the cells $C_1,\dots,C_r$ of π respectively, and for $k=1,\dots,r$ let

$$y_k(v) = \prod_{j\neq k} \frac{(c(v)-c(v_j))}{(c(v_k)-c(v_j))} = \begin{cases} 1, & \text{if } v\in C_k; \\ 0, & \text{otherwise.} \end{cases}$$

The functions $y_1,\dots,y_r$ are the characteristic functions of $C_1,\dots,C_r$ respectively. They all belong to U because U contains the constant functions $c(v_j)$. Moreover, they are linearly independent in U. Since each cell of π is contained in a cell of π_i, for $i=1,\dots,m$, we can express the x_i as a linear combination of $y_1,\dots,y_r$. As a result, the y_j's form a basis of U and U is equal to $F(V,\pi)$. □

5.2 Lemma. *Let X be a graph and let π be a partition of $V(X)$. Then π is equitable if and only if $F(V,\pi)$ is $\mathcal{A}$-invariant.*

Proof. Suppose that x is the characteristic function of the subset S, a cell of π of $V(X)$. If $i\in V$ then $(\mathcal{A}x)(i)$ equals the number of vertices in S adjacent to i. So $\mathcal{A}x$ is constant on the cells of π if and only if the number of neighbours in S of a vertex i is determined by the cell of π in which i lies. Now the result follows at once. □

Equivalently, for undirected graphs, we can define a partition π to be equitable if and only if $F(V,\pi)$ is $\mathcal{A}$-invariant. For directed graphs, there is a choice of the algebra $\langle A\rangle$ or $\langle A, A^T\rangle$ used to define equitable partitions.

If π and ρ are partitions of V and each cell of ρ is contained in a cell of π, we say that ρ is a *refinement* of π, and write $\rho\le\pi$. This shows that the set of all partitions of V forms a partially ordered set, but in fact it is even a lattice. The meet $\pi \wedge \rho$ of two partitions π and ρ is the partition whose cells are all the nonempty pairwise intersections of cells of π with cells of ρ. The join $\pi \vee \rho$ is most easily defined as the meet of all the partitions σ such that $\pi \leq \sigma$ and $\rho \leq \sigma$. Alternatively, define a graph with vertex set V, where two vertices are adjacent if and only if they lie in the same cell of π or of ρ; the connected components of this graph are the cells of $\pi \vee \rho$.

The meet of two equitable partitions need not be equitable. An example is given in Figure 4, which is the meet of π_2 and π_3 of the Petersen graph in Figure 3. However the join must be. We will derive this from the next result, the proof of which is left as an exercise.

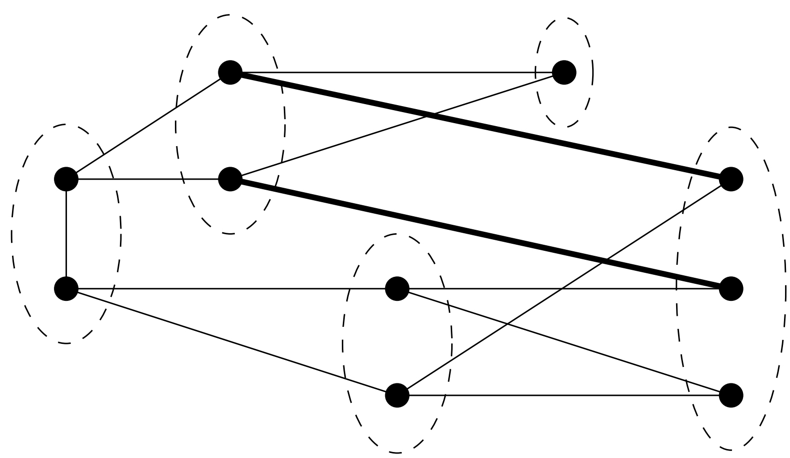

Figure 4: Meet of π_2 and π_3 is not equitable

5.3 Lemma. *Let π and ρ be partitions of the set V. Then $F(V, \pi \vee \rho) = F(V, \pi) \cap F(V, \rho)$.* □

5.4 Corollary (McKay [22, Lemma 5.3]). *Let π and ρ be equitable partitions of the graph X. Then $\pi \vee \rho$ is equitable.*

Proof. Assume $V = V(X)$. As π and ρ are equitable, both $F(V, \pi)$ and $F(V, \rho)$ are $\mathcal{A}$-invariant. Now the above lemma implies that $F(V, \pi \vee \rho)$ is $\mathcal{A}$-invariant and so $\pi \vee \rho$ is equitable. □

One important consequence is that any partition π has a unique maximal equitable refinement—it is the join of all the equitable partitions that refine π.

If π is a partition of V such that each cell is fixed by $\mathrm{Aut}(X)$ then the set of equitable refinements of π is fixed by $\mathrm{Aut}(X)$, and therefore the coarsest equitable refinement of π is fixed by $\mathrm{Aut}(X)$. (For example, we might choose π to be the partition of the vertices of X by valency, which is clearly invariant under $\mathrm{Aut}(X)$. If π', the coarsest equitable refinement of π, is discrete then it follows that $\mathrm{Aut}(X)$ must be the identity group.) If π is a partition with cells $C_1, \ldots, C_r$ and $v \in C_i$, define the *profile* of v to be $(n_1, \ldots, n_r; i)$, where n_j is the number of neighbours v has in C_j. Let π_1 be the partition such that two vertices are in the same cell if and only if their profiles (relative to π) are equal. Then π_1 is a refinement of π which is invariant under any automorphism that fixes each cell of π. It can be computed from π in polynomial time, hence it follows that by repeatedly computing refinements in this manner, we can determine π' in polynomial time. So it is not surprising that equitable partitions play an important role in practical graph isomorphism packages such as nauty. Further details will be found in [7].

6. Quotients

Any subset S of V can be represented by its characteristic function, which takes the value 1 on each element of S and 0 on each element not in S. Because a partition of V is a set whose elements are sets, we can represent a partition π by its *characteristic matrix*, which is the $|V| \times |\pi|$ matrix with the characteristic functions of the cells of π for columns.

6.1 Lemma. *Let A be the adjacency matrix of X and let π be a partition of $V(X)$ with characteristic matrix D. Then π is equitable if and only if there is a matrix B such that $AD = DB$.*

Proof. The column space of D can be identified with $F(V, \pi)$. Hence π is equitable if and only if the column space of D is A-invariant. This holds if and only if, for each column x of D, the vector Ax is a linear combination of columns of D and so can be written as Db for some vector b. □

The columns of the characteristic matrix of a partition are linearly independent, therefore there is at most one matrix B such that $AD = DB$. Alternatively, note that $D^T D$ is invertible and diagonal with entries being the sizes of the cells, and that $B = (D^T D)^{-1} D^T AD$, thus is determined by A and D. If π is equitable then B must be the adjacency matrix of X/π.

6.2 Theorem. *Let X be a graph with adjacency matrix A and let π be an equitable partition of X. Then the characteristic polynomial of $A(X/\pi)$ divides the characteristic polynomial of A.*

Proof. Suppose that $v = |V(X)|$ and $m = |\pi|$. Let $x_1, \dots, x_v$ be a basis for $\mathbb{R}^v$ such that $x_1, \dots, x_m$ are the distinct columns of the characteristic matrix of π and $x_{m+1}, \dots, x_v$ are each orthogonal to $x_1, \dots, x_m$. Let $D_1 = (x_1, \dots, x_m)$ and $D_2 = (x_{m+1}, \dots, x_v)$. Then the matrix representing $\mathcal{A}$ relative to this basis is

$$\begin{pmatrix} (D_1^T D_1)^{-1} & 0 \\ 0 & (D_2^T D_2)^{-1} \end{pmatrix} \begin{pmatrix} D_1^T \\ D_2^T \end{pmatrix} A \begin{pmatrix} D_1 & D_2 \end{pmatrix}.$$

Since $AD_1 = D_1 B$ and $D_1^T D_2 = 0$, the matrix has the block diagonal form

$$\begin{pmatrix} B & 0 \\ 0 & Y \end{pmatrix} \tag{6.1}$$

where B is the adjacency matrix of X/π. Since the characteristic polynomial of the matrix in (6.1) is the characteristic polynomial of A, the result follows. □

Note that it follows from this that, if the characteristic polynomial of $\mathcal{A}$ is irreducible over the rationals, then the only equitable partition of X is the discrete partition.

