

ALGEBRAIC COMBINATORICS

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To Gillian

Preface

There are people who feel that a combinatorial result should be given a “purely combinatorial” proof, but I am not one of them. For me the most interesting parts of combinatorics have always been those overlapping other areas of mathematics. This book is an introduction to some of the interactions between algebra and combinatorics. The first half is devoted to the characteristic and matchings polynomials of a graph, and the second to polynomial spaces. However anyone who looks at the table of contents will realise that many other topics have found their way in, and so I expand on this summary.

The characteristic polynomial of a graph is the characteristic polynomial of its adjacency matrix. The matchings polynomial of a graph G with n vertices is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k p(G, k) x^{n-2k},$$

where $p(G, k)$ is the number of k -matchings in G , i.e., the number of subgraphs of G formed from k vertex-disjoint edges. These definitions suggest that the characteristic polynomial is an algebraic object and the matchings polynomial a combinatorial one. Despite this, these two polynomials are closely related and therefore they have been treated together. In developing their theory we obtain as a by-product a number of results about orthogonal polynomials. The number of perfect matchings in the complement of a graph can be expressed as an integral involving the matchings polynomial. This motivates the study of moment sequences, by which we mean sequences of combinatorial interest which can be represented as the sequence of moments of some measure.

To be brief, if not cryptic, a polynomial space is obtained by associating an inner product space of “polynomials” to a combinatorial structure. The combinatorial structure might be the set of all k -subsets of a set of v elements, the symmetric group on n letters or, if the reader will be generous, the unit sphere in \mathbb{R}^n . Given this set-up it is possible to derive bounds on the sizes of “codes” and “designs” in the structure. The derivations are very simple and apply to a wide range of structures. The resulting bounds are often classical—the simplest and best known is Fisher’s inequality from design theory.

Polynomial spaces are perhaps impossibly general. We distinguish one important family which corresponds, when the underlying set is finite, to Q -polynomial association schemes. The latter have a well-developed theory, thanks chiefly to work of Delsarte. Our approach enables us to rederive and extend much of this work. In summary, the theory of polynomial spaces provides an axiomatisation of many of the applications of linear algebra to combinatorics, along with a natural way of extending the theory of Q -polynomial association schemes to the case where the underlying set is infinite.

From this discussion it is clear that to make sense of polynomial spaces, some feeling for association schemes is required. Hence I have included a reasonably thorough introduction to this topic. To motivate this in turn, I have also included chapters on strongly regular and distance-regular graphs. Orthogonal polynomials arise naturally in connection with polynomial spaces and distance-regular graphs, and thus form a connecting link between the two parts of this book.

My aim has been to write a book which would be accessible to beginning graduate students. I believe it could serve as a text for a number of different courses in combinatorics at this level, and I also hope that it will prove interesting to browse in. The prerequisites for successful digestion of the material offered are:

Linear algebra: Familiarity with the basics is taken for granted. The spectral decomposition of a Hermitian matrix is used more than once. The theory is presented in Chapter 2. Positive semi-definite matrices appear. A brief summary of the relevant material is included in the appendix.

Combinatorics: The basic language of graph theory is used without preamble, e.g., spanning trees, bipartite graphs and chromatic number. Once again some of this is included in the appendix. Generating functions and formal power series are used extensively in the first half of the book, and so there is a chapter devoted to them.

Group theory: The symmetric group creeps in occasionally, along with automorphism groups of graphs. The orthogonal group is mentioned by name at least once.

Ignorance: By which I mean the ability to ignore the odd paragraph devoted to unfamiliar material, in the trust that it will all be fine at the end.

I have not been able to draw up a dependence diagram for the chapters which would not be misleading. This is because there are few chains of argument extending across chapter boundaries, but many cases where the

material in one chapter motivates another. (For example it should be possible to get through the chapter on association schemes without reading the preceding chapter on distance-regular graphs. However these graphs provide one of the most important classes of association schemes.)

By way of compensation for the lack of this traditional diagram, I include some suggestions for possible courses.

- (1) **The matchings polynomial and moment sequences:**
1–3, 4.1–2, 4.4, 5.1–2, 5.6, 6, 7, 8.1–3, 9.
- (2) **The characteristic polynomial:**
1.1, 2, 3, 4.1–4, 5.1–4, 5.6, 6, 8.
- (3) **Strongly regular graphs, distance-regular graphs and association schemes:**
2, 5.1–2, 8, 10–13.
- (4) **Equitable partitions and codes in distance-regular graphs:**
2, 5.1–2, 5.6, 8.1–2, 11, 12.
- (5) **Polynomial spaces:**
2, 8.1–2, 10, 12.1–4, 13.1, 13.6, 14–16.

In making these suggestions I have made no serious attempt to consider the time it would take to cover the material indicated. On the basis of my own experience, I think it would be possible to cover at most three pages per hour of lectures. On the other hand, it would be easy enough to pare down the suggestions just made. For example, Chapter 3 covers formal power series and generating functions and depending on the backgrounds of one's victims, this might not be essential in (1) and (2).

