Covers of graphs and equiangular tight frames

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Outline

1. Strongly regular graphs from equiangular tight frames
   - Spherical t-designs
   - Strongly regular graphs from 3-designs
2. Tight fusion frames from covers
   - Drackns
   - Equi-isoclinic subspaces

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A graph from a set of equiangular line

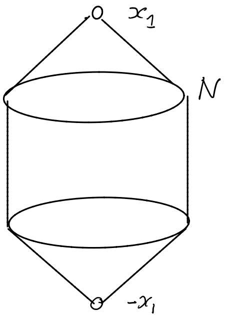
A set of n lines in $\mathbb{R}^d$ determines $2n$ unit vectors

$$V = \{\pm x_1, \ldots, \pm x_n\}.$$

If the lines are equiangular, say $\langle x_i, x_j \rangle = \pm\alpha$, we can construct a graph with vertex set V by defining two vectors in V to be adjacent if their inner product is α. (We assume $\alpha > 0$.)

This graph is regular with valency $n - 1$.

A switching graph



Spherical designs

Definition

Let Ω denote the unit sphere in $\mathbb{R}^d$. A finite subset Φ of Ω is a spherical t-design if, for any polynomial function of degree at most t on the sphere,

$$\frac{1}{\Phi}\sum_{u\in\Phi} f(u) = \int_\Omega f\,d\mu.$$

The degree of Φ is the number of different values taken by the inner product of two distinct points.

Equivalently if the degree of f is at most t, the average of f over Φ equals its average over the sphere.

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- If Φ is a 2-design and $-\Phi = \Phi$, then it is a 3-design.

Spherical 2-designs from strongly regular graphs

Suppose X is a strongly regular graph on n vertices with valency, and assume $\lambda \neq k$ is an eigenvalue of X with multiplicity d. If A and $\overline{A}$ are the adjacency matrices of X and its complement, then

$$E = \frac{d}{n}\left(I + \frac{\lambda}{k}A - \frac{\lambda+1}{n-1-k}\overline{A}\right)$$

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Conversely, each spherical 2-design with degree two gives rise to a strongly regular graph.

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Spherical 3-designs from equiangular tight frames

Suppose the vectors $x_1, \ldots, x_n$ form an equiangular tight frame in $\mathbb{R}^d$. Since the set $V = \{\pm x_1, \ldots, \pm x_n\}$ is closed under multiplication by -1 and since

$$\sum_i x_i x_i^T = \frac{n}{d} I,$$

we conclude that the points in V form a spherical 3-design.

Theorem

The vectors x_j in V such that $\langle x_1, x_j \rangle = \alpha$ form a spherical 2-design with degree two.

A proof of the theorem

First

$$x_1 x_1^T + \sum_{j=2}^{n} x_j x_j^T = \frac{n}{d} I.$$

Second, set $P = I = x_1 x_1^T$. Then $P x_1 x_1^T P = 0$ and so

$$\sum_{j=2}^{n} P x_j x_j^T P = \frac{n}{d} P.$$

It follows that, after normalizing, the vectors x_j for $j = 2, \ldots, n$ form a 2-design in $x_1^{\perp}$.

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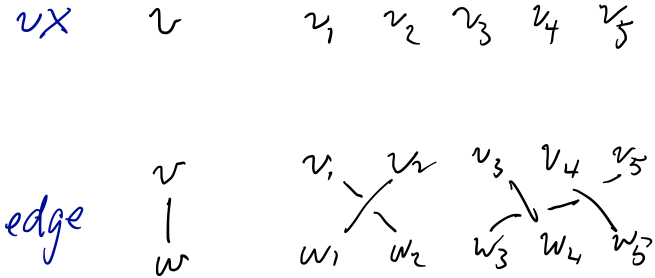
Covers

We construct a cover of a graph with index r by replacing each vertex by a set of r vertices, and each edge by a set of r vertex-disjoint edges. The sets of r vertices are the fibres of the cover.



Examples of covers

- The graph we constructed at the start from a set of n equiangular lines is a cover of K_n with index two.
- The cube is a 2-fold cover of K_4.
- The line graph of the Petersen graph is a 3-fold cover of K_5.

Characterizing drackns

We are only concerned with covers of K_n, but we insist on a number of special properties:

1. The cover should be connected with diameter three.
2. Two vertices are at distance three if and only if they lie in the same fibre.
3. There is a constant c_2 such that two vertices in the cover at distance two have exactly c_2 common neighbours.

If these conditions hold, we have a drackn—a distance-regular antipodal cover of K_n.

Eigenvalues for drackns

- A drackn has exactly four eigenvalues

$$n-1 > \theta > -1 > \tau$$

  where $\theta\tau = -n+1$.
- The matrix E_τ representing orthogonal projection onto the τ-eigenspace is the Gram matrix of a spherical 2-design.
- The image under E_τ of a fibre is a regular simplex, spanning a subspace of $\ker(A-\tau I)$ with dimension $r-1$.

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Isoclinic subspaces

Definition

Two subspaces U and V of $\mathbb{R}^d$ with dimension s are isoclinic with parameter λ if the projection onto V of the unit sphere in U is the sphere in V centered at the origin with radius λ. A collection of subspaces of the same dimension is equi-isoclinic if each of subspaces is isoclinic with the same parameter.

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Lemma

Matrices P and Q represent projections onto isoclinic subspaces with parameter λ if and only if $QPQ = \lambda P$.

(So $\langle P, Q \rangle = \lambda s$.)

Equi-isoclinic subspaces from drackns

Theorem

Let X be an antipodal distance-regular cover of K_n with index r and with least eigenvalue τ of multiplicity d. Then the images of a fibre in $\ker(A - \tau I)$ are an equi-isoclinic family of subspaces with dimension $r - 1$ in $\mathbb{R}^d$.

Fusion frames from drackns

Theorem

If $P_1, \ldots, P_n$ *are the projections onto a set of equi-isoclinic subspaces with dimension* s *and parameter* λ *in* $\mathbb{R}^d$*, and* $\lambda < s/d$*, then*

$$n \leq \frac{d - d\lambda}{s - d\lambda};$$

If equality holds,

$$\sum_{j=1}^{n} P_j = \frac{ns}{d} I_d.$$

Equality holds for the projections from a drackn.

A second class of fusion frames from drackns

Theorem

Let $P_1, \ldots, P_n$ be the projections onto a set of equi-isoclinic subspaces with dimension s and parameter λ in $\mathbb{R}^d$, and set $Q = I - P_1$. Then for $j = 2, \ldots, n$, the $n-1$ matrices $(1-\lambda)^{-1}QP_jQ$ are a tight fusion frame in $\mathbb{R}^{d-s}$.

The Proof

Proof.

Since $QP_1Q = 0$ and $\sum_{j=1}^{n} P_j = \frac{ns}{d} I$, it follows that

$$\sum_{j=2}^{n} QP_jQ = \frac{ns}{d} Q.$$

As $P_jP_1P_j = \lambda P_j$, we have

$$(QP_jQ)^2 = QP_j(I - P_1)P_jQ = Q(P_j - P_jP_1P_j)Q = (1 - \lambda)QP_jQ.$$

The result follows. □

The End(s)