Suppose $S \subseteq V(X)$. The *distance partition* of X relative to S is the partition with cells $C_0, \dots, C_r$, where C_i is the set of vertices in X at distance i from S. The integer r is the *covering radius* of S. We say that S is a *completely regular* subset of X if its distance partition is equitable. For example, the leftmost cell of each of the four partitions in Figure 3 are four different completely regular subsets of the Petersen graph. The *ball of radius* r about a vertex u of X consists of all vertices in X at distance at most r from u. The *packing radius* of S is the maximum integer e such that the balls of radius e about the vertices in S are pairwise disjoint. A subset with packing radius e is also called an *e-code*. If S has packing radius e and covering radius r then $e \leq r$. We call S a *perfect code* if $r = e$. (This terminology is consistent with the standard usage in coding theory.)

The Johnson graph, $J(n, k)$ has as vertices all the k-subsets of a n-set, where two k-subsets are adjacent if and only if they intersect in $k-1$ elements. Any single vertex in $J(n, k)$ is a perfect code. Furthermore, when $n = 2k$ and k is odd, any pair of disjoint k-subsets form a perfect $\frac{(k-1)}{2}$-code. No further perfect codes are known in the Johnson graphs. As a result, Delsarte [8, p55] states that "it is tempting to conjecture that such codes do not exist"; this question is still open.

6.3 Theorem. *Let X be a regular graph. If X has a perfect 1-code then -1 is an eigenvalue of $\mathcal{A}$.*

Proof. Assume that the valency of X is k. Suppose that C is a perfect code in X, and that $c = |C|$. Let π be the partition with C and $V \setminus C$ as its cells. A vertex in C has no neighbours in C and k neighbours in $V \setminus C$. A vertex in $V \setminus C$ is adjacent to exactly one vertex in C, and to $k-1$ vertices in $V \setminus C$. Therefore π is equitable, and the adjacency matrix of the quotient is

$$\begin{pmatrix} 0 & k \\ 1 & k-1 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$x^2 - (k-1)x - k = (x-k)(x+1),$$

whence Theorem 6.2 implies that -1 must be an eigenvalue of $\mathcal{A}$. □

Algebras

In the next two sections, we derive results concerning some matrix algebras, and apply them to equitable partitions. The following section provides some information about one of these algebras for trees.

7. Algebras

As we have seen, finding the equitable partitions of a graph is equivalent to finding the $\mathcal{A}$-invariant subalgebras of $F(V)$ that contain the constant functions. If Y is such a subalgebra then Y is invariant under all elements of the algebra generated by $\mathcal{A}$. But there is a larger algebra that we can work with. Let $\mathcal{J}$ denote the square matrix with all entries equal to 1 (and order determined by the context.) A subspace of $F(V)$ is $\mathcal{J}$-invariant if and only if it contains all constant functions, and therefore a subspace $F(V, \pi)$ is invariant under $\mathcal{A}$ and $\mathcal{J}$ if and only if it is $\mathcal{A}$-invariant.

The operator $\mathcal{J}$ is a polynomial in $\mathcal{A}$ if and only if X is a connected regular graph. (This result was first observed by A. J. Hoffman, for a proof see Biggs [3, Prop. 3.2].) Hence, if X is not regular, then $\langle \mathcal{A}, \mathcal{J} \rangle$ is strictly larger than $\langle \mathcal{A} \rangle$. In particular, if $\langle \mathcal{A}, \mathcal{J} \rangle$ is equal to the algebra of all linear operators on $F(V)$, then no proper non-zero subspace of $F(V)$ is $\langle \mathcal{A}, \mathcal{J} \rangle$-invariant, and consequently the only equitable partition of X is the discrete partition. (And X has no non-identity automorphisms.)

In fact, under comparatively mild conditions, $\langle \mathcal{A}, \mathcal{J} \rangle$ is equal to the algebra of all linear operators on $F(V)$. We introduce some machinery to let us justify this claim. If $S \subseteq V(X)$, let $\mathcal{J}_S$ be the operator on $F(V)$ defined by

$$(\mathcal{J}_S f)(i) = \begin{cases} \sum_{j \in S} f(j), & \text{if } i \in S; \\ 0, & \text{otherwise.} \end{cases}$$

7.1 Theorem. *Let $\mathcal{A}$ be the adjacency operator of the graph X and let S be a subset of $V(X)$ with characteristic function σ. Let W be the subspace of $F(V)$ spanned by the functions $\mathcal{A}^r \sigma$. Then the algebra generated by $\mathcal{A}$ and $\mathcal{J}_S$ is isomorphic to the direct sum of the algebra of all linear operators on W and the algebra generated by the restriction of $\mathcal{A}$ to W^-.*

Proof. The space $F(V)$ is a direct sum of eigenspaces of $\mathcal{A}$; let $\sigma_1, \ldots, \sigma_m$ denote the distinct non-zero projections of σ on the eigenspaces of $\mathcal{A}$. As the vectors σ_i lie in distinct eigenspaces of $\mathcal{A}$, they have distinct eigenvalues and therefore, for each i, there is a polynomial p_i such that $p_i(\mathcal{A})\sigma_i = \sigma_i$ and $p_i(\mathcal{A})\sigma_j = 0$ when $j \neq i$. This also shows that the vectors σ_i all lie in W. Because $\sigma = \sum_i \sigma_i$, we see that W is contained in the span of the vectors σ_i and therefore these vectors form a basis for W.

Since $\sigma^T \sigma_i = \sigma_i^T \sigma_i^T$ we have

$$p_i(\mathcal{A})\mathcal{J}_S p_j(\mathcal{A})\sigma_r = p_i(\mathcal{A})\,\sigma\sigma^T p_j(\mathcal{A})\sigma_r = \begin{cases} \|\sigma_j\|^2 \sigma_i, & \text{if } r = j; \\ 0, & \text{otherwise.} \end{cases}$$

This implies that the m^2 operators $p_i(\mathcal{A})\mathcal{J}_S p_j(\mathcal{A})$ span the space of all linear operators on W.

As $\sigma \in W$, each vector in W^- is orthogonal to σ and therefore $\mathcal{J}_S$ acts on W^- as the zero operator. □

7.2 Corollary. *Let $\mathcal{A}$ be the adjacency operator of the graph X, let S be a subset of $V(X)$ with characteristic function σ and let ρ be the partition of V with S and $V \setminus S$ as its cells. If no eigenvector of $\mathcal{A}$ is orthogonal to σ then the discrete partition is the only equitable refinement of ρ.*

Proof. If W^- is non-zero then it contains an eigenvector for $\mathcal{A}$, this eigenvector is necessarily orthogonal to σ. Thus the hypothesis implies that $W = F(V)$, and therefore that $\mathcal{A}$ and $\mathcal{J}_S$ generate the algebra of all linear operators on $F(V)$. Consequently no non-zero proper subspace of $F(V)$ is invariant under $\mathcal{A}$ and $\mathcal{J}_S$. This implies that there is no equitable refinement of the partition with cells S and $V \setminus S$. □

If $i \in V(X)$ and f is an eigenvector for $\mathcal{A}$ such that $f(i) = 0$ then the restriction of f to $V \setminus i$ is an eigenvector for $\mathcal{A}(X \setminus i)$, with the same eigenvalue. Consequently Corollary 7.2 implies that, if $\mathcal{A}(X)$ and $\mathcal{A}(X \setminus i)$ have no eigenvalue in common, no non-identity automorphism of X fixes i.

If $\mathcal{A}$ has no eigenvector orthogonal to $\mathbf{1}$ then Corollary 7.2 implies that the discrete partition is the only equitable partition of X. We leave it as an exercise to show that, if the characteristic polynomial of $\mathcal{A}$ is irreducible over the rationals, then $\mathcal{A}$ and $\mathcal{J}$ have no common eigenvector, and consequently X has no non-identity automorphisms. This is a classical result, first appearing in [25]. Another proof follows from Theorem 6.2.

We present an extension of Theorem 7.1.

