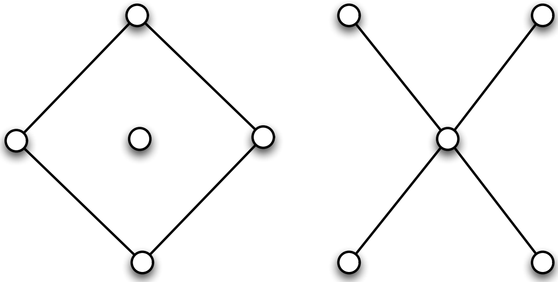
Awful Graphs

Chris Godsil

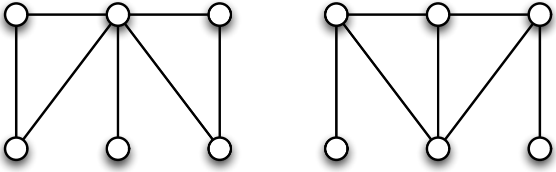
Waterloo, July 2009: Tutte Seminar

Outline

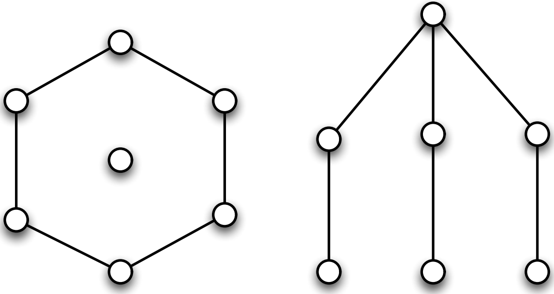
Cospectral Graphs



Connected Cospectral Graphs



Cospectral Graphs with Cospectral Complements



A Proof

The adjacency matrices of these graphs are of the form

$$\begin{pmatrix} 0 & B_i \\ B_i^T & 0 \end{pmatrix}$$

where

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof, ctd.

If $Q = \frac{1}{2}J_4 - I_4$, then

$$QQ^T = Q^2 = I,$$

so Q is orthogonal. Also

$$QB_1 = B_2,$$

from which it follows that

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & B_1 \\ B_1^T & 0 \end{pmatrix} \begin{pmatrix} Q^T & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & B_2 \\ B_2^T & 0 \end{pmatrix}.$$

This proves our graphs are cospectral. (Since $Q\mathbf{1} = \mathbf{1}$, it also follows that their complements are cospectral.)

Weighted Adjacency Matrices

After you have worked with the adjacency matrix for while, it might occur to you that we could choose scalars a, b and c, and consider matrices of the form

$$aI + bA + cJ.$$

Theorem

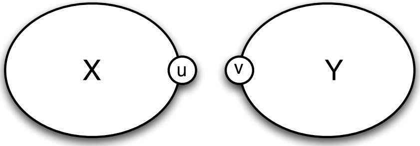
Let X_1 and X_2 be cospectral graphs and let a, b and c be scalars with $b \neq 0$. Then $\overline{X}_1$ and $\overline{X}_2$ are cospectral if and only if the matrices

$$aI + bA(X_1) + cJ, \quad aI + bA(X_2) + cJ$$

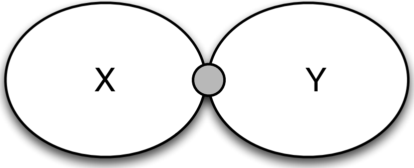
are similar.

Some Graphs. . .

If X and Y are as shown:



and Z is their 0-sum:



. . . and their Characteristic Polynomial

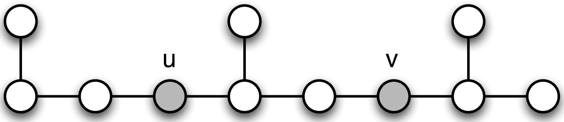
then:

Lemma

$$\phi(Z,t) = \phi(X\backslash u,t)\phi(Y,t) + \phi(X,t)\phi(Y\backslash v,t) - t\phi(X\backslash u,t)\phi(Y\backslash v,t)$$

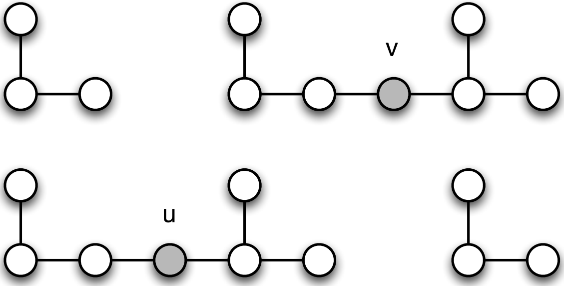
A Tree and Two Vertices

There is no automorphism of the following tree T that maps the vertex u to the vertex v...



