Chapter 3

Free Groups

We allow our groups to be infinite. Recall that a monoid is a set with an associative multiplication and an identity, e.g., the set of all endomorphisms of a graph, or any group.

3.1 Reduced Walks

We consider walks in directed graphs. A walk is a sequence of vertices, and a walk is *reduced* if it does not contain any subsequence of the form vwv. If wvw is a subwalk of the walk α, the operation of replacing it by w is an *elementary reduction*. The reverse operation is *elementary exapnsion*. Two walks α and β are *equivalent* if one can be obtained from the other by a sequence of elementary reductions and expansions. We use $[\alpha]$ to denote the equivalence class of α.

3.1.1 Theorem. *If α and β are equivalent reduced walks in X, they are equal.*

Assume $v \in V(X)$ and define a graph on the reduced walks in X that start at v, where two walks are adjacent if one is a maximal subwalk of the other. (This is a graph even if X is not.)

3.1.2 Theorem. *The graph on the reduced walks in X based at v is a tree.*

Proof. If α is reduced walk, let $t(\alpha)$ denote its last vertex. If $\alpha_1, \ldots, \alpha_k$ is a sequence of reduced walks with consecutive terms adjacent, then the sequence of vertices

$$t(\alpha_1), \dots, t(\alpha_k)$$

determines the sequence of reduced walks, and is equal to α_k. It follows that two sequences of reduced closed walks with the same starting and finishing points must be equal. □

We call this tree the *walk tree* of X based at v, and denote it by $T(X, v)$.

3.1.3 Theorem. *The reduced closed walks that start at the vertex v in the graph X form a group, the elements of which induce a semiregular group of automorphisms of $T(X, v)$.*

Proof. If X is a graph, then each reduced closed walk at v has a multiplicative inverse. The multiplication operation is concatenation followed by reduction, and the difficulty is to show that this operation is associative.

If β is a reduced walk starting at v and α is a closed walk on v, then the product $\alpha\beta$ is a (reduced) walk starting at v. We see that reduced walks β and γ (starting at v) are adjacent if and only $\alpha\beta$ and $\alpha\gamma$ are adjacent. Hence left multiplication by α gives an automorphism of the tree on reduced walks from v; not that this automorphism determines α and if $\alpha \neq 1$, it does not fix a vertex or an edge of the tree. □

The group formed by the reduced closed walks at v is called the *fundamental group* of X based at v, and is denoted $\pi_1(X, v)$. For each vertex w of X, the reduced walks that start at v and finish at w are an orbit for $\pi_1(X, v)$ and the quotient of the walk tree over this orbit partition is X.

3.2 Cayley Graphs

Let M be a monoid and let $\mathcal{C}$ be a subset of M. The directed Cayley graph $X(M, \mathcal{C})$ has vertex set M and arc set consisting of the pairs (g, cg) for $g \in M$ and $c \in \mathcal{C}$. If both c and c^{-1} belong to $\mathcal{C}$, then we have an edge $\{g, cg\}$ for each g in M.

If $e \in \mathcal{C}$, then there is a loop on each vertex of $X(G, \mathcal{C})$, but we will normally assume that $e \notin \mathcal{C}$. We view an edge as a pair of oppositely directed arcs, so $\{a, b\}$ is an edge if and only if ba^{-1} and ab^{-1} both lie in C. If

$$\mathcal{C}^{-1} = \{c^{-1} : c \in \mathcal{C}\}$$

then X is a graph if and only if $\mathcal{C} = \mathcal{C}^{-1}$. We do not require $\mathcal{C}$ to generate G; note that $\mathcal{C}$ generates M if and only if $X(M, \mathcal{C})$ is weakly connected. Each element of M determines an endomorphism of X. Note that a monoid may contain idempotent elements apart from the identity, and these will give rise to loops if they belong to $\mathcal{C}$.

If S is a subset of the group G and $S \cap S^{-1} = \emptyset$, we say that G is the free group on the set S if the Cayley graph $X(G, S \cup S^{-1})$ is a tree.