7.3 Theorem. *Let X be a connected graph and let $\sigma_1, \ldots, \sigma_m$ be characteristic vectors of subsets $S_1, \ldots, S_m$ of $V(X)$. Let W be the space spanned by all vectors of the form $A^r \sigma_i$. Then the algebra $\mathcal{M}$ generated by $\mathcal{A}$ and $\mathcal{J}_{S_i}$ $(i = 1, \ldots, m)$ is isomorphic to the direct sum of the algebra of all linear operators on W and the algebra generated by the restriction of $\mathcal{A}$ to W^-.*

Proof. Suppose y is an eigenvector for $\mathcal{A}$ in W. If $\sigma_i^T y = 0$ for each i then

$$\sigma_i^T \mathcal{A}^r y = 0$$

for each i and so $y \in W^- \cap W$; hence $y = 0$. Thus we may assume that $\sigma_i^T y \neq 0$ for some i.

We now show that W is an irreducible $\mathcal{M}$-module. Suppose that y lies in a submodule W_1 of W with $\sigma_i \in W_1$. As X is connected, for any j there is an integer r such that

$$\sigma_j^T \mathcal{A}^r \sigma_i > 0.$$

Consequently $\sigma_j \in W_1$, for all j, and therefore $W_1 = W$.

Next, let L be a linear endomorphism of W that commutes with each element of $\mathcal{M}$. We aim to show that L is a scalar multiple of the identity map. For any vector σ, if

$$L\sigma\sigma^T = \sigma\sigma^T L$$

then $L\sigma$ and σ must be parallel; i.e, σ is a right eigenvector for L. Similarly σ^T and $\sigma^T L$ are parallel and therefore σ^T is a left eigenvector. If $L\sigma = c\sigma$ and $\sigma^T L = c\sigma^T$ then

$$c\sigma^T \sigma = \sigma^T L \sigma = d\sigma^T \sigma,$$

whence we see that σ and σ^T have the same eigenvalue.

Suppose now that $L\sigma_i = 0$. Because $\mathcal{A}$ and L commute we than have $L\mathcal{A}^r \sigma_i = 0$ for all r and therefore, for any j, $\sigma_j^T L \mathcal{A}^r \sigma_i = 0$. Since L commutes with $\mathcal{J}_{S_j} = \sigma_j \sigma_j^T$, from above, $\sigma_j^T L = c\sigma_j^T$ for some c. It follows that $c\sigma_j^T A^r \sigma_i = 0$ for all r, and hence $c = 0$. Thus we have proved that $L\sigma_i = 0$ for all i.

Assume next that $L\sigma_i = c_i \sigma_i$ for $i = 1, \ldots, m$ and $c_i \neq 0$ for any i. Then

$$c_i \sigma_j^T \mathcal{A}^r \sigma_i = \sigma_j^T \mathcal{A}^r L \sigma_i = c_j \sigma_j^T \mathcal{A}^r \sigma_i,$$

for all values of r. This implies that c_i is independent of i; denote its common value by c.

If $y \in W$ and $\mathcal{A}y = \theta y$ then

$$\theta L y = L\mathcal{A}y = \mathcal{A}Ly$$

which shows that each eigenspace of $\mathcal{A}$ in W is fixed by L. Suppose that $y_1, \ldots, y_r$ are the non-zero projections of σ_i on the eigenspaces of $\mathcal{A}$ in W. Then

$$\sigma_i = y_1 + \cdots + y_r$$

and so

$$c\sigma_i = L\sigma_i = Ly_1 + \cdots + Ly_r.$$

¿From this it follows that $Ly_j = cy_j$ for all j.

To complete the proof that L is a scalar operator, it suffices to show that each eigenspace of $\mathcal{A}$ in W is spanned by projections of the vectors σ_i. Assume by way of contradiction that this is false. Then we would have an eigenspace U and a vector u in U orthogonal to the projection of each vector σ_i onto U. This implies that $\sigma_i^T u = 0$ for all i and hence $u = 0$.

Summing up, we have shown that W is an irreducible $\mathcal{M}$-module, and that the only linear endomorphisms of W that commute with $\mathcal{M}$ are the scalar operators. By Schur's theorem [19, Theorem 5.3], it follows that $\mathcal{M}$ acts on W as the algebra of all linear endomorphisms. Because W contains each vector σ_i, we see that $\mathcal{J}_{S_i}$ acts on W^- as the zero operator. This completes the proof. □

The condition that $\langle \mathcal{A}, \mathcal{J} \rangle$ is the algebra of all linear operators of $F(V)$ is clearly very strong. Nonetheless it often holds. Some evidence for this is provided by the following table. We believe that the proportion of graphs on n vertices such that $\langle \mathcal{A}, \mathcal{J} \rangle$ is the full matrix algebra tends to 1 as n increases. (This belief is based on the results in the table and on experiments with random graphs on up to 20 vertices.)

$\|V\|$	No. of graphs	No. of graphs with $\langle \mathcal{A}, \mathcal{J} \rangle = M_{n\times n}(\mathbb{R})$	No. of asymmetric graphs	No. of asymmetric & connected graphs
6	78	4	4	4
7	522	46	76	72
8	6996	1370	1848	1786
9	154354	48457	67502	65726

Figure 5 contains all four graphs of order six with the property that $\langle \mathcal{A}, \mathcal{J} \rangle$ generates the algebra of all linear operators of $F(V)$.

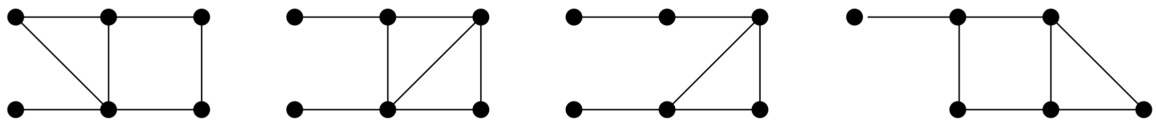

Figure 5: Graphs of order 6 with $\langle \mathcal{A}, \mathcal{J} \rangle = M_{n\times n}(\mathbb{R})$

The proofs of the main results in this section (Theorem 7.1 and Theorem 7.3) are based on ideas from Laffey [20].

8. Walks Again

Two graphs X and Y are *cospectral* if $A(X)$ and $A(Y)$ are similar. The following result implies that cospectral graphs have the same closed walk generating function.

8.1 Lemma. *If A and B are symmetric matrices, then the following are equivalent:*

i. *A and B are similar.*

ii. *A and B have the same characteristic polynomial.*

iii. *$tr(A^n) = tr(B^n)$, for all $n \geq 0$.* □

We let $W(X,t)$ denote the generating function for walks in the graph X, enumerated by their length. If $S \subseteq V(X)$ then $W_S(X,t)$ is the generating function for the walks in X that start at a vertex in S, enumerated by length. If S and T are subsets of $V(X)$ then $W_{S,T}(X,t)$ is the generating function for the walks in X that start at a vertex in S and finish in T. In particular, $W_i(X,t)$ is the generating function for the walks in X that start at i and $W_{i,i}(X,t)$ is the generating function for the walks in X that start and finish at the vertex i.

The next two results can be derived from identities in [17].

8.2 Lemma. *Let X and Y be cospectral graphs. Then their complements $\overline{X}$ and $\overline{Y}$ are cospectral if and only if $W(X,t) = W(Y,t)$.* □

8.3 Lemma. *Let i and j be vertices in the graph X. Then $X\setminus i$ and $X\setminus j$ are cospectral if and only if $W_{i,i}(X,t) = W_{j,j}(X,t)$. Their complements are cospectral as well if and only if we also have $W_i(X,t) = W_j(X,t)$.* □

If σ is the characteristic vector of S and the space spanned by the vectors $\mathcal{A}^r\sigma$ has dimension $s+1$, we say that S has *dual degree* s. The *covering radius* of S is the least integer r such that each vertex of X is at distance at most r from some vertex in S. It follows that $(I+A)^0\sigma, \dots, (I+A)^r\sigma$ are linearly independent with increasing number of non-zero entries, thus $r \le s$. As the diameter of X is the maximum value for the covering radius of a vertex, this inequality is a generalization of the well known fact that the diameter of X is bounded above by the number of distinct eigenvalues of X, less 1. The terms covering radius and dual degree come from coding theory, where they refer to the covering radius and dual degree of subsets of the n-cube or,more generally, of the Hamming graph $H(n,q)$. For more information see, e.g., [11, Chapter 11].