Two Subgraphs

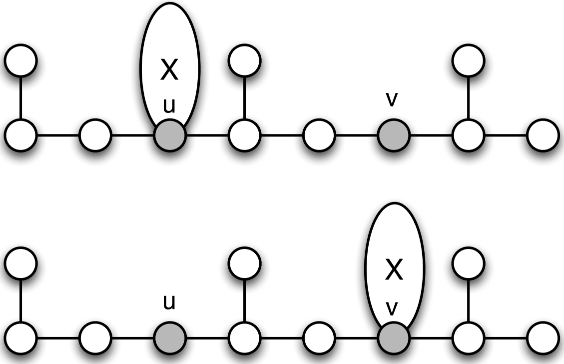
... but $T \setminus u$ and $T \setminus v$ are isomorphic:



Hence $\phi(T \setminus u, t) = \phi(T \setminus v, t)$.

Constructing Cospectral Trees

It follows that, for any tree X, the two trees below are cospectral and are not isomorphic:



Almost all Trees are Cospectral

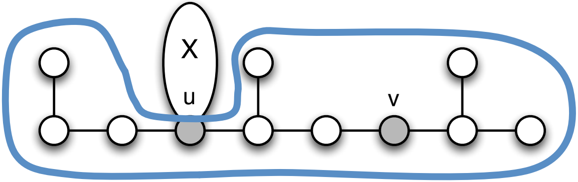
Theorem (Allen Schwenk)

The proportion of trees on n vertices that are determined by their characteristic polynomial goes to zero as $n \to \infty$.

Allen Schwenk

A Limb of a Tree

Schwenk proved that almost all trees contain a given limb, for example, the following on 11 vertices:



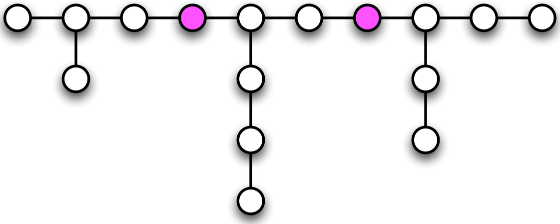
Zhu and Wilson: A Table

Size	Number	**A**	**L**
8	23	0.087	0
9	47	0.213	0
10	106	0.075	0
11	235	0.255	0.0255
12	551	0.216	0.0109
13	1301	0.319	0.0138
14	3159	0.261	0.0095
15	7741	0.319	0.0062
16	19320	0.272	0.0035
17	48629	0.307	0.0045
18	123867	0.261	0.0019
19	317955	0.265	0.0014
20	823065	0.219	0.0008
21	2144505	0.213	0.0005
22	5623756	0.177	0.00028
23	14828074	0.168	0.00019

A Quote: 2008

graphs at all sizes. Both the Laplacian and its normalised counterpart show a decreasing trend, suggesting that for larger trees the fraction which are cospectral in these matrices could be negligible. The trend for the adjacency

McKay's Limbs: 1977



Walk Matrices

Definition

Let A be the adjacency matrix of the graph X on n vertices and let $\mathbf{1}$ denote the all-ones vector of length n. The walk matrix of X is the $n \times n$ matrix

$$\begin{pmatrix} \mathbf{1} & A\mathbf{1} & \dots & A^{n-1}\mathbf{1} \end{pmatrix}.$$

Automorphisms

The automorphism group of X is (isomorphic to) the group of permutation matrices P that commute with A.

Lemma

If $P \in \mathrm{Aut}(X)$ then P fixes each column of W. □

Proof.

