Homomorphisms and Cores of Vertex-Transitive Graphs

Chris Godsil

Combinatorics & Optimization
University of Waterloo

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Outline

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Definition

Let X and Y be graphs with vertex sets $V(X)$ and $V(Y)$ respectively.

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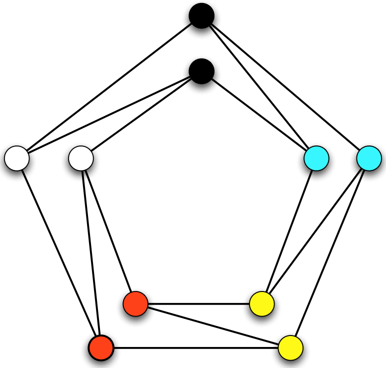
A homomorphism f from X to Y is a map from $V(X)$ to $V(Y)$ such that whenever u and v are adjacent vertices in X, the vertices $f(u)$ and $f(v)$ are adjacent in Y.

Our graphs do not have loops, so the images of adjacent vertices under a homomorphism are always distinct.

Trivial Examples

- Any isomorphism from X to Y is a homomorphism.
- If X is a subgraph of Y then the identity map from $V(X)$ to $V(Y)$ is a homomorphism.

A Better Example



Fibers

Definition

If $f : X \to Y$ is a homomorphism and $y \in V(Y)$, the preimage $f^{-1}(y)$ is the fibre of f at y.

Thus f is an isomorphism if its fibres are either single vertices or empty.
More importantly, the subgraph of X induced by a fibre is an independent set (or coclique).

Colourings

Since the fibres of a homomorphism are cocliques, any homomorphism f defined on X determines a partition of $V(X)$ into cocliques—one for each vertex in the image of f.

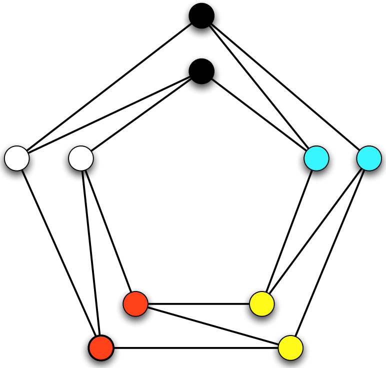
Hence a graph X has a proper colouring with m colours if and only if there is a homomorphism from X to the complete graph K_m.

Foldings

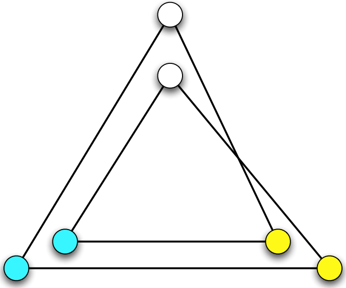
Definition

If u and v are two vertices at distance two in X, and Y is the graph we get by identifying u and v, then there is a homomorphism from X to Y. This a simple folding. We say f is folding if it is a composition of simple foldings.

Example



Not an Example



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- Homomorphisms
- Cores

2 Vertex-Transitive Graphs
- Introduction
- Core-Complete Graphs
- Cubelike Graphs

Equivalence

Definition

Two graphs X and Y are homomorphically equivalent if there are homomorphisms f and g such that

$$f : X \to Y, \quad g : Y \to X.$$

If X and Y are homomorphically equivalent, we write $X \leftrightarrow Y$. We will also find it useful to use $X \to Y$ to denote that there is a homomorphism from X to Y.

Examples

$X \leftrightarrow K_2$ if and only if X is bipartite and has at least one edge.

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$X \leftrightarrow K_m$ if and only if $\chi(X) = \omega(X) = m$.

Equivalence Classes

Let $[X]$ denote the homomorphic equivalence class of X.

Lemma

Let Y_1 and Y_2 be graphs in $[X]$ with the least possible number of vertices. Then $Y_1 \cong Y_2$.

Proof.

Suppose $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_1$.

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- $gf : Y_1 \to Y_1$.
- Since $Y_1 \leftrightarrow gf(Y_1)$, the map gf is surjective.

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- Similarly fg must be surjective.