Let S be a subset of $V(X)$ with characteristic function σ. Define two vertices i and j of X to be equivalent if

$$(\mathcal{A}^r\sigma)(i) = (\mathcal{A}^r\sigma)(j), \qquad r \ge 0.$$

This determines a partition of V, which we call the partition *induced by* S. The partition induced by V itself is the *walk partition*, since two vertices i and j are in the same cell if and only the generating function for the walks in X that start at i is equal to the generating function for the walks in X that start at j. The walk partition ω is a refinement of the partition of $V(X)$ by valency. It is refined in turn by the coarsest equitable partition of X that refines the partition with S and its complement as its cells. This shows that the dual degree of X is a lower bound on the number of cells in this partition.

It follows from the proof of Theorem 7.1 that the $\mathcal{A}$-module generated by σ is an irreducible module for the algebra $\langle \mathcal{A}, \mathcal{J}_S \rangle$. This module has a basis consisting of vectors $\mathcal{A}^r\sigma$, and so the matrix M representing the action of $\mathcal{A}$ on this module is determined by the inner products

$$\langle \mathcal{A}^r\sigma, \mathcal{A}\,\mathcal{A}^s\sigma \rangle, \qquad r, s \ge 0.$$

As $\langle \mathcal{A}^r\sigma, \mathcal{A}\,\mathcal{A}^s\sigma \rangle = \langle \mathbf{1}, \mathcal{A}^{r+s+1}\sigma \rangle$ it follows that M is determined by the numbers $\sigma^T A^r \sigma$, i.e., by the number of walks in X that start in finish in S and have length r, for each non-negative integer r.

8.4 Theorem. *Let X be a graph on n vertices and let S be a subset of $V(X)$ with dual degree $n-1$. Then the walk generating functions $W_{i,S}(X,t)$, for $i \in V(X)$, determine X.*

Proof. Let σ be the characteristic function of S. As the dual degree of S is $n-1$, it follows from the proof of Theorem 7.1 that the matrices

$$A_{r,s} := A^r\sigma\sigma^T A^s, \quad 1 \le r, s \le n$$

form a basis for the algebra of all $n \times n$ matrices. But

$$A^r \sigma\sigma^T A^r = A^r \sigma(A^s \sigma)^T,$$

consequently the matrices $A_{r,s}$ are determined by the given walk generating functions. As these matrices form a basis, we can write A as a linear combination of them. The coefficients in this linear combination are determined by the inner products

$$\langle A, A_{r,s}\rangle = \langle A, A^r \sigma\sigma^T A^s\rangle = \sigma^T A^{r+s+1}\sigma.$$

Hence these coefficients are determined by the walk generating function $W_{S,S}(X,t)$; since this equals $\sum_{i\in S} W_{i,S}(X,t)$, the theorem follows. □

If $W_{i,S} = W_{j,S}$ then

$$\langle e_i - e_j, \mathcal{A}^r \sigma\rangle = 0$$

for all non-negative integers r. But if the dual degree of S is $n-1$ then the vectors $\mathcal{A}^r\sigma$ span $\mathbb{R}^n$, consequently in this case $e_i - e_j$ must zero, and therefore i and j are equal. Note also that $W_{i,S}(X,t)$ is determined by its first n coefficients. We can order the set of all real generating functions by writing $F(t) > G(t)$ if the first non-zero coefficient of $F(t) - G(t)$ is positive. The point of Theorem 8.4 is that, under the given hypothesis, the graph X is determined by the ordered set of walk generating functions. Thus it provides an effective isomorphism test.

9. Trees

Let D be the diagonal matrix whose i-th diagonal entry is the valency of the i-th vertex of the graph X. If X is connected then 0 is a simple eigenvalue of $A - D$ with eigenvector $\mathbf{1}$; it follows from the spectral decomposition that the all-ones matrix J is a polynomial in $A - D$. Hence, when X is connected, the algebra generated by A and J is contained in the algebra generated by A and D. If X is a tree we can say more about the latter algebra. What we do say has not been published before.

First we need some more notation. A walk in X is *closed* if its first and last vertices are the same; it is *irreducible* if any two vertices at distance two in the sequence are distinct. If X is a tree then any two vertices are joined by exactly one irreducible walk, and the length of this walk is the distance between the two vertices.

Let A be the adjacency matrix of X, and let $p_r(A)$ denote the matrix with ij-entry equal to the number of irreducible walks of length r from i to j in X. Define the generating function $\Phi(X, t)$ by

$$\Phi(X, t) = \sum_r p_r(A)t^r.$$

Note that $p_0(A) = I$ and $p_1(A) = A$. As a walk of length two is either closed or irreducible, $p_2(A) = A^2 - D$. Our next result implies that $p_r(A)$ is a polynomial in A and D, with degree r in A.

9.1 Theorem. *Let A be the adjacency matrix of X. We have*

$$(t^2(D - I) - tA + I)\Phi(X, t) = (1 - t^2)I. \tag{9.1}$$

Proof. The coefficient of t^r in (9.1) is

$$(D - I)p_{r-2}(A) - Ap_{r-1}(A) + p_r(A);$$

with the understanding that $r \geq 0$ and $p_r(A) = 0$ when $r < 0$.

Suppose that i and j are vertices of X. Assume further that $r \geq 3$. Then $(Ap_{r-1}(A))_{i,j}$ equals the number of walks of length r from i to j that are formed by a walk of length 1 from i to one of its neighbours, ℓ say, followed by an irreducible walk of length $r - 1$ from ℓ to j. This class of walks includes all irreducible walks of length r from i to j. It also contains the walks that start (i, ℓ, i), followed an irreducible walk of length $r - 2$ from i to j with second vertex distinct from ℓ, that is, an irreducible walk of length $r - 2$ from i to j with second vertex not equal to ℓ. The number of these walks is $((D - I)p_{r-2}(A))_{i,j}$.

Accordingly we have proved that the coefficient of t^r in $(t^2(D - I) - tA + I)\Phi(X, t)$ is zero when $r \geq 3$. We have

$$\Phi(X, t) = I + tA + t^2(A^2 - D) + t^3M,$$

for some matrix M (with entries formal power series in t). Now

$$(I - tA + t^2(D - I))(I + tA + t^2(A^2 - D) + t^3M) = I + t^2(A^2 - D - A^2 + D - I) + t^3N,$$

where the entries of N are, again, formal power series in t. It follows from this that the left and right sides of (9.1) are equal as generating functions. $\square$

9.2 Corollary. *Let A be the adjacency matrix of the graph X, and let D be the diagonal matrix of valencies of X. Then $p_r(A)$ is a polynomial in A and D with degree r in A.* □

If X has vertex set V, let X_i denote the graph with the same vertex set, where two vertices are adjacent in X_i if and only if they are at distance i in X. (Thus $X_1 = X$.) Let A_i be the adjacency matrix of X_i, and let A_0 denote the identity matrix of order $|V| \times |V|$. If r is greater than the diameter of X then X_r has no edges and $A_r = 0$. We call the A_i's the *distance matrices* of X.

9.3 Corollary. *If X is a tree with diameter d then $\langle A, D\rangle$ contains $A_0, \dots, A_d$.*

Proof. If X is a tree then two vertices are at distance r if and only if there is a unique irreducible walk of length r joining them. Therefore $A_r = p_r(A)$ and

$$\sum_r A_r t^r = (1-t^2)(I - tA + t^2(D-I))^{-1}$$

and therefore the matrices A_r are polynomials in A and D. □

The remaining results in this section have no bearing on the main topic of these notes, but may still be interesting.

If X is a graph on n vertices, $W(X,t)$ denotes the $n \times n$ matrix with ij-entry equal to the generating function for the walks in X from i to j, counted by length. We call $W(X,t)$ the *walk generating function* of X. It is not hard to show that, if A is the adjacency matrix of X, then

$$W(X,t) = (I - tA)^{-1}.$$

Our next result shows that, for regular graphs, $\Phi(X,t)$ is determined by $W(X,t)$.