The following result is known as Sabidussi's lemma:

3.2.1 Lemma. *If the group G acts regularly on the graph X, then X is a Cayley graph for G.* □

3.2.2 Theorem. *If the group G acts semiregularly on the connnected graph X, then G acts regularly on some contraction of X.*

Proof. Since G is semiregular, each orbit admits a bijection onto G. Since X is connected, its quotient over the orbit partition is connected and so X contains a subgraph H that is connected, acyclic and contains exactly one vertex from each orbit. As the action of G is semiregular, the distinct translates Hg for g in G partition $V(X)$. It follows that G acts regularly on the graph we get by contracting each translate to a vertex. □

3.2.3 Corollary. *Let X be a graph, let v be a vertex of X and let T be a spanning tree for the connected component of X that contains v. Let $\mathcal{C}$ be the set of arcs coming from the chords of T. Then $\pi_1(X, v)$ is free on $\mathcal{C}$.*

Proof. In the previous section we say that $\pi_1(X, v)$ acts semiregularly on a tree T, and so there is a contraction of T on which $\pi_1(X, v)$ acts regularly. Since T is a tree, this contraction is also a tree, and it follows that $\pi_1(X, v)$ is freely generated by the chords of T. □

3.2.4 Theorem. *If $\mathcal{F}$ is a free group, then all subgroups of $\mathcal{F}$ are free.*

Proof. Assume $\mathcal{F}$ is free relative to the generating set $\mathcal{C}$ and set $X = X(\mathcal{F}, \mathcal{C})$. If $H \leq \mathcal{F}$, then H acts semiregularly on X and therefore it acts regularly on a contraction of X. Since $\mathcal{F}$ is free, X is a tree and, since X is tree, each contraction of X is a tree. Hence X has a Cayley graph which is a tree, and therefore H is free. □

Consider the case where $|\mathcal{C}| = d$ and $|\mathcal{F} : H| = m$. Then the tree T contains exactly $m - 1$ edges, and so there are $md - m + 1$ edges joining vertices in T to vertices not in T. Therefore the quotient tree has degree $md - m + 1$.

3.2.5 Corollary. *If H is a subgroup of index m in the free group on d generators, then H is free on $md - m + 1$ generators.* □

3.3 Free Groups and Initial Objects

An object A in a category is *initial* if for each object B, there is a unique arrow $A \to B$. The trivial group in the category of groups provides an example. Dually we have *terminal objects*, and the trivial group in the category of groups is again an example.

Let k be a fixed positive integer. Let $\mathcal{C}$ be the category whose objects are ordered pairs (G, σ) where G is a group and σ is a sequence of k elements from G. An arrow from (G, σ) to (H, τ) is a homomorphism $\phi : G \to H$ such that $\phi(sg_i) = \tau_i$ for $i = 1, \ldots, k$. The free group $\mathcal{F}_k$ on k generators is an initial object in this category. This means that there are elements $x_1, \ldots, x_k$ in $\mathcal{F}$ and, for each object (G, σ), there is a unique homomorphism $\phi : \mathcal{F}_k \to G$ such that $\phi(x_i) = \sigma_i$. Additionally, if there is an arrow ρ from (G, σ) to (H, τ), and the composition of this with the arrow from $\mathcal{F}$ to (G, σ) is equal to the arrow from (H, τ). Of course now you will need to look up the definitions of category and initial object, but the effort will be repaid on many occasions.

In more concrete terms, $\mathcal{F}_k$ is free on $S = \{x_1, \ldots, x_k\}$ if for each group G and each map $f : S \to G$, there is a homomorphism from $\mathcal{F}_k$ to G that agrees with f on S.

We offer another example. Let X and Y be two fixed graphs. Consider the category whose objects are triples (Z, f, g), where Z is a graph and f and g are homomorphisms from Z to X and Y respectively and where the arrows are graph homomorphisms compatible with the maps to X and Y. Then a terminal object in this category is the direct product of X and Y.

3.4 Covers

A graph homomorphism $f : X \to Y$ is a *covering map* if it is a *local isomorphism*, that is, the restriction of f to the neighbours of a vertex u is an isomorphism from $N_X(u)$ to $N_Y(f(u))$.

3.4.1 Lemma. *The fundamental group $\pi_1(X, a)$ acts semiregularly on $T(X, a)$. Its orbits are the fibres of a covering map from $T(X, a)$ to X.*

Proof. Suppose α and β are reduced walks in X from a to b. Then $\alpha\beta^{-1}$ is a closed walk in X and $(\alpha\beta^{-1})\beta = \alpha$. Thus two reduced walks from a are in the same orbit of $\pi_1(X, a)$ if and only if they end at the same vertex. □

The tree $T(X, a)$ is the *universal cover* of X, based at a.

If we view the edges of our graphs as pairs of oppositely directed arcs and if we label the arcs of X, then this determines a labelling of the arcs of $T(X, a)$. For the moment, this does not matter.

3.4.2 Lemma. *If X is connected and X covers a tree T, then X is isomorphic to T.* □

Consider the category whose objects are the covers of a fixed graph F; thus the objects are pairs (Y, f) where Y is a graph and f is a local isomorphism from Y to X; the arrows are local isomorphisms. We construct a product for this category.

