# Association Schemes

Lecture Notes

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# Chapter 1

# Introduction

Let $X$ be a graph with diameter $d$. We define the distance graphs, $X_r$ for $r = 1, \dots d$ of $X$ by

$$V(X_r) = V(X), \quad E(X_r) = \{(u,v) : \mathrm{dist}_X(u,v) = r\}.$$

Note that $X_1 = X$.

We denote by $\mathrm{Aut}(X)$ the automorphism group of $X$. We can view this as the set of permutation matrices $P$ that commute with $A(X)$. Note that if $P \in \mathrm{Aut}(X)$, then $P \in \mathrm{Aut}(X_r)$ for $r = 1, \dots d$.

Set $A_r = A(X_r)$ and $A_0 = I$. Then $P \in \mathrm{Aut}(X)$ if and only if $P$ commutes with each element of $\{A_0, \dots, A_d\}$. It follows that $P \in \mathrm{Aut}(X)$ if and only if $P$ commutes with each element of the matrix algebra $\langle A_0, A_1, \dots, A_d\rangle$, generated by $\{A_0, A_1, \dots, A_d\}$. Roughly speaking, if $\langle A_0, \dots, A_d\rangle$ is large, then $\mathrm{Aut}(X)$ will be small. How small can the algebra be?

**Claim:** $\dim\langle A_0, \dots, A_d\rangle \geq d + 1$

What if this bound is tight? Then for all $i, j$, $A_i A_j \in \mathrm{span}\{A_0, \dots, A_d\}$. This implies that $A_i A_j$ is symmetric and hence that $A_i A_j = A_j A_i$.

**Definition.** An *association scheme* $\mathcal{A}$ is a set of $n \times n$ 01-matrices, $A_0, \dots, A_d$ such that

(i) $A_0 = I$

(ii) $\sum_i A_i = J$

(iii) $A_i A_j = A_j A_i$ for all $i, j$

(iv) $A_i A_j \in \mathrm{span}\{A_0, \dots, A_d\}$

(v) $A_i^T \in \{A_0, \dots, A_d\}$.

The structure is *symmetric* if $A_1, \ldots, A_d$ are symmetric. The algebra, $\langle A_0, \ldots, A_d \rangle$ is called the *Bose-Mesner algebra* of the scheme, and is denoted $\mathbb{C}[\mathcal{A}]$. This is a commutative algebra of matrices.

Recall the Schur product of matrices $M$ and $N$ (of the same size) is given by

$$(M \circ N)_{i,j} = M_{i,j} N_{i,j}.$$

The identity for this is $J$.

**Claim:** The Bose-Mesner algebra is closed under the Schur product.

*Proof.* $\{0, A_0, A_1, \ldots, A_d\}$ is Schur-closed. $\square$

*Exercise.* Any Schur-closed subspace of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ has a 01-basis.

**Example.**

(a) $d = 1$: $X_1 = K_n$, $A_1 = J - I$

(b) $d = 2$: Strongly regular graphs,

- $C_5$
- $L(K_n)$

(c) Johnson scheme. $V$ is all $k$-subsets of $\{1, \ldots, v\}$, $\alpha$ and $\beta$ are adjacent in $X_r$ if $|\alpha \cap \beta| = k - r$

(d) $C_n$

(e) Hamming scheme, $H(n, q)$.

A permutation group $G$ on a set $V$ is *generously transitive* if for pairs of points, $u, v \in V$ there is a permutation $g \in G$ such that $u^g = v$ and $v^g = u$.

**Example.**

(a) $\mathrm{Aut}(C_n)$

(b) Cayley graphs for abelian groups of odd order. Choose a subset $\mathcal{C}$ of $G$ such that $0 \notin \mathcal{C}$ and $-\mathcal{C} = \mathcal{C}$. Then $X(G, \mathcal{C})$ has vertex set $G$ and $E(X) = \{uv : v - u \in \mathcal{C}\}$.

(c) $\mathrm{Sym}(v)$ acting on the $\binom{v}{k}$ subset of $V = \{1, \ldots, v\}$ with size $k$. If $G$ is a permutation group os $V$, its orbitals are the orbits of acting on $V \times V$. We see that $G$ is transitive if and only if the diagonal $\{(u, u) : u \in V\}$ is a single orbit.