9.4 Corollary. *If X is a regular graph with valency k then*

$$\Phi(X,t) = \frac{1-t^2}{1+(k-1)t^2} W(X, t(1+(k-1)t^2)^{-1}).$$ □

Suppose that X has exactly c connected components. It follows from Theorem 9.1 that each entry of $\Phi(X,t)$ is the product of a formal power series in t with the polynomial $1-t^2$, given this it can be shown that $(1-t^2)^c$ divides $\det \Phi(X,t)$.

9.5 Lemma. *Let X be a graph. Then*

$$\det(I - tA + t^2(D - I)) = 1 - t^2$$

if and only if X is a tree.

Proof. We first show that the result holds when X is a tree. We proceed by induction on the number of vertices, noting that it can be easily verified when X has two vertices. Suppose X has at least three vertices. Suppose i is a vertex in X with valency 1, adjacent to a vertex j. with valency at least 2. Let Y be the graph got by deleting i from X. Let D_1 and A_1 be respectively the valency and adjacency matrices of Y. If

$$M := I - tA + t^2(D - I)$$

then $M_{i,i} = 1$, $M_{i,j} = -t$ and $M_{i,r} = 0$ if $r \neq i, j$. If we add t times column i of M to column j and delete row and column i, the resulting matrix is $I - tA_1 + t^2(D_1 - I)$, and its determinant is equal to $\det(M)$.

Now we prove the converse. By Theorem 9.1,

$$\Phi(X, t) = (1 - t^2)(I - tA + t^2(D - I))^{-1}.$$

If $\det(I - tA + t^2(D - I)) = (1 - t^2)$, this implies that the entries of $\Phi(X, t)$ are polynomials in t and therefore there is an upper bound on the length of an irreducible walk in X. Consequently X has no cycles. From the first part of the proof, it follows that $\det(I - tA + t^2(D - I))$ is $(1 - t^2)^c$, where c is the number of components of X; this forces us to conclude that X is a tree. □

It is left an exercise to show that $\det(I - tA + t^2(D - I))$ divides $(1 - t^2)^n$ if and only if X is a forest.

Finally, we note that differentiating both sides of (9.1) with respect to t yields, after some manipulation, that

$$(I - tA + t^2(D - I))\Phi'(X, t)(I - tA + t^2(D - I)) = (t^2 + 1)A - 2tD.$$

Here $\Phi'(X, t)$ is the derivative of $\Phi(X, t)$ with respect to t. If X is a forest then $\Phi'(X, 1)$ is the distance matrix of X.

Compact Graphs

Compact graphs are a class of graphs with the property that, if a non-identity automorphism exists then we can find one in polynomial time. Equitable partitions provide a useful tool to study them with. Our treatment follows [13], with some adjustments occasioned by interesting work of Evdokimov, M. Karpinski and I. Ponomarenko [9].

10. A Convex Polytope

Let $\mathcal{M}$ be a matrix algebra, for example, the adjacency algebra of a graph. We define $\mathcal{S}(\mathcal{M})$ to be the set of all doubly stochastic matrices that commute with each element of $\mathcal{M}$. When $\mathcal{M} = \langle A \rangle$, we will usually write $\mathcal{S}(A)$ rather than $\mathcal{S}(\mathcal{M})$. The set $\mathcal{S}(\mathcal{M})$ is convex and contains the permutation matrices that commute with each element of $\mathcal{M}$. If G and H are two matrices in $\mathcal{S}(\mathcal{M})$ and $F = tG + (1-t)H$ for some t in the interval $[0,1]$ then $F_{i,j} \neq 0$ if either $G_{i,j} \neq 0$ or $H_{i,j} \neq 0$. From this we see that any permutation matrix in $\mathcal{S}(\mathcal{M})$ is an extreme point. Slightly modifying Evdokimov et al, we define $\mathcal{M}$ to be *compact* if all the extreme points of $\mathcal{S}(\mathcal{M})$ are permutation matrices. Note that when $\mathcal{M}$ is finitely generated the set $\mathcal{S}(\mathcal{M})$ is polyhedral—if $\mathcal{M}$ is generated by $A_1, \ldots, A_r$ then $\mathcal{S}(\mathcal{M})$ consists of the matrices F such that

$$\begin{aligned} F &\geq 0, \\ A_i F = F A_i, &\quad i = 1, \ldots, r \\ F\mathbf{1} &= \mathbf{1}, \\ F^T\mathbf{1} &= \mathbf{1}. \end{aligned}$$

As $\mathcal{S}(\mathcal{M})$ is clearly bounded, it will therefore be a convex polytope.

We say that a graph X is compact if its adjacency algebra is compact. In this case the terminology, and many of the basic results, are due to Tinhofer [30].

Why is this property significant? One reason is that it is possible to find extreme points of $\mathcal{S}(A)$ in polynomial time. If all extreme points of $\mathcal{S}(A)$ belonged to $\mathrm{Aut}(A)$ then we would be able to decide, in polynomial time, whether a graph admitted an non-identity automorphism. In general, no such algorithm is known.

The complete graphs are compact, a fact which is equivalent to the statement that any doubly stochastic matrix is a convex combination of permutation matrices. Cycles and trees are compact graphs; the disjoint union of two connected non-isomorphic k-regular graphs is not compact. (For proofs of these assertions, see [30].) The complement of a compact graph is compact, as is easily shown. The next result provides further evidence that being compact is a strong condition.

10.1 Lemma. *A compact regular graph is vertex-transitive.*

Proof. Suppose X is compact and regular, with adjacency matrix A. Then $\frac{1}{|V|}\mathcal{J} \in \mathcal{S}(A)$ and, as X is compact, we may assume that

$$\mathcal{J} = \sum_i a_i P_i$$

where $a_i \geq 0$ and P_i is an automorphism of X, for all i. Assume $1 \in V(X)$. Since all entries in J are equal to 1 it follows that, for each r, there is a permutation matrix in the above sum, P say, such that $P_{1,r} = 1$. Hence, for each vertex r of X, there is an automorphism of X that maps 1 to r, and thus $\mathrm{Aut}(X)$ acts transitively on $V(X)$ as claimed. □

Lemma 10.1 is not the full truth, we have stated it here because the argument used in its proof is important. We now offer a more precise result, from [13], but with a new proof. Note that a permutation group on a set V is *generously transitive* if, given any two points in X, there is a permutation that swaps them.

10.2 Lemma. *If X is regular, connected and compact, then* $\mathrm{Aut}(X)$ *is generously transitive.*

Proof. Let $G = \mathrm{Aut}(X)$. As X is compact, the matrices in G span the same subspace as the matrices in $\mathcal{S}(A)$. As X is regular and connected, and matrix which commutes with A commutes with J, and therefore its row and columns sums are all equal. Hence the real span of the matrices in $\mathcal{S}(A)$ is equal to $C(A)$, and consequently the centralizer of G in $M_{n\times n}(\mathbb{R})$ is $C(C(A)) = \langle A\rangle$, [28, Theorem 39.3], which is commutative. Hence G is multiplicity free, with rank equal to the number of eigenvalues of A [2, Example 2.1]. Also, $\langle A\rangle$ is an association scheme—since it is generated by A and $A = A^T$, all matrices in $\langle A\rangle$ are symmetric and so all orbitals of G are symmetric. Hence G is generously transitive. □

11. Equitable Partitions

Let π be an equitable partition of X, and let D be the characteristic matrix of π. Let P be defined by

$$P = D(D^T D)^{-1} D^T.$$

Then P is symmetric, $P^2 = P$ and P and D have the same column space. Hence P is the matrix representing orthogonal projection onto the column space of D, i.e., onto $F(V, \pi)$. Note that $D^T D$ is a diagonal matrix, with i-th diagonal entry equal to the size of the i-th cell of π, and that P is a block-diagonal matrix with each diagonal block a multiple of the 'all-ones' matrix. In particular, all entries of P are non-negative. Because $\mathbf{1} \in F(V, \pi)$ it follows that $P\mathbf{1} = \mathbf{1}$, which implies that each row of P sums to 1. Similarly each column sums to 1 and therefore P is a doubly stochastic matrix.