If (Y, f) and (Z, g) are covers of X, we define $Y \vee Z$ to be the subgraph of $Y \times Z$ induced by the vertices

$$\{(y, z) : f(y) = g(z)\}.$$

(Equivalently it is the preimage relative to the map $(f, g) : Y \times Z \to X \times X$ of the diagonal if $X \times X$.) You may show that $Y \vee Z$ covers Y, Z, and X.

Nowsuppose that Y and T both cover X and T is a tree. Then any connected component of $Y \vee T$ covers T, and so it must be isomorphic to T. Therefore T covers Y and you may show that T is an initial object in the category of covers of X. We say that T is the *universal cover* of X. We note that the walk tree is a tree that covers X, so each graph has a universal cover.

3.5 Explicit Examples

We present three constructions of free groups; they occur more frequently than one might expect.

The Linear Fractional Group $PGL(2, \mathbb{C})$

For the first example, we work in the linear fractional group over $\mathbb{C}$: if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

define the map τ on $\mathbb{C} \cup \infty$ by

$$\tau_A(z) = \frac{az+b}{cz+d}.$$

Note that τ_A is invertible if and only if A is. Now set

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and let α and β denote τ_A and τ_B respectively.

We claim that $\Gamma = \langle \alpha, \beta \rangle$ is free on $\{\alpha, \beta\}$. To see this observe that any nonzero power of α maps the interior of the unit circle $|z| = 1$ to the exterior, and any nonzero power of β maps the exterior to the nonzero elements of the interior. Using this it is not hard to show that no word of the form

$$\alpha^{k_1} \beta^{\ell_1} \cdots \alpha^{k_{r-1}} \beta^{\ell_{r-1}} \alpha^r$$

with all exponents positive can be trivial. Now use conjugacy and inverses to cover the remaining cases. Accordingly no nontrivial reduced word in α and β is equal to the identity, and hence Γ is free on $\{\alpha, \beta\}$.

It also follows that the matrices A and B generate a subgroup of $GL(2, \mathbb{C})$ that is free on $\{A, B\}$.

The Orthogonal Group $O(3, \mathbb{R})$

We outline a construction of a free group on two generators as a subgroup of the group of real orthogonal matrices of order 3×3. The matrices are

$$A = \frac{1}{5} \begin{pmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad B = \frac{1}{5} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{pmatrix}.$$

These matrices are orthogonal, with determinant 1. Our claim is that there is no non-empty word in A, A^{-1}, B, B^{-1} that is equal to I.

Consider the matrices

$$5A,\ 5A^{-1},\ 5B,\ 5B^{-1}.$$

We view these matrices as matrices over $GF(5)$, and note that over this field all four matrices have rank one. Their column spaces are spanned respectively by the vectors

$$\begin{pmatrix}3\\-4\\0\end{pmatrix},\ \begin{pmatrix}3\\4\\0\end{pmatrix},\ \begin{pmatrix}0\\3\\-4\end{pmatrix},\ \begin{pmatrix}0\\3\\4\end{pmatrix}.$$

and we have

$$\begin{aligned}\operatorname{im}(5A) &= \ker(5A^{-1})^{\perp},\\ \operatorname{im}(5A^{-1}) &= \ker(5A)^{\perp},\\ \operatorname{im}(5B) &= \ker(5B^{-1})^{\perp},\\ \operatorname{im}(5B^{-1}) &= \ker(5B)^{\perp}.\end{aligned}$$

Using this you may show that the only reduced word in these these matrices that is equal to I is the trivial word, and hence the matrices A and B geberate a free group.

The existence of such free subgroups in the 3-dimensional orthogonal group leads to the Banach-Tarski paradox. (See `https://www.math.ucla.edu/~tao/preprints/Expository/banach-tarski.pdf` for more details.)

The technique we have used to establish freeness in this and in the previous example is sometimes referred to as a "ping-pong" argument.

Subdirect Product

If G is a finite group and $a, b \in G$, we refer to (G, a_1, a_2) as a triple. Two triples (G, a_1, a_2) and (H, b_1, b_2) are isomorphic if there is an isomorphism from G to H that sends (a_1, a_2) to (b_1, b_2). Let $\mathcal{S}$ be a sequence of triples (G, a_1, a_2) that contains exactly one triple from each possible isomorphism class, let α_1 be the sequence of group elements occuring as a_1 in a triple and let α_2 be the sequence of group elements occuring as a_2. Prove that $\langle \alpha, \beta \rangle$ is free. [The simplest approach here is to prove that this group is initial in the appropriate category.]