The Colin de Verdière Number

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ABSTRACT

These notes provide an introduction to some properties of the Colin de Verdière number of a graph. They are heavily dependent on a survey by Van der Holst, Lovász and Schrijver. There are two possible novelties. We make use of matrix perturbation theory, and offer an alternative interpretation of the so-called strong Arnold hypothesis.

1. Perturbation

The following discussion summarizes Theorems II.5.4 and II.6.8 of Kato [].

Let A and H be real symmetric $n \times n$ matrices. We concern ourselves with the eigenvalues of $A + tH$, for small values of t; we will see that it is possible to view these as perturbations of the eigenvalues of A. Assume θ is an eigenvalue of A with multiplicity m, and let P be the projection on to the associated eigenspace. Then there is a matrix-valued function $P(t)$ such that

(a) $P(0) = P$.
(b) $P(t)$ is a real analytic function of t, and a projection.
(c) The column space of $P(t)$ is invariant under $A + tH$.

Note that $\mathrm{rk}(P) = m$. As $\mathrm{rk}(P(t)) = \mathrm{tr}(P(t))$, it follows that $\mathrm{rk}(P(t))$ is a continuous integer-valued function. Therefore

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(d) $\operatorname{rk} P(t) = m$.

From (c) it follows that the column space of $P(t)$ is a sum of eigenspaces of $A + tH$. These eigenspaces can be viewed as arising by splitting the θ-eigenspace of A.

The eigenvalues associated with these eigenspaces are analytic functions $\theta_1(t), \ldots, \theta_k(t)$ such that $\theta_i(0) = \theta$. If U is a matrix whose columns form an orthonormal basis for the columns of $P = P(0)$, then $P = UU^T$ and the derivatives $\theta_1'(0), \ldots, \theta_k'(0)$ are the eigenvalues of $U^T HU$. The dimension of the $\theta_i(t)$ eigenspace equals the dimension of the $\theta_i'(0)$-eigenspace of $U^T HU$.

1.1 Lemma. *Let Q be a symmetric matrix and suppose that the columns of the matrix U for an orthonormal basis for $\ker(Q)$. If K is symmetric then the corank of $Q + tK$ equals the corank of Q for all sufficiently small values of t if and only if $U^T KU = 0$.*

Proof. The matrix UU^T is the orthogonal projection onto $\ker(Q)$. As $U^T U = I$, we see that $U^T KU = 0$ if and only if $PKP = 0$. □

2. The Strong Arnold Hypothesis

If A and B are two matrices of the same order, we use $A \circ B$ to denote their Schur product, which is defined by the condition

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j}.$$

If X is a graph on n vertices, we define a *generalized Laplacian* for X to be a symmetric matrix Q such that $Q_{u,v} < 0$ if u and v are adjacent vertices in X and $Q_{u,v} = 0$ is u and v are distinct and not adjacent. (There are no constraints on the diagonal entries of Q.) Examples are the usual Laplacian and $-A$, where A is the adjacency matrix of X. Note that we have not assumed that the least eigenvalue of Q is simple, although this will hold if X is connected, by Perron-Frobenius.

We associate two spaces of symmetric matrices to each generalized Laplacian Q. Let $\mathcal{N}_Q$ denote the space of symmetric $n \times n$ matrices H such that

$$H \circ I = H \circ Q = 0$$

and let $\mathcal{K}_Q$ denote the space of symmetric $n \times n$ matrices K such that

$$QK = 0.$$

We say that Q satisfies the *Strong Arnold Hypothesis* if $\mathcal{N}_Q \cap \mathcal{K}_Q = 0$. This is often abbreviated to SAH. If θ is an eigenvalue of Q, we say that its associated eigenspace satisfies the SAH if $Q - \theta I$ does.

To give a very small example, suppose that $\ker Q$ has dimension one, and is spanned by a vector x. Any symmetric matrix H such $QH = 0$ must be a multiple of xx^T, but $(xx^T)_{i,i} = (x_i)^2$ and if $I \circ xx^T = 0$ then $x = 0$. Hence, if $\dim \ker Q = 1$, then Q satisfies the SAH.