11.1 Lemma. *Let X be a graph and let π be a partition of $V(X)$. Then π is equitable if and only if the matrix P representing orthogonal projection onto $F(V, \pi)$ lies in $\mathcal{S}(A)$.*

Proof. The entries of P are non-negative, as we saw above. Because $\mathbf{1} \in F(V, \pi)$ it follows that $P\mathbf{1} = \mathbf{1}$ and, because P is symmetric, it is therefore a doubly stochastic matrix. The column space of P is $F(V, \pi)$, which is A-invariant. Hence each column of AP lies in $F(V, \pi)$ and therefore $PAP = AP$. Accordingly

$$PA = (AP)^T = (PAP)^T = PAP = AP,$$

which proves that A and P commute and therefore $P \in \mathcal{S}(A)$. Conversely, if P and A commute then

$$AP = APP = PAP,$$

since P represents the orthogonal projection onto $F(V, \pi)$, $AP \in F(V, \pi)$. As a result, $F(V, \pi)$ is A-invariant, and by Lemma 5.2, π is equitable. □

11.2 Lemma. *If X is a compact graph, each equitable partition is an orbit partition.*

Proof. Let π be an equitable partition of the compact graph X, and let P be the matrix representing orthogonal projection on $F(V(X), \pi)$. Then $P \in \mathcal{S}(A(X))$, and hence P is a convex combination of automorphisms of X. Arguing as in the proof of Lemma 10.1, it follows that the cells of π are the orbits of some group of automorphisms of X. □

One consequence of this is that the line graph of K_n, $L(K_n)$, is not compact when $n \geq 7$—the spanning subgraph of K_n formed by a 3-cycle and and $(n-3)$-cycle, with its complement, determines an equitable partition of $L(K_n)$ with two cells that is not an orbit partition.

But a stronger statement is true. Evdokimov, M. Karpinski and I. Ponomarenko [9] have recently proved that the Johnson graph $J(v,k)$ is compact if and only if $k = 1$ or $v = 4$ and $k = 2$. This answers a question raised in an earlier draft of this work, where we asked whether the Petersen graph is compact. Their argument is based on the following result.

11.3 Lemma. *If X is a compact regular graph then* $\mathrm{Aut}(X)$ *contains automorphisms that map each vertex of X to an adjacent vertex.*

Proof. Suppose that X has valency k and adjacency matrix A. Then $\frac{1}{k}A$ belongs to $\mathcal{S}(A)$, hence is a convex combination of automorphisms. Each of these automorphisms maps any vertex to and adjacent vertex. □

It is a simple exercise using this to determine the compact Johnson graphs. Evdokomov et al also show that the n-cube is compact if and only if $n \geq 4$.

We have shown that each equitable partition of X determines an element of $\mathcal{S}(A)$. There is a converse to this. If $M \in \mathcal{S}(A)$, let X_M denote the directed graph with vertex set $V(X)$ and with arc from i to j if and only if $M_{i,j} \neq 0$.

11.4 Theorem. *Let X be a graph and suppose that $M \in \mathcal{S}(A)$. Then the strong components of X_M form an equitable partition of X.*

Proof. Let Y denote X_M and assume that C is a subset of $V(Y)$ such that there is no arc (u,v) with $u \in C$ and $v \notin C$. Then the sum of the entries of M in the rows corresponding to C is $|C|$, whence the sum of the entries in the submatrix of M with rows and columns indexed by C is again $|C|$. But this implies that if $v \notin C$ and $u \in C$ then $(M)_{vu} = 0$, and therefore there are no arcs in Y from a vertex not in C to a vertex in C. It follows that, as M is doubly stochastic, we may write in block-diagonal form as

$$M = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_r \end{pmatrix}$$

where $M_1, \ldots, M_r$ are doubly stochastic matrices and the graphs X_{M_i} are strongly connected.

Let π be the partition whose cells are the vertex sets of the strong components of Y, and let P be the matrix representing orthogonal projection on $F(V(X), \pi)$. We aim to show that P is a polynomial in M; since P is doubly stochastic it will follow that $P \in \mathcal{S}(A)$.

Assume that Y is a strongly connected directed graph on m vertices with adjacency matrix M_i. Then the matrix $\frac{1}{2}(M_i + I)$ is doubly stochastic and, for some s,

$$N = (\frac{1}{2}(M_i + I))^s > 0.$$

Choose $1 \geq c > 0$ maximal such that $N - \frac{c}{m}J \geq 0$. Now, we want to show by induction that for all $k \geq 1$,

$$N^r \geq (1 - (1-c)^k)\frac{1}{m}J.$$

This is true when $k = 1$. Suppose the inequality holds for all $k \leq t$. Then

$$\begin{aligned}(N - c\frac{1}{m}J)(N^t - (1-(1-c)^t)\frac{1}{m}J) &= N^{t+1} - (1-(1-c)^{t+1})\frac{1}{m}J \\ &\geq 0.\end{aligned}$$

Since $1 \geq c > 0$, we have

$$\lim_{t\to\infty} N^t \geq \lim_{t\to\infty}(1-(1-c)^t)\frac{1}{m}J = \frac{1}{m}J.$$

Because N^t is doubly stochastic for all $t \geq 0$, we have $\lim_{t\to\infty} N^t = \frac{1}{m}J$. Consequently, any matrices that commute with M_i would commute with $\frac{1}{m}J$, that is, $\frac{1}{m}J \in C(C(M_i))$. In [28], Theorem 39.3 states that any matrix in $C(C(M_i)) = \langle M_i \rangle$; as $\frac{1}{m}J \in C(C(M_i))$ it follows that $J = q_i(M_i)$ for some polynomial q_i.

Accordingly, for $i = 1, \ldots, r$ there is a polynomial q_i such that

$$q_i(M_i) = \frac{1}{|V(Y_i)|}\mathcal{J};$$

It is now a straightforward exercise to show that, if $q := \prod_i q_i$, then $q(M)$ is the matrix representing orthogonal projection P on $F(V(Y), \pi)$. □

11.5 Corollary. *The discrete partition is the only equitable partition of X if and only if $\mathcal{S}(A) = \{I\}$.*

To complete this section, we comment further on the recent interesting work by Evdokimov, M. Karpinski and I. Ponomarenko [9]. A *cellular algebra* is a matrix algebra which contains I and J and is closed under Schur multiplication. (Cellular algebras are also known as coherent algebras, and Schur multiplication is often referred to as Hadamard multiplication.) It is not too hard to show that an automorphism of a graph X commutes with each elements of the cellular algebra generated by $A(X)$. Evdokimov et al define a graph X to be *weakly compact* if the cellular algebra generated by its adjacency matrix is compact. Every compact graph must be weakly compact. However Evdokomov et al show that if $\mathrm{Aut}(X)$ is a regular permutation group, or has an abelian regular subgroup of index two, then X is weakly compact. Since the automorphism group of a compact regular graph must be generously transitive, it follows that there are many weakly compact graphs that are not compact. Their works indicates that the class of weakly compact graphs is at least as nice as the class of compact graphs. In particular, they show that the automorphism group of a weakly compact graph can be computed in polynomial time.

We can define a partition π of $V(X)$ to be equitable relative to the cellular algebra $\mathcal{C}$ generated by $A(X)$ if $F(V,\pi)$ is invariant under $\mathcal{C}$. With this definition, Lemma 11.1 and Theorem 11.4 can be extended without further work to weakly compact graphs.

Distance Regular Graphs

The material in these last three sections follows the treatment from [11, Chapters 11 & 13]. (Fairly closely, which may not be a surprise.)

12. Definitions

We define a graph X to be *distance regular* if

(a) it is connected,
(b) for each vertex u, the distance partition ∂_u of X relative to u is equitable, and
(c) the quotient X/∂_u is independent of u.

In fact condition (c) can be replaced by the assumption that X is regular, although this is not trivial to prove. (See [18].) We consider one family of examples. The vertices of the *Johnson graph* $J(v,k)$ are the k-subsets of a fixed set of size k; two k-subsets are adjacent if their intersection has size $k-1$. The easiest way to see this graph is distance regular is to show that the distance partition relative to a vertex u is also the orbit partition for the stabilizer of u in $\mathrm{Aut}(J(v,k))$.