$P\mathbf{1} = \mathbf{1}$ and therefore:

$$PA^r\mathbf{1} = A^rP\mathbf{1} = A^r\mathbf{1}.$$

□

Asymmetric Graphs

Corollary

If $\operatorname{rk}(W) = n$ *then* X *is asymmetric.* □

Proof.

If P is an automorphism then $PW = W$. If W is invertible, then $P = I$. □

Awfulness

Definition

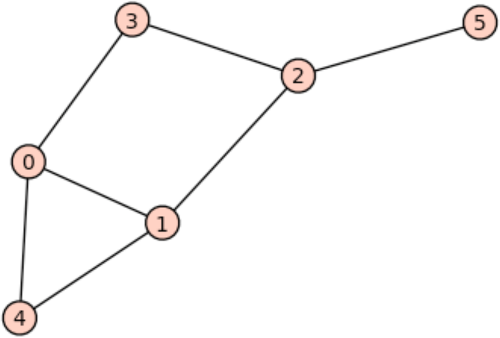
A graph X is awful if its walk matrix is invertible.

Exercise

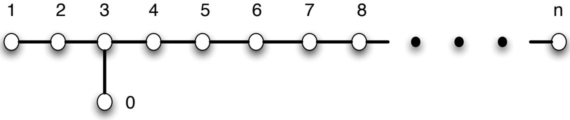
A graph is awful if and only if its complement is.

The Scapegoat

An Example



More Examples



A Sequence

6	7	9	10	15	19	21	22	25	27	30	31
34	37	39	42	45	46	49	51	54	55	57	61
66	67	69	70	75	79	81	82	85	87	90	91
94	97	99	102	105	106	109	111	114	115	117	121

Awful Algebra

Theorem (Godsil)

Let X be a graph with adjacency matrix A. The following claims are equivalent:

- *X is awful.*
- *The matrices A and J generate the algebra of all $n \times n$ matrices.*
- *The matrices $A^i J A^j$ $(0 \leq i, j < n)$ are a basis for the space of $n \times n$ matrices.*

A Conjecture

Conjecture

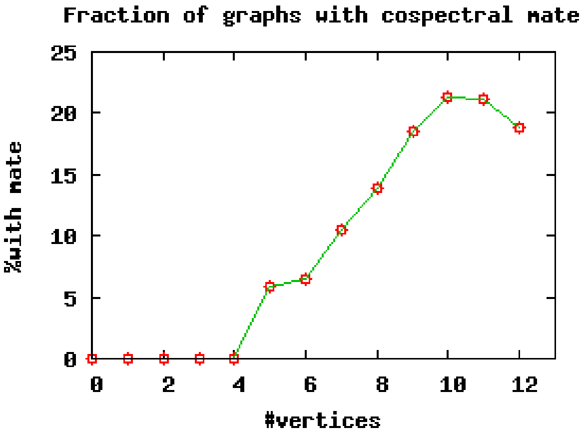
Almost all graphs are awful.

Data

#vxs	#graphs	#asymmetric	#awful
6	156	8	8
7	1044	152	92
8	12346	3696	2332
9	2744668	135004	85036

(Computations carried out with Fidel Barrera-Cruz)

Brouwer and Spence's Computations



Walk Equivalence

Definition

Let X and Y be graphs with walk matrices W_X and W_Y respectively. We say that X and Y are walk equivalent if

$$W_X^T W_X = W_Y^T W_Y.$$

Note that

$$(W^T W)_{i,j} = \mathbf{1}^T A^{i+j} \mathbf{1}.$$

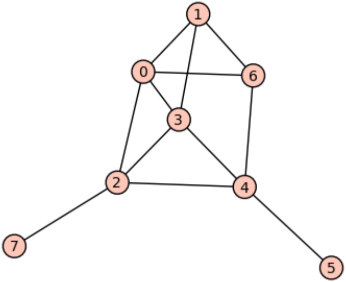
Hence any two k-regular graphs on n vertices are walk equivalent.