We now describe a second version of the SAH. The space of symmetric $n \times n$ matrices is an inner product space, relative to the bilinear form

$$\langle A, B \rangle = \mathrm{tr}(AB).$$

We have the following:

2.1 Lemma. *The SAH holds for Q if and only if $\mathcal{N}_Q^\perp + \mathcal{K}_Q^\perp$ is the space of all symmetric $n \times n$ matrices.* □

Clearly $\mathcal{N}_Q$ consists of the symmetric matrices H such that $H_{u,v} = 0$ whenever $uv \in E(X)$. Hence $H \in \mathcal{N}_Q^\perp$ if and only if $Q + tH$ is a generalized Laplacian for all sufficiently small values of t.

To characterize $\mathcal{K}_Q^\perp$, we need a preliminary result.

2.2 Lemma. *Let Q be a symmetric matrix with corank m, and let U be a matrix whose columns form an orthonormal basis for $\ker Q$. Then a symmetric matrix K satisfies $QK = 0$ if and only if there is a symmetric matrix B such that $K = UBU^T$.*

Proof. The stated condition is sufficient, we prove that it is also necessary. If $QK = 0$ then the column space of K lies in $\ker Q$. As K is symmetric, there is a matrix U_1 whose columns lie in the column space of K and a symmetric matrix B_1 such that $K = U_1 B_1 U_1^T$. There is a matrix, R say, such that $U_1 = UR$ and therefore $K = U(RB_1R^T)U^T$, as required. □

It follows that $\mathcal{K}_Q^\perp$ consists of the matrices H such that $\langle UBU^T, H \rangle = 0$ for all symmetric matrices B. As

$$\langle UBU^T, H \rangle = \mathrm{tr}(UBU^T H) = \mathrm{tr}(U^T HUB) = \langle U^T HU, B \rangle,$$

we see that $H \in \mathcal{K}_Q^\perp$ if and only if $U^T HU$ is orthogonal to all symmetric matrices B. But this implies that $U^T HU = 0$ and therefore $\mathcal{K}_Q^\perp$ consists of the symmetric matrices H such that $U^T HU = 0$. Using Lemma 1.1, we conclude that $H \in \mathcal{K}_Q^\perp$ if and only if $Q + tH$ has the same corank as Q, for all sufficiently small values of t.

3. Quadratic Rank

The *quadratic map* q from $\mathbb{R}^m$ to $\mathbb{R}^{\binom{m+1}{2}}$ maps a vector $(u_1, \ldots, u_m)$ to the vector

$$(u_i u_j)_{i \le j}.$$

If U is an $n \times m$ matrix then $q(U)$ denotes the $n \times \binom{m+1}{2}$ matrix we get by applying q to each row of U. The *quadratic rank* of U is the rank of $q(U)$.

The quadratic rank of U is less than $\binom{m+1}{2}$ if and only if the columns of $q(U)$ are linearly independent. This happens if and only if there are scalars $b_{i,j}$, not all zero, such that for each row of B we have

$$\sum_{i,j:i \le j} b_{i,j} u_i u_j.$$

Equivalently, there is an $m \times m$ symmetric matrix B such that $u^T B u = 0$. (Note that, if $i \neq j$ then $B_{i,j} = b_{i,j}/2$.) In other terms, the quadratic rank of U is less than $\binom{m+1}{2}$ if and only if the rows of U lie on a homogeneous quadric. It follows that, if R is an invertible $m \times m$ matrix then U and UR have the same quadratic rank. Consequently the quadratic rank of U is a property of its column space, rather than of the matrix itself.

If $v_1^T, \ldots, v_n^T$ are the rows of U then the quadratic rank of U is the dimension of the space spanned by the matrices vv^T. In particular, the quadratic rank of U is $\binom{m+1}{2}$ if and only if the matrices $v_i v_i^T$ span the space of all $m \times m$ symmetric matrices.

Let X be a graph on n vertices. A *representation* of X in $\mathbb{R}^m$ is a map ρ from $V(X)$ into $\mathbb{R}^m$. Usually this map is chosen so that the geometry of the image of $V(X)$ reveals some information about X, but this is not a requirement of the definition.