**Theorem 1.1.** *Suppose $G$ is a transitive permutation group on $V$. Then the non-diagonal orbitals of $G$ are graphs if and only if $G$ is generously transitive.*

**Corollary 1.2.** *If $G$ is generously transitive its orbitals form an association scheme.*

**Theorem 1.3.** *If $\Gamma$ is a set of permutations on $V$, then the set of matrices that commute with each element of $\Gamma$ is a matrix algebra which is closed under Schur product.*

*Proof.* Exercise. □

$M$ is normal if and only if $M$ and $M^*$ commute.

**Theorem 1.4.** *A matrix is normal if and only if it is unitarily diagonalizable, e.g.*

$$M = LDL^*$$

*where $L$ is unitary and $D$ is diagonal.*

Note that $M$ is normal if and only if $\langle M, M^* \rangle$ is commutative.

The Bose-Mesner algebra of an association scheme is commutative and $*$-closed. Therefore, all matrices in $\mathbb{C}[\mathcal{A}]$ are normal.

**Theorem 1.5.** *If $U \leq \mathbb{C}^v$ and $U$ is $\mathbb{C}[\mathcal{A}]$-invariant, then so is $U^\perp$.*

*Proof.* If $y \in U$ and $x \in U^\perp$, then

$$0 = \langle x, Ay \rangle = \langle A^* x, y \rangle.$$

Hence $U^\perp$ is $\mathbb{C}[\mathcal{A}]$-invariant. □

How do we find $U$? Suppose $U$ is an eigenspace for $A$ in $\mathbb{C}[\mathcal{A}]$ with eigenvalue $\lambda$. If $BA = AB$ and $u \in U$, then

$$ABu = BAu = \lambda Bu$$

and so $Bu \in \ker(A - \lambda I)$.

**Corollary 1.6.** *$\mathbb{C}^v$ has a basis consisting of eigenvectors for $\mathbb{C}[\mathcal{A}]$.*

An eigenspace for $\mathbb{C}[\mathcal{A}]$ is a subspace on which each matrix in $\mathbb{C}[\mathcal{A}]$ acts as a scalar, and is maximal with this property. If $z$ is an eigenvector for $\mathbb{C}[\mathcal{A}]$ then for each matrix $M$ we have

$$Mz = \lambda_M z.$$

The map $M \mapsto \lambda_M$ is an element of the dual space $\mathbb{C}[\mathcal{A}]$. The dimension of the dual is $d+1$. If $V$ is a vector space and $V = U_1 \oplus U_2$ then there is an idempotent linear map $P : V \to V$ with image $U_1$ and kernel $U_2$.

**Remark.** A linear map $L : V \to V$ fixes $U_1$ and $U_2$ if and only if $L$ and $P$ commute.

We aim to show that the projections onto the eigenspaces of $\mathbb{C}[\mathcal{A}]$ lie in $\mathbb{C}[\mathcal{A}]$.

# Chapter 2

# Spectral decomposition

Suppose $A$ is normal. Then $A = LDL^*$ ($D$ diagonal, $L$ unitary). We can write $D$ as

$$\sum_i \lambda_i D_i$$

where $D_i$ is diagonal, $D_i^2 = D_i$, $\sum D_i = I$ and $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $D$. So

$$A = \sum_{i=0}^{d} \lambda_i L D_i L^*$$

where $\lambda_0, \ldots, \lambda_d$ are eigenvalues of $A$. Set $E_i = LD_iL^*$. Then

$$A = \sum_{i=0}^{d} \lambda_i E_i.$$

Here,

$$\begin{aligned} E_i^2 &= E_i, \\ E_i E_j &= 0, \\ \sum E_i &= I, \\ AE_i &= \lambda E_i. \end{aligned}$$

Thus the columns of $E_i$ are the eigenvectors for $A$ with eigenvalue $\lambda_i$. If

$$p_i(t) = \prod_{j \neq i} \frac{t - \lambda_j}{\lambda_i - \lambda_j}$$

then $p_i(A) = E_i$. Assume $\mathcal{A} = \{A_0, \ldots, A_d\}$ and

$$A_i = \sum_j p_i(j) E_j^{(i)}$$

is the spectral decomposition of $A_i$. Then

$$I = \prod_{i=0}^{d} \left( \sum_j E_j^{(i)} \right). \tag{2.1}$$

Note that if $E$ and $F$ are commutative idempotents, then $EF$ and $FE$ are idempotents. So (2.1) expresses $I$ as a sum of commuting idempotents. Also if $E$ and $F$ are commuting idempotents, then

$$F = EF + (I - E)F$$

expresses $F$ as a sum of idempotents. An idempotent $F$ in $\mathbb{C}[\mathcal{A}]$ is *minimal* if it can be written as a sum $F_1 + F_2$ of non-zero idempotents from $\mathbb{C}[\mathcal{A}]$. Minimal idempotents are orthogonal.