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PROOF.

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- $gf : Y_1 \to Y_1$.
- Since $Y_1 \leftrightarrow gf(Y_1)$, the map gf is surjective.
- Similarly fg must be surjective.
- Therefore f and g are isomorphisms.

$\square$

The Core of a Graph

Definition

The core of X is any graph in $[X]$ with the minimal number of vertices.

The core of X will be denoted by $X^\bullet$. Note that it is an induced subgraph of X.

Cores

Lemma

If Y is a graph such that any endomorphism of Y is an automorphism, then Y is the core of any graph in $[Y]$.

If Y has no proper endomorphisms, we will say that it is a core.

Examples

- Complete graphs.
- Odd cycles.
- Colour-critical graphs, e.g., the previous two classes or, for variety, the circulant on 13 vertices with connection set $\{1, 5, 8, 12\}$.

Retracts

A subgraph Y of X is a retract of X if there is a homomorphism $f : X \to Y$ such that $f \restriction V(Y) = 1_Y$. We say that f is a retraction.

Lemma

The core of a graph is a retract.

Retracts and Foldings

Lemma

If X is connected than any retraction of X is a folding.

In particular, if $f : X \to Z$ is a retraction then there vertices u and v at distance two in X such that $f(u) = f(v)$.

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2. What is the core of $L(K_n)$?

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VERTICES The 280 partitions of $\{1,\ldots,9\}$ into three triples.

EDGES Two partitions are adjacent if they are skew: each cell in one partition contains one point from each of the cells in the second.

The Same Question

What is the core of $\mathcal{P}(3 \times 3)$?

Some Theory

Theorem

If X is a vertex-transitive graph then:

1. *$X^\bullet$ is vertex transitive.*
2. *$|V(X^\bullet)|$ divides $|V(X)|$.*

Corollary

If X is connected, vertex transitive and cubic, then $X^\bullet$ is vertex transitive with valency 1, 2 or 3:

Valency=1 $X^\bullet = K_2$ and X is bipartite.

Valency=2 $X^\bullet$ is an odd cycle (and $\chi(X) = 3$).

Valency=3 $X = X^\bullet$.

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- We see that if $f : X \to X^\bullet$, then the restriction of f to an induced 5-cycle must be injective.
- Since each path in X with length two lies in a 5-cycle, f cannot identify two vertices at distance two.
- Since f is a folding, we conclude that the Petersen graph is a core. :-)

A Generalization

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One consequence of this is that if X is triangle-free with diameter two and no two distinct vertices have the same neighbors, then X is a core.

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Assume $X = L(K_n)$ and $Z = X^\bullet$. We claim that either $X^\bullet = X$ or $X^\bullet$ is a complete graph. Asume by way of contradiction that $Z = X^\bullet$ is not complete and let z_1 and z_2 be two vertices in Z at distance two.

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- If $Z \neq X$, we may assume there is a retraction $f : X \to Z$ such that $f(x_1) = f(x_2)$ for some pair of vertices x_1 and x_2 at distance two in X.
- There is an automorphism γ of X that maps z_1 to x_1 and z_2 to x_2.
- The composition of γ followed by f is an endomorphism of Z that maps z_1 and z_2 to the same vertex. But since Z has no proper endomorphisms, this is impossible.

$L(K_{2m})^\bullet$

If $n = 2m$, then K_n has a 1-factorization. The image in $L(K_n)$ of a 1-factor of K_n is a coclique of size m and we have a homomorphism

$$L(K_{2m}) \to K_{m-1}.$$

From this we conclude that $L(K_{2m})^\bullet = K_{m-1}$.

$L(K_{2m+1})^\bullet$

Suppose $n = 2m+1$. Then $|V(L(K_{2m+1})| = m(2m+1)$, which is not divisible by $2m$. So although $L(K_{2m+1})$ contains cliques of size $2m$, they cannot be cores. Since $\omega(L(K_{2m+1})) = 2m$, the core of $L(K_{2m+1})$ cannot be a clique of size $2m+1$ or more. We conclude that $L(K_{2m+1})^\bullet = L(K_{2m+1})$.