It is often convenient to describe ρ by an $n \times m$ matrix, U say, with rows indexed by $V(X)$. Then $\rho(v)$ is the v-row of U. Eigenspaces of a generalized Laplacian Q provide useful representations—simply choose U to be a matrix whose columns form an orthogonal basis for the given eigenspace.

Suppose ρ is a representation of a graph X and let U be the matrix with rows

$$\rho(v),\ v \in V(X), \quad \rho(v) - \rho(w),\ uv \in E(X).$$

We define the quadratic rank of ρ to be the quadratic rank of U. Note that if X has e vertices and e edges then any representation of X has quadratic rank at most $v + e$.

If the quadratic rank of ρ is less than $\binom{m+1}{2}$ then there is an $m \times m$ symmetric matrix B such that

$$\rho(u)^T B \rho(u) = 0, \quad \forall u \in V(X) \tag{1}$$

and

$$(\rho(u) - \rho(v))^T B(\rho(u) - \rho(v)) = 0, \quad \forall uv \in E(X). \tag{2}$$

Given (1), we see that (2) is equivalent to the condition

$$\rho(u)^T B \rho(v) = 0, \quad \forall uv \in E(X).$$

We can summarize our deliberations thus:

3.1 Lemma. *Let ρ be a representation of X in $\mathbb{R}^m$. The quadratic rank of ρ is less than $\binom{m+1}{2}$ if and only if there is a non-zero homogeneous quadric which contains the image of each vertex of X and the lines that join the images of each pair of adjacent vertices.* □

Finally we give the connection to the SAH.

3.2 Theorem. *Let Q be a generalized Laplacian for X and let θ be an eigenvalue of Q with multiplicity m. Then the θ-eigenspace of Q satisfies the SAH if and only if the associated representation has quadratic rank $\binom{m+1}{2}$.*

Proof. Let U be an $n \times m$ matrix whose columns form an orthonormal basis for the θ-eigenspace of Q and let ρ be the corresponding representation. The SAH fails for this eigenspace if and only if there is a non-zero symmetric matrix B such that $I \circ (UBU^T) = 0$ and $(Q - \theta I) \circ UBU^T = 0$.

The matrix B defines a projective quadric. We have $I \circ (UBU^T) = 0$ if and only if the image of each vertex of X lies on this quadric. We have $(Q - \theta I) \circ UBU^T = 0$ if and only if $\rho(u)^T B \rho(v) = 0$ for each edge uv. Hence B exists if and only if the SAH fails and, as we saw above, B exists if and only if the quadratic rank of ρ is less than $\binom{m+1}{2}$. □

The rank of matrix is equal to the largest integer k such that the determinant of some $k \times k$ submatrix is non-zero. Given this, it is not hard to see that a small perturbation of a matrix cannot increase its rank. Further, if the columns of a matrix are linearly independent then a small perturbation does not change its rank. It follows that if an $n \times m$ matrix U has quadratic rank $\binom{m+1}{2}$, then any small perturbation of U has quadratic rank $\binom{m+1}{2}$.

We note that the quadratic rank is defined for any subspace of $\mathbb{R}^n$, not just for eigenspaces. There is one simple but useful consequence of this.

3.3 Lemma. *Suppose W is a subspace of $\mathbb{R}^n$ with dimension m and quadratic rank $\binom{m+1}{2}$. If W_1 is a subspace of W with dimension k, its quadratic rank is $\binom{k+1}{2}$.* □

4. A Minor-Monotone Parameter

Let X be a graph and let $\mathcal{Q}$ denote the set of all generalized Laplacians Q such that:

(a) $\lambda_1(Q)$ is simple.
(b) The λ_2-eigenspace of Q satisfies the SAH.

The *Colin de Verdière* number of X is the maximum multiplicity of λ_2, over all matrices in $\mathcal{Q}$. We denote it by $\mu(X)$.