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Contents

Preface	vii
Contents	xi
1 The Matchings Polynomial	1
1.1 Recurrences	1
1.2 Integrals	4
1.3 Rook Polynomials	7
1.4 The Hit Polynomial	10
1.5 Stirling and Euler Numbers	11
1.6 Hit Polynomials and Integrals	13
Exercises	14
Notes and References	16
2 The Characteristic Polynomial	19
2.1 Coefficients and Recurrences	19
2.2 Walks and the Characteristic Polynomial	22
2.3 Eigenvectors	23
2.4 Regular Graphs	25
2.5 The Spectral Decomposition	27
2.6 Some Further Matrix Theory	30
Exercises	33
Notes and References	36
3 Formal Power Series and Generating Functions	37
3.1 Formal Power Series	37
3.2 Limits	38
3.3 Operations on Power Series	40
3.4 Exp and Log	41
3.5 Non-linear Equations	43
3.6 Applications and Examples	45
Exercises	48
Notes and References	49

4	Walk Generating Functions	51
4.1	Jacobi's Theorem	51
4.2	Walks and Paths	55
4.3	A Decomposition Formula	57
4.4	The Christoffel-Darboux Identity	59
4.5	Vertex Reconstruction	61
4.6	Cospectral Graphs	65
4.7	Random Walks on Graphs	68
	Exercises	70
	Notes and References	72
5	Quotients of Graphs	75
5.1	Equitable Partitions	75
5.2	Eigenvalues and Eigenvectors	77
5.3	Walk-Regular Graphs	79
5.4	Generalised Interlacing	82
5.5	Covers	84
5.6	The Spectral Radius of a Tree	86
	Exercises	87
	Notes and References	90
6	Matchings and Walks	93
6.1	The Path-Tree	93
6.2	Tree-Like Walks	98
6.3	Consequences of Reality	100
6.4	Christoffel-Darboux Identities	104
	Exercises	108
	Notes and References	109
7	Pfaffians	113
7.1	The Pfaffian of a Skew Symmetric Matrix	113
7.2	Pfaffians and Determinants	114
7.3	Row Expansions	117
7.4	Oriented Graphs	119
7.5	Orientations	121
7.6	The Difficulty of Counting Perfect Matchings	123
	Exercises	125
	Notes and References	127

8	Orthogonal Polynomials	131
8.1	The Definitions	131
8.2	The Three-Term Recurrence	133
8.3	The Christoffel-Darboux Formula	135
8.4	Discrete Orthogonality	137
8.5	Sturm Sequences	140
8.6	Some Examples	144
	Exercises	145
	Notes and References	146
9	Moment Sequences	149
9.1	Moments and Walks	150
9.2	Moment Generating Functions	153
9.3	Hermite and Laguerre Polynomials	156
9.4	The Chebyshev Polynomials	158
9.5	The Charlier Polynomials	160
9.7	Sheffer Sequences of Polynomials	164
9.7	Characterising Polynomials of Meixner Type	165
9.8	The Polynomials of Meixner Type	167
	Exercises	171
	Notes and References	174
10	Strongly Regular Graphs	177
10.1	Basic Theory	177
10.2	Conference Graphs	180
10.3	Designs	184
10.4	Orthogonal Arrays	187
	Exercises	188
	Notes and References	192
11	Distance-Regular Graphs	195
11.1	Some Families	195
11.2	Distance Matrices	197
11.3	Parameters	198
11.4	Quotients	200
11.5	Imprimitive Distance-Regular Graphs	201
11.6	Codes	205
11.7	Completely Regular Subsets	208
11.8	Examples	212
	Exercises	215
	Notes and References	217

12 Association Schemes	221
12.1 Generously Transitive Permutation Groups	221
12.2 p 's and q 's	223
12.3 P - and Q -Polynomial Association Schemes	228
12.4 Products	230
12.5 Primitivity and Imprimitivity	232
12.6 Codes and Anticodes	236
12.7 Equitable Partitions of Matrices	241
12.8 Characters of Abelian Groups	243
12.9 Cayley Graphs	245
12.10 Translation Schemes and Duality	247
Exercises	251
Notes and References	255
13 Representations of Distance-Regular Graphs	261
13.1 Representations of Graphs	261
13.2 The Sequence of Cosines	263
13.3 Injectivity	264
13.4 Eigenvalue Multiplicities	267
13.5 Bounding the Diameter	270
13.6 Spherical Designs	272
13.7 Bounds for Cliques	276
13.8 Feasible Automorphisms	278
Exercises	279
Notes and References	282
14 Polynomial Spaces	285
14.1 Functions	285
14.2 The Axioms	287
14.3 Examples	289
14.4 The Degree of a Subset	291
14.5 Designs	293
14.6 The Johnson Scheme	295
14.7 The Hamming Scheme	297
14.8 Coding Theory	299
14.9 Group-Invariant Designs	300
14.10 Weighted Designs	301
Exercises	302
Notes and References	305

15	<i>Q</i>-Polynomial Spaces	307
15.1	Zonal Orthogonal Polynomials	307
15.2	Zonal Orthogonal Polynomials: Examples	309
15.3	The Addition Rule	311
15.4	Spherical Polynomial Spaces	313
15.5	Harmonic Polynomials	315
15.6	Association Schemes	316
15.7	<i>Q</i> -Polynomial Association Schemes	319
15.8	Incidence Matrices for Subsets	323
15.9	$J(v, k)$ is <i>Q</i> -Polynomial	326
	Exercises	328
	Notes and References	330
16	Tight Designs	333
16.1	Tight Bounds	333
16.2	Examples and Non-Examples	336
16.3	The Grassman Space	338
16.4	Linear Programming	341
16.5	Bigger Bounds	343
16.6	Examples	345
	Exercises	347
	Notes and References	349
	Appendix: Terminology	353
	Index of Symbols	355
	Index	359