Let $E_0, \dots, E_m$ be the set of minimal idempotents.

**Claims:**

(a) The idempotents span $\mathbb{C}[\mathcal{A}]$,

(b) they are linearly independent (hence $m = d$),

(c) $\operatorname{col}(E_i)$ is an eigenspace for $\mathbb{C}[\mathcal{A}]$,

(d) $E_i \in \mathbb{C}[\mathcal{A}]$ for all $i$.

So the Bose-Mesner algebra has two bases, $A_0, \dots, A_d$ and $E_0, \dots, E_d$.

The constant vectors are an invariant subspace for $\mathbb{C}[\mathcal{A}]$, the corresponding projection is $\frac{1}{v}J$; we denote this by $E_0$.

Recall that

(i) $A_0 = I$,

(ii) $\sum_i A_i = J$,

(iii) $A_i A_j = A_j A_i$ for all $i, j$,

(iv) $A_i^T \in \mathcal{A}$,

(v) $A_i A_j \in \operatorname{span}(\mathcal{A})$.

We also have

(i) $E_0 = \frac{1}{v}J$,

(ii) $\sum_i E_i = I$,

(iii) $E_i \circ E_j = E_j \circ E_i$ for all $i, j$,

(iv) $E_i^T \in \mathcal{A}$,

(v) $E_i \circ E_j \in \operatorname{span}(E_0, \ldots, E_d)$.

The idempotents $E_i$ are Hermitian. There are scalars $p_i(j)$ such that

$$A_i = \sum_{j=0}^{d} p_i(j) E_j$$

and scalars $q_j(i)$ such that

$$E_j = \frac{1}{v} \sum_i q_j(i) A_i.$$

We call the numbers $p_i(j)$ the *eigenvalues of the association scheme* and the $q_j(i)$ the *dual eigenvalues of the association scheme*. We also have scalars $p_{i,j}(k)$ such that

$$A_i A_j = \sum_k p_{i,j}(k) A_k$$

called *intersection numbers* and scalars $q_{i,j}(k)$ such that

$$E_i \circ E_j = \frac{1}{v} \sum_k q_{i,j}(k) E_k$$

called *Krein parameters*. We define the matrix of eigenvalues by

$$(P)_{r,s} = (p_s(r)).$$

Similarly we define $Q$ by $PQ = vI$.

Let $v_i$ be th valency of $X_i$. We have $v_i = p_i(0)$. Also, $m_i = \operatorname{tr}(E_i) = \operatorname{rank}(E_i)$. Note that

$$A_i E_j = p_i(j) E_j \tag{2.2}$$

$$E_j \circ A_i = \frac{1}{v} q_j(i) A_i \tag{2.3}$$

Denote by $\operatorname{sum}(M)$ the sum of the entries of $M$. Taking the trace of (2.2) we get

$$p_i(j) m_j = \operatorname{tr}(p_i(j) E_j) = \operatorname{tr}(A_i E_j) = \operatorname{sum}(A_i^T \circ E_j) = \operatorname{sum}(A_i \circ \bar{E}_j) = \overline{q_j(i)} v_i.$$

Hence,

$$\frac{p_i(j)}{v_i} = \frac{\overline{q_j(i)}}{m_j}. \tag{2.4}$$

The set of $m \times n$ matrices over $\mathbb{C}$, $\mathrm{Mat}_{m\times n}(\mathbb{C})$ is an inner product space, with inner product

$$\langle M, N\rangle = \mathrm{tr}(M^* N).$$

Note that

$$\mathrm{tr}(M^* N) = \mathrm{sum}(\overline{M} \circ N).$$

The idempotents, $A_0, \dots, A_d$ form an orthogonal basis for $\mathbb{C}[\mathcal{A}]$; similarly, $E_0, \dots, E_d$ is an orthogonal basis. Define

$$\Delta_v = \begin{pmatrix} v_0 & & \\ & \ddots & \\ & & v_d \end{pmatrix} \quad \text{and} \quad \Delta_m = \begin{pmatrix} m_0 & & \\ & \ddots & \\ & & m_d \end{pmatrix}.$$

Then we can rewrite (2.4) in matrix form:

$$P\Delta_v^{-1} = \Delta_m^{-1} Q^*. \tag{2.5}$$

Since $PQ = vI$, this is equivalent to

$$P\Delta_v^{-1} P^* = v\Delta_m^{-1}. \tag{2.6}$$

# Chapter 3

# Strongly regular graphs

A strongly regular graph is one of the graphs in association schemes with two classes. An association scheme with $d$ classes is *primitive* if $X_1, \ldots, X_d$ are connected. For strongly regular graphs $X$ the scheme is primitive if $X$ and $\overline{X}$ are connected. The only imprimitive strongly regular graphs are the graphs $mK_n$ (with $m, n > 1$).