4.1 Theorem. *If $e \in E(X)$ then $\mu(X \setminus e) \le \mu(X)$.*

Proof. Let Y denote $X \setminus e$ and let Q be a generalized Laplacian for Y such that λ_2 has multiplicity equal to $\mu(Y)$. Let Ξ be the adjacency matrix of e, viewed as a subgraph of X with $|V(X)|$ vertices. By Lemma 2.1, we can write Ξ as a sum $N + K$ where $N \in \mathcal{N}_Q^\perp$ and $K \in \mathcal{K}_Q^\perp$.

Therefore $Q+tK$ is a generalized Laplacian for X when $t \ne 0$. As $K \in \mathcal{K}_Q^\perp$, it follows from Lemma 1.1 that $\lambda_2(Q + tK) = \lambda_2(Q)$ has multiplicity equal to $\mu(Y)$ whenever t is small enough. By our remarks at the end of '`quadrk`', the representations of X associated with $\lambda_2(Q + tK)$ and $\lambda_2(Q)$ have the same quadratic rank, and so the SAH holds for $Q + tK$.

Since the multiplicity of $\lambda_2(Q + tK)$ is constant for small t, we also see that $\lambda_1(Q + tK)$ is simple. □

4.2 Theorem. *If $e \in E(X)$ then $\mu(X/e) \le \mu(X)$.*

Proof. Suppose $e = 12$ and

$$Q(X/e) = \begin{pmatrix} a & b^T \\ b & Q_1 \end{pmatrix}.$$

We assume that $\lambda_2(Q(X/e)) = 0$. Let Y be the graph $K_1 \cup (X/e)$ and let $Q = Q(Y)$ be a generalized Laplacian for Y. We may assume $Q(Y)$ has the form

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & a & b^T \\ 0 & b & Q_1 \end{pmatrix},$$

where $\epsilon > 0$, and will be restricted further shortly. Let Ξ be the matrix

$$\Xi = \begin{pmatrix} 0 & -1 & c^T \\ -1 & 0 & -c^T \\ -c & 0 & 0 \end{pmatrix}.$$

Here c is a non-positive vector such that $Q + \Xi$ is a generalized Laplacian for X. So, for example:

$$c_i = \begin{cases} 0, & \text{if } 1 \not\sim i \text{ and } 2 \not\sim i; \\ b_i, & \text{if } 1 \sim i \text{ and } 2 \not\sim i; \\ 0, & \text{if } 1 \not\sim i \text{ and } 2 \sim i; \\ b_i/2, & \text{otherwise.} \end{cases}$$

As before, $\Xi = N + K$, where $N \in \mathcal{N}_Q^\perp$ and $K \in \mathcal{K}_Q^\perp$. Consider the matrix pencil $Q+tK$. For small values of t we know that the rank of $Q+tK$ does not change, and the SAH holds for $\ker(Q+tK)$. Choose some positive value of t that works, and assume that ϵ was chosen so that $\epsilon < t^2/(1-t)$. If we multiply the first row and colum of $Q + tK$ by $(1-t)/t$, we get the matrix

$$Q' = \begin{pmatrix} \epsilon(1-t)^2/t^2 & t-1 & (1-t)c^T \\ t-1 & a & b^T - tc^T \\ (1-t)c & b-tc & Q_1 \end{pmatrix}$$

This operation does not change the rank, and it is not hard to see that SAH holds for $\ker(Q')$ and that $\lambda_1(Q')$ is simple.

Let Q'' be the matrix we get from Q' by subtracting its first row from its second, and the first column from the second. We observe that Q'' is a generalized Laplacian for X (at last), and that its rank is equal to the rank of Q'. We have to show that the SAH holds.

Let U be the $n \times m$ matrix whose columns form a basis for $\ker(Q')$, and let M be the elementary matrix we get by adding the first row of I to its second row. Thus $Q'' = MQ'M^T$ and the columns of $M^{-1}U$ are a basis for $\ker(Q'')$. Since the SAH holds for Q', the space of all $m \times m$ symmetric matrices is spanned by the matrices

$$u_i u_i^T, \quad (u_i - u_j)(u_i - u_j)^T, \quad i \in V(X),\ ij \in E(X)$$

where u_i is the i-th row of U. Let v_i denote the i-th row of $M^{-1}U$.

