Linear Fractional Maps

Chris Godsil

Tuesday 28^th^ July, 2020

We investigate "linear" graphs formed by 1-sums. There are some digressions.

1 1-*Sums*

Our basic building block is a triple (X, a, b) consisting of a graph X and two distinct vertices a and b. We refer to these as 2-*rooted graphs*. We may vary the graph, but the root vertices will always be a and b. Let X and Y be 2-rooted graphs and let Z be the graph we get by identifying b in X with a in Y. Then

$$\begin{aligned}\phi(Z,t) &= \phi(X\setminus b,t)\phi(Y,t)+\phi(X,t)\phi(Y\setminus a,t)-t\phi(X\setminus b,t)\phi(Y\setminus a,t)\\ &= \phi(X\setminus b,t)\phi(Y,t)+(\phi(X,t)-t\phi(X\setminus b,t))\phi(Y\setminus a,t).\end{aligned}$$

We view Z as rooted at a (in X) and b (in Y).

To make our formulas look less horrible, we will use X, X_a, X_{ab} to respectively denote $\phi(X,t)$, $\phi(X\setminus a,t)$, $\phi(X\setminus\{a,b\},t)$. Then we have

$$Z = X_bY + (X - tX_b)Y_a$$

and in addition

$$Z_a = X_{ab}Y + (X_a - tX_{ab}Y_a).$$

We combine these two identities:

$$\begin{pmatrix} Z\\ Z_a\end{pmatrix} = \begin{pmatrix} X_b & X-tX_b\\ X_{ab} & X_a - tX_{ab}\end{pmatrix}\begin{pmatrix} Y\\ Y_a\end{pmatrix}. \tag{1.1}$$

We define

$$\Psi_{a,b}(X) := \begin{pmatrix} X_b & X-tX_b\\ X_{ab} & X_a - tX_{ab}\end{pmatrix}$$

and note that

$$\Psi_{a,b}(X) := \begin{pmatrix} X_b & X\\ X_{ab} & X_a\end{pmatrix}\begin{pmatrix} 1 & -t\\ 0 & 1\end{pmatrix},$$

whence

$$\det(\Psi_{a,b}(X)) = X_aX_b - XX_{ab}.$$

Here $X_aX_b - XX_{ab}$ is nonnegative and determines the walk generating function $W_{a,b}(X,t)$. [See e.g. (AC, p.53)]

Also

$$\Psi_{a,b}(X)\begin{pmatrix} t\\ 1\end{pmatrix} = \begin{pmatrix} X\\ X_a\end{pmatrix}$$

and hence

$$\begin{pmatrix} Z\\ Z_a\end{pmatrix} = \Psi_{a,b}(X)\Psi_{a,b}(Y)\begin{pmatrix} t\\ 1\end{pmatrix}.$$

If we form Z by chaining together k copies of (X, a, b), then

$$\begin{pmatrix} Z \\ Z_a \end{pmatrix} = \Psi_{a,b}(X)^k \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Taking X to be K_2 we deduce that

$$\begin{pmatrix} \phi(P_n, t) \\ \phi(P_{n-1}, t) \end{pmatrix} = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

As an example, if $X = K_3$,

$$\Psi = \begin{pmatrix} t^2 - 1 & t \\ t & -1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} Z_b & Z \\ Z_{ab} & Z_a \end{pmatrix} = \Psi_{a,b}(X) \begin{pmatrix} Y_b & Y \\ Y_{ab} & Y_a \end{pmatrix}.$$

EXERCISE: prove that if a and b are cospectral in X and Z is obtained by chaining copies of X, then a and b are cospectral in Z.

2 *Linear Fractional Maps*

Let $\mathbb{F}$ be a field (possibly infinite). We adjoin an element ∞ to $\mathbb{F}$, satisfying the rules you were not allowed to use in Calculus, for example

$$\frac{1}{\infty} = 0, \quad \frac{a\infty}{b\infty} = \frac{a}{b}, \quad \infty + a = \infty.$$

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we define the map L_A on $\mathbb{F} \cup \infty$ by

$$L_A(z) = \frac{az+b}{cz+d}.$$

This map is invertible if A is, and the maps L_A for invertible A form a group, the *linear fractional group*. If $\lambda \neq 0$, then $L_{\lambda A} = L_A$ and thus we have an action of the projective general linear group $PGL(2,\mathbb{F})$ on $\mathbb{F} \cup \infty$ (which we can view as the projective line).

Observe that

$$A \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}.$$

We can identify the projective line with the 1-dimensional subspaces of $\mathbb{F}^2$. The span of the vector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is the point ∞ and, if $b \neq 0$, the span of

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

is a/b.

We see that

$$L_A(\infty) = \frac{a\infty + b}{c\infty + d} = \frac{a}{c}$$

and so L_A fixes ∞ if and only if $c = 0$; similarly it fixes 0 if and only if $b = 0$. The map $z \mapsto z^{-1}$ swaps 0 and ∞.

The group of linear fractional maps is generated by the maps

$$z \mapsto z + b, \quad z \mapsto az \ (a \neq 0), \quad z \mapsto z^{-1},$$

as you might well show.

If $(u, v, w, x) \in (\mathbb{F} \cup \infty)^4$, its *cross-ratio* is

$$\frac{(u-w)(v-x)}{(u-x)(v-w)}.$$

Linear fractional maps preserve cross-ratio; in fact there is a linear fractional map sending a 4-tuple α to a 4-tuple β if and only α and β have the same cross-ratio.

EXERCISE. Prove that the linear fractional maps form a 3-transitive group of permutations of points of the projective line. [Hint: use the generators]

3 *Linear Fractional Maps on Graphs*

We combine the previous two sections by working with linear fractional maps over the field of rational functions $\mathbb{C}(t)$, or perhaps over its completion, the Laurent series in t.

The point is that $\Psi_{a,b}(X)$ can be viewed as a linear fractional map that sends the rational function

$$\frac{\phi(Y,t)}{\phi(Y \setminus a, t)}$$

to

$$\frac{\phi(Z,t)}{\phi(Z \setminus a, t)}$$

If Z is formed by chaining k copies of X,

$$W_{a,b}(Z,t) = W_{a,b}(X,t)^k.$$