A graph X is *distance transitive* if, given two ordered pairs (u,u') and (v,v') such that

$$\mathrm{dist}(u,u') = \mathrm{dist}(v,v')$$

there is an automorphism a of X such that $ua = v$ and $u'a = v'$. Every distance transitive graph is distance regular, but the converse is far from true. We note one class of examples, *Latin square graphs*. Let L be an $n\times n$ Latin square, which we view as a set of n^2 ordered triples

$$(i,j,L_{i,j}),\quad 1\le i,j\le n.$$

(Here $L_{i,j}$ denotes the ij-entry of L.) Let $X(L)$ be the graph with these triples as vertices, where two triples are adjacent if they agree in some coordinate. Note that the definition of a Latin square implies that two triples can agree in at most one coordinate. The graph $X(L)$ has n^2 vertices, valency $3(n-1)$ and diameter 2. It is distance regular, but we leave the proof of this for the reader. There are examples of Latin squares L of order as low as 8 such that $X(L)$ has trivial automorphism group. (For details, see [27].)

A *Steiner triple system* is a collection of 3-subsets, called blocks, of a v-set such that every 2-subset of the v-set occurs in exactly one of the blocks. The line graph of a Steiner triple system has the blocks being its vertices, and two blocks are adjacent if and only if their intersection is non-empty. The line graphs of a Steiner triple system are distance regular with diameter two, which are also called *the strongly regular graphs*. Mendelsohn has shown that every finite group is the automorphism group of a finite Steiner triple system and a strongly regular graph, [23, 24].

If X is a connected graph, let X_r denote the graph with the same vertex set as X, but with two vertices adjacent in X_r if and only if they are at distance r in X. Let A_r denote the adjacency matrix of X_r and let A_0 denote the identity matrix. Let J denote the matrix with all entries equal to 1. If X is connected with diameter d, then the following conditions hold (as the reader may verify).

1. $A_0 = I$,
2. $\sum_{r=0}^{d} A_r = J$,
3. $A_r^T = A_r$.

Furthermore, the connected graph X is distance regular if and only if the following holds [11],

4. $A_r A_s$ belongs to the real span of $A_0, \dots, A_d$, for all r and s.

This shows that the matrices form an *association scheme with d classes*. For background on association schemes see, e.g., [8,2,4,11].

12.1 Theorem. *Let X be a distance regular graph. Then the walk generating function $W_{i,j}(X,t)$ is determined by the distance between i and j in X.*

Proof. It follows from Axiom 4 above that A_1^m lies in the span of $A_0, \dots, A_d$, for all non-negative integers m. Hence the ij-entry of A_1^m is determined by the distance between i and j in X. □

13. Cosines

Suppose X is a distance regular graph with diameter d and let u_θ be the representation belonging to the eigenvalue θ of $\mathcal{A}$. As we noted at the end of the last section, the walk generating function $W_{i,j}(X,t)$ is determined by the distance between i and j in X. One consequence of this is that $W_{i,i}(X,t)$ is independent of the choice of the vertex i, and therefore X is walk-regular, in the sense used at the end of Section 3. This means that all vectors $u_\theta(i)$ have the same length, and hence that the geometry of the points $u_\theta(i)$, for i in $V(X)$, is determined by the inner products $\langle u_\theta(i), u_\theta(j)\rangle$. Hence we define the *cosine*

$w_\theta(r)$ to be the cosine of the angle between $u_\theta(i)$ and $u_\theta(j)$, where i and j are two vertices at distance r in X.

Let ∂ be the distance partition of X relative to some vertex. As X is distance regular, this partition is equitable. Let $\mathcal{B}$ denote the adjacency operator for the quotient X/∂. Then w_θ is a function on the vertices of X/∂, with the following important property.

13.1 Theorem. *Let θ be an eigenvalue of the distance-regular graph X and let ∂ be the distance partition of X relative to some vertex. Then w_θ is an eigenvector for the adjacency operator of X/∂.*

Proof. Assume ∂ is the distance partition relative to the vertex 1 of X, with cells $\partial_0, \dots, \partial_d$, and let i be a vertex at distance r from 1. Let $B_{r,s}$ denote the number of vertices in ∂_s adjacent to a given vertex in ∂_r; note that this is 0 if $|r-s|>1$. Thus the matrix B with rs-entry $B_{r,s}$ is tridiagonal. Let u be the representation of X afforded by the eigenspace belonging to θ. Then

$$\theta u(i) = \sum_{j\sim i} u(j).$$

A neighbour of i is at distance $r-1$, r or $r+1$ from 1. Hence, if we take the inner product of each side of the equality with $u(1)$ and then divide by the square of the length of $u(1)$, we find that

$$\theta w_\theta(r) = B_{r,r-1} w_\theta(r-1) + B_{r,r} w_\theta(r) + B_{r,r+1} w_\theta(r+1).$$

This proves our claim. □

One consequence of this theorem is that each eigenvalue of $\mathcal{A}$ must be an eigenvalue of $\mathcal{B}$. (It follows from Theorem 6.2 that each eigenvalue of $\mathcal{B}$ is an eigenvalue of $\mathcal{A}$.) This implies in turn that the eigenvalues of $\mathcal{A}$ are determined by the quotient X/∂. Since the number of vertices in X/∂ is usually much smaller than the number of vertices of X, this is somewhat surprising. Theorem 13.1 also implies that the function on $V(X)$ that maps each vertex in ∂_r to $w_\theta(r)$ is an eigenvector for $\mathcal{A}$ contained in $F(V,\partial)$.

The cosines belonging to an eigenvalue of a distance regular graph satisfy a number of useful identities. Assume that X has valency k. As

$$\theta u_\theta(i) = \sum_{j\sim i} u_\theta(j)$$

we have

$$\theta = \sum_{j \sim i} \frac{\langle u_\theta(j), u_\theta(i) \rangle}{\langle u_\theta(i), u_\theta(i) \rangle} = k w_\theta(1)$$

and therefore

$$w_\theta(1) = \frac{\theta}{k} \leq 1. \tag{13.1}$$

14. Feasibility

Let u be the representation belonging to the eigenvalue θ of the distance regular graph X, and let E be the matrix representing orthogonal projection on the eigenspace belonging to θ. If i and j are vertices of X at distance r then

$$E_{i,j} = \langle u(i), u(j) \rangle = w(r) \langle u(i), u(i) \rangle.$$

Let m be the multiplicity of θ. As $E^2 = E$, each eigenvalue of E is 0 or 1, hence $\operatorname{tr} E$ is equal to the rank of E, which is m. On the other hand

$$\operatorname{tr} E = \sum_{i \in V} \langle u(i), u(i) \rangle.$$

But X is walk-regular, whence $\langle u(i), u(i) \rangle$ is independent of i and therefore

$$\langle u(i), u(i) \rangle = \frac{m}{|V|}.$$

Consequently $E_{i,j} = w(r) m / |V|$.

As $E^2 = E$ and $E = E^T$, we have

$$\sum_{j \in V} (E_{1,j})^2 = E_{1,1}. \tag{14.1}$$

Let v_r denote the number of vertices at distance r from a fixed vertex in X. Then (14.1) can be rewritten as

$$\sum_r v_r \frac{w(r)^2 m^2}{|V|^2} = \frac{m}{|V|},$$

from which it follows that

$$\sum_r v_r w(r)^2 = \frac{|V|}{m}. \tag{14.2}$$

It can be shown that the integers v_r are determined by the quotient of X with respect to the distance partition of a vertex. Given this, (14.2) implies that the multiplicities of the eigenvalues of X can be determined from this quotient.

This leads to the most important feasibility condition for the existence of distance-regular graphs. We do not discuss this here, but simply refer the reader to [4,11]. Instead we consider a related result, which constrains the possible automorphisms of distance-regular graphs. The result is due to G. Higman, although the first published account is in Cameron [6]. (And our treatment follows Section 13.8 of [11].)

Let X be a distance-regular graph with diameter d. If $a \in \mathrm{Aut}(X)$, let $v_r(a)$ denote the number of vertices i of X such that ia is at distance r from i. An algebraic integer is a complex number that satisfies a monic polynomial with integer coefficients.