We have $v_1 = u_1$, $v_2 = u_1 - u_2$ and $v_1 - v_2 = u_2$. Hence the SAH holds. As Q'' and Q' are congruent, it follows from Sylvester's law of inertia that $\lambda_1(Q'')$ is simple. □

The previous two results combine to give the most important property of the Colin de Verdière number:

4.3 Corollary. *If Y is a minor of X then $\mu(Y) \leq \mu(X)$.*

5. Properties

We derive some further relations between the Colin de Verdière number of a graph and its subgraphs. We begin with a technical result.

5.1 Lemma. *If an eigenspace of X contains two eigenvectors with disjoint supports, then the SAH hypothesis fails.*

Proof. Suppose x and y are vectors and U is the matrix

$$U = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

As the Schur product of the columns of U is zero, it follows that $\operatorname{qrk}(U) = 2$. By '`sahsub`', we deduce that the SAH fails for any subspace that contains x and y. □

We now determine $\mu(K_n)$ and $\mu(\overline{K_n})$. It is easy to see that $\mu(K_1) = 0$. Suppose $n \geq 2$. Then $-J$ is a generalized Laplacian for K_n with $\lambda_2(Q) = 0$ having multiplicity $n-1$. Here $\mathcal{N}_Q^\perp$ is the space of all symmetric matrices, and so the SAH holds by Lemma 2.1. (Conversely, it is not too hard to show that $\mu(X) = |V(X)| - 1$ if and only if $X = K_n$.)

Next we consider $X = \overline{K_n}$. Here $\mathcal{N}_Q^\perp$ is the space of diagonal matrices. Suppose Q is a generalized Laplacian for X such that λ_2 is simple. We may assume without loss that the associated eigenvector is e_1, the first standard basis vector. Then

$$\mathcal{K}_Q^\perp = \{H : e_1^T H e_1 = 0\} = \{H : H_{1,1} = 0\}.$$

Thus $\mathcal{K}_Q^\perp$ has codimension 1 in the space of symmetric matrices, and so Lemma 2.1 again yields that the SAH holds. Hence $\mu(X) \geq 1$.

Suppose now that λ_2 has multiplicity at least two. We may assume that the eigenspace contains e_1 and e_2, whence '`dis-sup`' yields that the SAH fails.

5.2 Lemma. *If X has at least one edge, then $\mu(X)$ equals the maximum value of $\mu(Y)$, where Y ranges over the components of X.*

Proof. By '`edge-del`', $\mu(X) \geq \mu(C)$, where C runs over the components of X.

Let Q be a generalized Laplacian for X that realises $\mu(X)$, with $\lambda_2(Q) = 0$. Exactly one component Y of X has least eigenvalue equal to λ_1; all other components must have least eigenvalue no less than zero. If two distinct components of X have eigenvalue zero then the kernel of Q contains two vectors with disjoint support, and the SAH fails.

If some component of X has least eigenvalue zero, then since a component is connected, its least eigenvalue is simple and $\mu(X) = 1$. As X contains an edge, some component C has $\mu(C) \geq 1$, and the lemma follows.

Otherwise 0 must be an eigenvalue of Y and $\mu(X) = \mu(Y)$. $\square$

Suppose Q is a generalized Laplacian for X and ρ is the representation on some eigenspace of Q with eigenvalue θ and dimension m. Let u be a vertex of X and let W be the subspace of the eigenspace spanned by the eigenvectors that vanish on u. The restriction to $V(X) \setminus u$ of any of these eigenvectors is an eigenvector for $Q(X \setminus u)$, with eigenvalue θ. This shows that the multiplicity of θ as an eigenvalue of $Q(X \setminus u)$ is at least $m - 1$. It follows from spectral decomposition that the dimension is $m - 1$ if and only if $\rho(u) \neq 0$. (See Lemma 8.13.1 in Godsil & Royle.)