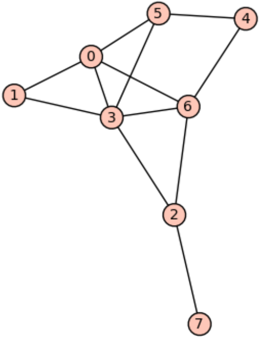
Cospectral Graphs

Lemma

If two graphs X and Y are cospectral, then their complements are cospectral if and only if they are walk equivalent.

Two Walk-Equivalent Awful Graphs





Walk-Equivalent Awful Graphs

Theorem (Wang & Xu)

If X and Y are walk-equivalent awful graphs, then $Q = W_X W_Y^{-1}$ is orthogonal, $Q^T A_X Q = A_Y$ and $Q\mathbf{1} = \mathbf{1}$.

So walk-equivalent awful graphs are cospectral with cospectral complements.

Example

$$Q = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Rationality

The matrix Q is clearly rational. The experimental evidence seems to indicate that usually $2Q$ is integral. If $2Q$ is integral, the structure of Q is known (Wang & Xu). Up to permutation equivalence we have:

Structure I

Assume

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then we may have one of the matrices

$$\frac{1}{2}\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} A & B & 0 \\ B & 0 & A \\ 0 & A & B \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} A & B & 0 & 0 \\ B & 0 & A & 0 \\ 0 & A & 0 & B \\ 0 & 0 & B & A \end{pmatrix}, \ldots$$

Structure IIa

Otherwise we have one of

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$I + P^2 + P^3 - P^4, \quad P^7 = I.$$

Structure IIb

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}$$

$$I + Q_1 + Q_2 + Q_3 - Q_1 Q_2 Q_3, \quad Q_i^2 = I,\; Q_i Q_j = Q_j Q_i.$$

Generating the Unitary Group

Theorem (Godsil)

If X is awful and A is its adjacency matrix, then the matrices

$$\exp(iAt), \quad \exp(iJt) \qquad (t \geq 0)$$

generate a dense subgroup of the unitary group.

The operators $\exp(iAt)$ determine a continuous quantum walk.

The Unitary Group

The unitary group $U(n)$ consists of all complex matrices Q such that $Q^*Q = I$. The relevant property of the unitary group is that, if U and V are "very small" matrices such that

$$I + U, \quad I + V$$

are unitary, then $I + (UV - VU)$ is unitary.
This implies that the vector space spanned by the "very small" matrices is closed under the Lie product

$$[U, V] := UV - VU.$$

(We can make this rigorous if we replace "very small" by "tangent space at I".)

Showing we have Generators

- The tangent space at I in the unitary group consists of all skew Hermitian matrices, that is, the matrices H such that $H^* = -H$.
- We can show that if X is awful, the Lie algebra generated by A and J consists of all real $n \times n$ matrices.
- Using this we can prove that iA and iJ generate the Lie algebra of skew Hermitian matrices.
- Since the unitary group is connected, it follows that our matrix exponentials generate a dense subgroup of it.

Sparser Generators?

Let D denote the diagonal matrix of valencies of X. If X is connected then $\ker(A - D)$ is spanned by $\mathbf{1}$. By spectral decomposition, it follows that $J = \mathbf{1}\mathbf{1}^T$ is a polynomial in $A - D$.

Lemma

If X is awful, then A and D generate the algebra of all $n \times n$ matrices. □

This Week's News, I

We can extend the concept of awfulness to subsets of $V(X)$. If b is the characteristic vector of a subset S of $V(X)$, call it awful if the matrix

$$\begin{pmatrix} b & Ab & \dots & A^{n-1}b \end{pmatrix}$$

is invertible. This is the original notion if $S = V(X)$. If S is awful then A and bb^T generate the full matrix algebra.

This Week's News, II

If $S \subseteq V(X)$ and $R \subseteq V(Y)$ such that

$$W_{X,S}^T W_{X,S} = W_{Y,R}^T W_{Y,R}$$

we say that S and R are walk equivalent. If X and Y are cospectral and S and R are awful subsets, and

$$Q := W_{Y,R} W_{X,S}^{-1}$$

then $Q A_X Q^{-1} = A_Y$. If S and T are walk equivalent as well, then Q is orthogonal.

Awful vertices seem to be common.

That's It!