14.1 Theorem (G. Higman). *Let X be a distance-regular graph with diameter d and let θ be an eigenvalue of X with multiplicity m. If $a \in \mathrm{Aut}(X)$ then*

$$\frac{m}{|V(X)|}\sum_{r=0}^{d} v_r(a)w_\theta(r)$$

is an algebraic integer.

Proof. Let $u = u_\theta$ be the representation belonging to θ. If $a \in \mathrm{Aut}(X)$ then

$$a_\theta = U_\theta^T \hat{a} U_\theta.$$

Given that $\hat{a}$ and $E_\theta = U_\theta U_\theta^T$ commute, it is easy to show that

$$a_\theta^r = U_\theta^T (\hat{a})^r U_\theta.$$

Now $a^m = 1$ for some integer m, whence $a_\theta^m = 1$ and hence each eigenvalue of a_θ is an m-th root of unity, and an algebraic integer. Accordingly, $\mathrm{tr}(a_\theta)$ is a sum of algebraic integers, and so is itself an algebraic integer.

To complete the proof, we show that the sum in the statement of the theorem is just the trace of a_θ. We have

$$\mathrm{tr}(a_\theta) = \mathrm{tr}(U_\theta^T \hat{a} U_\theta) = \mathrm{tr}(U_\theta U_\theta^T \hat{a}) = \mathrm{tr}(E_\theta \hat{a}).$$

To compute the last trace, we compute the ii-entry of $E_\theta \hat{a}$. This is the i-th entry of $E_\theta \hat{a} e_i$. But

$$E_\theta \hat{a} e_i = E_\theta e_{ia^{-1}}$$

and therefore

$$(E_\theta \hat{a})_{i,i} = (E_\theta)_{i,ia^{-1}} = \langle u(i), u(ia^{-1})\rangle.$$

This implies that

$$\mathrm{tr}(a_\theta) = \sum_i \langle u(i), u(ia^{-1})\rangle = \sum_r v_r(a)\frac{m}{|V|}w_\theta(r),$$

from which the theorem follows. □

Following G. Higman, Cameron [6] uses Theorem 14.1 to show that the Moore graph on 3250 vertices cannot be vertex-transitive, if it exists.

Theorem 14.1 can also be used to that show there is no automorphism of the line graph of K_n that maps each vertex to a distinct, non-adjacent vertex. Suppose such an automorphism a exists, that is $v_0(a) = v_1(a) = 0$ and $v_2(a) = |V(L(K_n))|$. It is sufficient to show that $m_\theta w_\theta(2)$ is not an algebraic integer for some eigenvalue θ of $L(K_n)$. The graph $L(K_n)$ is a strongly regular graph, from [11: Chapter 5], the eigenvalues of $L(K_n)$ are $k = 2(n-2)$, $\theta = (n-4)$ and $\tau = -2$ with multiplicities 1, $n-1$ and $\frac{n(n-3)}{2}$ respectively. We have

$$w_\theta(1) = \frac{\theta}{k} = \frac{n-4}{2(n-2)}.$$

The complement of $L(K_n)$ is also strongly regular, therefore,

$$w_\theta(2) = \frac{-\theta - 1}{v - 1 - k} = \frac{-2}{(n-2)},$$

and

$$m_\theta w_\theta(2) = \frac{-2(n-1)}{(n-2)}$$

which is not an algebraic integer when $n \geq 4$.

A similar argument shows that the block graph of a Steiner triple system cannot have an automorphism that maps each vertex to a distinct and non-adjacent vertex.

References

[1] L. Babai, D. Yu. Grigoryev and D. M. Mount, Isomorphism of graphs with bounded eigenvalue multiplicity, *Proc 14th ACM STOC* (1982), 310–324.

[2] E. Bannai and T. Ito, *Algebraic Combinatorics I.* (Benjamin/Cummings, Menlo Park, CA.) 1984.

[3] N. Biggs, *Algebraic Graph Theory*, 2nd Edition. (Cambridge U. P., Cambridge.) 1993.

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs.* (Springer, Berlin) 1989.

[5] R. A. Brualdi, Some applications of doubly stochastic matrices, *Linear Alg. Appl.*, **107** (1988), 77–100.

[6] P. J. Cameron, Automorphism groups of graphs, in: *Selected Topics in Graph Theory, Volume 2.* Edited by L. W. Beineke and R. J. Wilson. (Academic Press, London) 1983, pp. 89–127.

[7] D. G. Corneil and C. C. Gottleib, An efficient algorithm for graph isomorphism, *JACM* **17** (1970) 51-64.

[8] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Research Reports Supplements* 1973, No. 10.

[9] S. Evdokimov, M. Karpinski and I. Ponomarenko, Compact cellular algebras and permutation groups, preprint, 1996.

[10] C. D. Godsil, Graphs, groups and polytopes, in: *Combinatorial Mathematics.* Edited by D. A. Holton and Jennifer Seberry, Lecture Notes in Mathematics 686, (Springer, Berlin) 1978, pp. 157–164.

[11] C. D. Godsil, *Algebraic Combinatorics.* (Chapman and Hall, New York) 1993.

[12] C. D. Godsil, Equitable partitions, in: *Combinatorics, Paul Erdős is Eighty (Volume 1).* Edited by D. Miklós, V. T. Sós and T. Szőnyi, (János Bolyai Mathematical Society, Hungary) 1993.

[13] C. D. Godsil, Compact graphs and equitable partitions, *Linear Alg. Appl.*, to appear.

[14] C. D. Godsil, Eigenpolytopes of distance regular graphs, *Canadian J. Math.*, to appear.

[15] C. D. Godsil and W. J. Martin, Quotients of association schemes, *J. Combinatorial Theory, Ser. A,* **69** (1995), 185–199.

[16] C. D. Godsil and B. D. McKay, Feasibility conditions for the existence of walk-regular graphs, *Linear Algebra Appl.,* **30** (1980), 51–61.

[17] C. D. Godsil and B. D. McKay, Spectral Conditions for the reconsructibility of a graph, *J. Combinatorial Theory, Series B,* **30** (1981), 285–289.

[18] Chris D. Godsil and John Shawe-Taylor, Distance-regularised graphs are distance-regular or distance biregular, *J. Combinatorial Theory, Series B,* **43** (1987) 14–24.

[19] D. Gorenstein, *Finite Groups.* (Harper & Row, New York) 1968.

[20] T. J. Laffey, A basis theorem for matrix algebras, *Linear and Multilinear Algebra,* **8** (1980), 183–187.

[21] P. Lancaster and M. Tismenetsky, *The Theory of Matrices: with Applications*, 2nd edition. (Academic Press, New York) 1985.

[22] B. D. McKay, *Backtrack Programming and the Graph Isomorphism Problem.* M. Sc. Thesis, University of Melbourne, 1976.

[23] E. Mendelsohn, Every (finite) group is the group of automorphisms of a (finite) strongly regular graph, *Ars Combinatoria,* **6** (1978), 75–86.

[24] E. Mendelsohn, On the groups of automorphisms of Steiner triple and quadruple systems, *Journal of Combinatorial Theory. Series A,* **25** (1978), 97–104.

[25] A. Moshowitz, Graphs, groups and matrices, in: *Proc. 25th Summer Meeting Can. Math. Congress, Cong. Num. IV.* (Utilitas Math., Winnipeg) 1971.

[26] M. Petersdorf and H. Sachs, Spektrum und Automorphismengruppe eines Graphen, in *Combinatorial Theory and its Applications, III.* (North-Holland, Amsterdam.) pp. 891–907.

[27] K. Phelps, Latin square graphs and their automorphism groups, *Ars Combin.* **7** (1979), 273–299.

[28] V. V. Prasolov, *Problems and Theorems in Linear Algebra.* (A.M.S., Providence) 1994.

[29] H. Schreck and G. Tinhofer, A note on certain subpolytopes of the assignment polytope associated with circulant graphs, *Linear Algebra Appl.,* **111** (1988), 125–134.

[30] G. Tinhofer, Graph isomorphism and theorems of Birkhoff type, *Computing* **36** (1986), 285–300.

[31] G. Tinhofer, A note on compact graphs, *Discrete Applied Math.*, **30** (1991), 253–264.