5.3 Lemma. *Let Q be a generalized Laplacian for X, and let ρ be the representation on the λ_2-eigenspace of Q. Assume $\lambda_2(Q)$ has multiplicity m and let u be a vertex in X such that*

a) $\rho(u) \neq 0$.
b) *The images of u and its neighbours span $\mathbb{R}^m$.*

Then if the SAH holds for $Q(X \setminus u)$, it holds for Q.

Proof. By hypothesis, the vectors in $\ker(Q - \lambda_2 I)$ that do not have u in their support span a space W of dimension $m-1$, with quadratic rank $\binom{m}{2}$. Let $x_1, \ldots, x_{m-1}$ be a basis for W and let z be an eigenvector for Q with eigenvalue λ_2 such that $z_u = 1$.

Suppose the SAH fails for Q. Then there is a non-zero symmetric $m \times m$ matrix B such that

$$\rho(v)^T B \rho(w) = 0$$

if $v = w$ or $v \sim w$. As $\rho(u)$ is the first standard basis vector in $\mathbb{R}^m$ and as the images of u and its neighbours span $\mathbb{R}^m$, it follows that $\rho(u)^T B = 0$. Thus the first row and column of B are zero. If B is not zero the the SAH fails for $Q(X \setminus u)$. $\square$

5.4 Lemma. *If $u \in V(X)$ then $\mu(X \setminus u) \geq \mu(X) - 1$. If u is adjacent to each vertex in $V(X) \setminus u$ and $|V(X)| \geq 2$, then $\mu(X \setminus u) = \mu(X) - 1$.*

Proof. Suppose that Q is a generalized Laplacian for X that realizes the Colin de Verdière number of X. Let u be a vertex of X, let Q_u be the matrix we get by deleting the u-row and u-column from Q and let W be the space consisting of the λ_2-eigenvectors x of Y such that $x_u = 0$. It is easy to verify that deleting the u-coordinate from a vector in W gives an eigenvector for Q_u with eigenvalue $\lambda_2(Q)$. As the SAH holds for Q, it holds for W.

To complete the proof of the first claim, we show that if Q is a generalized Laplacian for X and W is a k-dimensional subspace of the λ_2-eigenspace which satisfies the SAH, then $\mu(X) \geq k$.

Suppose $\lambda_2(Q)$ has multiplicity m and let F be the projection onto the complement of W in the λ_2-eigenspace. Let U_1 be an $n \times k$ matrix whose columns are a basis for W. As the SAH holds for W, we have $F = N + K$, where $N \in \mathcal{N}_Q^\perp$ and $U_1^T K U = 0$. Let U be a $n \times n$ matrix with a basis of the λ_2-eigenspace as columns. Then $U^T N U = U^T(F - K)U$ has eigenvalue 0 with multiplicity k and 1 with multiplicity $m - k$. So, for small non-zero values of t, we find that $\lambda_2(Q + tN)$ has multiplicity k and satisfies the SAH.

Finally, suppose u is adjacent to all vertices in $X \setminus u$. By the previous result, we may assume that $X \setminus u$ is connected. Let Q' be a matrix realizing $\mu(X \setminus u)$ with $\lambda_2(Q') = 0$, and let z be an eigenvector of Q' with eigenvalue $\lambda_1 = \lambda_1(Q)$. We may choose z so that $z < 0$ and $\|z\| = 1$. Let Q be the generalized Laplacian given by

$$Q = \begin{pmatrix} \lambda_1^{-1} & z^T \\ z & Q' \end{pmatrix}.$$

Then $\ker Q$ contains $(\lambda_1, z)^T$ and all vectors of the form $(0, x)^T$, where $x \in \ker(Q')$. The SAH holds by the previous lemma.

The least eigenvalue of Q is simple, because X is connected. By interlacing, $\lambda_2(Q) = \lambda_2(Q')$. We conclude that $\mu(X) = \mu(X \setminus u) + 1$. □

5.5 Corollary. *Suppose C is a vertex cutset in X and let $Y_1, \ldots, Y_r$ be the components of $X \setminus C$. Then $\mu(X) \leq |C| + \max_i \mu(Y_i)$.*