**Example.** The following graphs are strongly regular.

(a) $C_5$ with parameters $(5, 2, 0, 1)$.

(b) The Petersen graph.

(c) $L(K_{n,n})$.

(d) $L(K_n)$ with parameters

$$v = \binom{n}{2}, \quad k = 2n - 4, \quad a = n - 2 \quad \text{and} \quad c = 4.$$

(e) Moore graphs of diameter two (strongly regular with girth five).

Suppose $\mathcal{A} = \{A_0, A_1, A_2\}$ with $A = A_1$. Then

$$A_1^2 = kA_0 + aA_1 + cA_2,$$

so

$$A^2 = kI + aA + c(J - I - A)$$

and

$$A^2 - (a - c)A - (k - c)I = cJ.$$

The eigenvalues of a strongly regular graph are $k$ and the roots of $t^2 - (a - c)t - (k - c)$. We denote these by $\theta$ and $\tau$ ($\theta > \tau$). The matrix of eigenvalues is

| | $A_0$ | $A_1$ | $A_2$ |
|---|---|---|---|
| $E_0$ | 1 | $k$ | $v-1-k$ |
| $E_1$ | 1 | $\theta$ | $-\theta-1$ |
| $E_2$ | 1 | $\tau$ | $-\tau-1$ |

Now use (2.6) to compute the multiplicities $m_0, m_1, m_2$,

$$1 + m_\theta + (v - 1 - m_\theta) = v, \quad k + m_\theta\theta + (v - 1 - m_\theta)\tau = 0.$$

**Computing $Q$ from $P$**

$$\begin{pmatrix} 1 & k & l \\ 1 & \theta & -\theta-1 \\ 1 & \tau & -\tau-1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & \theta/k & \frac{-(\theta+1)}{d} \\ 1 & \tau/k & \frac{-(\tau+1)}{d} \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{m_\theta\theta}{k} & \frac{-m_\theta(\theta+1)}{l} \\ 1 & \frac{m-\tau}{k} & -\frac{m_\tau(\tau-1)}{l} \end{pmatrix} \to \begin{pmatrix} & & \end{pmatrix}^T.$$

A scheme is *formally self dual* if $P = \overline{Q}$.

**Computing $p_{i,j}(k), q_{i,j}(k)$**

$$A_iA_j = \sum_r p_{i,j}(r)A_r$$

$$E_i \circ E_j = \frac{1}{v}\sum_r q_{i,j}(r)E_r$$

Here, $q_{i,j}(r)$ is an eigenvalue of the Hermitian matrix $E_i \circ E_j$, so it is real. Since $E_i \succcurlyeq 0$ and $E_j \succcurlyeq 0$ their Schur product is positive semidefinite. So $q_{i,j}(r) \geq 0$ (Krein condition). We have

$$A_iA_jE_s = p_i(s)p_j(s)E_s$$

and

$$\left(\sum_r p_{i,j}(r)A_r\right)E_s = \left(\sum_r p_{i,j}(r)p_r(s)\right)E_s$$

so

$$p_i(s)p_j(s) = \sum_r p_{i,j}(r)p_r(s), \quad (s = 0, \ldots, d).$$

$$\begin{aligned} A_iA_j &= \left(\sum_r p_i(r)E_r\right)\left(\sum_r p_j(r)E_r\right) \\ &= \sum_r p_i(r)p_j(r)E_r \\ (A_iA_j) \circ A_k &= \sum_r p_i(r)p_j(r)\,\frac{1}{v}\,\overline{q_r(k)}. \end{aligned}$$