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Godsil & Royle:

Block graphs of Steiner systems and orthogonal arrays are either cores, or their cores are complete.

Outline

Some Special Cayley Graphs

We consider Cayley graphs for $\mathbb{Z}_2^d$. Such a Cayley graph is specified by a subset $\mathcal{C}$ of the non-zero vectors in $\mathbb{Z}_2^d$, as follows

VERTICES: $\mathbb{Z}_2^d$.

EDGES: $u \sim v \iff v - u \in \mathcal{C}$.

The Cayley graph is connected if and only if $\mathcal{C}$ spans $\mathbb{Z}_2^d$. If $\mathcal{C}$ is a basis, then the graph is the d-cube.

Linear Algebra

Since we are working over the vector space $\mathbb{Z}_2^d$, it is natural to represent a connection set $\mathcal{C}$ by a $d \times |\mathcal{C}|$ matrix D over $\mathbb{Z}_2$ (with no zero columns). From this view point $\mathbb{Z}_2^d$ is the column space of D, and two vectors in $\mathbb{Z}_2^d$ are adjacent if their difference is a column of D.

Example

If we have

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Then D has rank four and our Cayley graph will have 16 vertices and valency 5.

Example,ctd

Applying elementary row operations to D will not change the graph. If we add each of the first four rows to the one below it, D becomes

$$D' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we see that the Cayley is graph obtained from the 4-cube by joining each pair of vertices at distance four: it is the Clebsch graph.

Free Cayley Graphs

We have the following particular case of an unpublished result, due to Nasrasr and Tardif:

Theorem

Let X be a graph on v vertices with d edges and incidence matrix D over $\mathbb{Z}_2$. Let $\mathcal{F}(X)$ denote the Cayley graph for $\mathbb{Z}_2^d$ with connection matrix D.

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If there is a graph homomorphism f from X into a Cayley graph Y for $\mathbb{Z}_2^k$, then there is a graph homomorphism from $\mathcal{F}(X)$ into Y whose underlying map is a $\mathbb{Z}_2$-linear map from $\mathbb{Z}_2^d$ to $\mathbb{Z}_2^k$.

Homomorphisms

- The subgraph of $\mathcal{F}(X)$ induced by the vectors in the standard basis is isomorphic to X. Thus $X \to \mathcal{F}(X)$.

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- If $X \to Y$ then $\mathcal{F}(X) \to \mathcal{F}(Y)$.
- If X is cubelike then $X \leftrightarrow \mathcal{F}(X)$.

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- $\chi(\mathcal{F}(C_{2r+1})) = 4$. (Payan.)
- Hence $\chi(Y) \geq 4$. Thus a cubelike graph cannot have chromatic number three!

Odd Cycles

- Since $C_{2r+1} \to K_3$, it follows that $\mathcal{F}(C_{2r+1}) \to \mathcal{F}(K_3)$.
- As $\mathcal{F}(K_3) = 2K_4$, we have $\chi(\mathcal{F}(C_{2r+1}) \le 4$.
- $\mathcal{F}(C_{2r+1})$ is distance transitive and its odd girth is $2r+1$. So if $r \ge 2$, its core cannot be complete, and therefore it is a core.

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- However $\mathcal{F}(K_5)$ is the complement of the Clebsch graph and $\chi(\mathcal{F}(K_5)) = 8$. Thus if X is cubelike and contains K_5, then $\chi(X) \geq 8$.
- $\mathcal{F}(K_n)$ is the "distance-1 or -2" graph of the $(n-1)$-cube.
- If n is a power of two, then $\mathcal{F}(K_n)^\bullet = K_n$ (easy); otherwise $\mathcal{F}(K_n)$ is a core (hard).

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2. Are there any more "excluded values" for the chromatic number of a cubelike graph? (Probably not.)
3. Is there a primitive vertex-transitive graph X which is not a core and whose core is not complete?
4. We can define free Cayley graphs over $\mathbb{Z}_3$ (using signed incidence matrices). About all we know is that the free Cayley graph for K_4 over $\mathbb{Z}_3$ has chromatic number seven.