$\mathrm{Mat}_{n\times n}(\mathbb{C})$ is an inner product space,

$$\langle M, N\rangle = \mathrm{tr}(M^*N) = \mathrm{sum}(\bar{M}\circ N).$$

Both $A_0, \dots, A_d$ and $E_0, \dots, E_d$ are orthogonal bases for $\mathbb{C}[\mathcal{A}]$. If $M \in \mathrm{Mat}_{v\times v}(\mathbb{C})$, its projection onto $\mathbb{C}[\mathcal{A}]$ is

$$\hat{M} = \sum_{i=0}^{d} \frac{\langle M, A_i\rangle}{\langle A_i, A_i\rangle} A_i = \sum_{j=0}^{d} \frac{\langle M, E_j\rangle}{\langle E_j, E_j\rangle} E_j.$$

In practice, $M$ will usually be of the form $xx^T$, where $x$ is the character vector of some subset $V$. In this case,

$$\langle M, A_i\rangle = \mathrm{tr}(xx^T A_i) = \mathrm{tr}(x^T A_i x) = x^T A_i x \geq 0$$

and

$$\langle M, E_j\rangle = \mathrm{tr}(xx^T E_j) = x^T E_j x \geq 0.$$

Hence, if $M = xx^T$, then $\hat{M} \geq 0$ and $\hat{M} \succcurlyeq 0$. If $\mathcal{I} \subseteq \{1, \dots, d\}$ then

$$X_{\mathcal{I}} = \bigcup_{i\in\mathcal{I}} X_i.$$

An $\mathcal{I}$-*clique* is a clique in $X_{\mathcal{I}}$; an $\mathcal{I}$-*coclique* is a coclique in $X_{\mathcal{I}}$. Suppose $C \subseteq V$. Then $C$ is an $\mathcal{I}$-clique if $x_C x_C^T \circ A_i = 0$ for $i \notin \mathcal{I}$. It is an $\mathcal{I}$-coclique if $x_C x_C^T \circ A_i = 0$ for $i \in \mathcal{I}$.

**Theorem 3.1.** *[Clique-coclique bound]*

*If $C$ is an $\mathcal{I}$-clique and $S$ is an $\mathcal{I}$-coclique, then $|C||S| \leq v$. If equality holds then*

$$x_C^T E_j x_C \cdot x_S^T E_j x_S = 0.$$

*Proof.* Set $M = x_C x_C^T$, $N = x_S x_S^T$. We have

$$\hat{M} = \sum_r \frac{\langle M, A_r\rangle}{\langle A_r, A_r\rangle} A_r = \sum_r \frac{x_C^T A_r x^C}{v v_r} A_r$$

$$\hat{N} = \cdots = \sum_j \frac{x_S^T E_j x_S}{m_j} E_j.$$

$$\hat{M}\circ\hat{N} = \frac{|C||S|}{v^2} I$$

$$\begin{aligned}\hat{M}\hat{N} &= \sum_j \frac{x_C^T E_j x_C x_S^T E_j x_S}{m_j^2} E_j \\ &\succcurlyeq (x_C^T E_0 x_C x_S E_0 x_S) E_0, \qquad (E_0 = \frac{1}{v} J) \\ &= \frac{|C|^2|S|^2}{v^3} J\end{aligned}$$

Further, we have

$$\frac{|C||S|}{v} = \text{sum}(\hat{M} \circ \hat{N}) = \text{tr}(\hat{M}\hat{N}) \geq \frac{|C|^2|S|^2}{v^2}$$

and so $|C||S| \leq v$. $\square$

**Theorem 3.2.** *If $M, N \succcurlyeq 0$ are matrices in $\mathbb{C}[\mathcal{A}]$ with $M \circ N = \alpha I$ then*

$$\frac{\text{sum}(M)}{\text{tr}(M)} \cdot \frac{\text{sum}(N)}{\text{tr}(N)} \leq v,$$

*and if equality holds, $MN = \beta J$.*

*Proof.* Exercise. $\square$

**Example** (Bound on $\alpha(X)$)**.** Set $A = A(X)$, let $\tau$ be the least eigenvalue of $A$ and set $N = A - \tau I$. Then $N \succcurlyeq 0$ and

$$\begin{aligned} \text{tr}(N) &= v(-\tau) \\ \text{sum}(N) &= vk + v(-\tau) \\ \Rightarrow |S| &\leq \frac{v(-\tau)}{k - \tau} = \frac{v}{1 - k/\tau} \quad \text{(ratio bound).} \end{aligned}$$

(a) Petersen graph; $\tau = -2, k = 3, v = 10$,

$$|S| \leq \frac{10}{1 + 3/2} = 4.$$

(b) Kneser graph $K_{v:k}$; $|V(K_{v:k})| = \binom{v}{k}$, valency $= \binom{v-k}{k}$, $\tau = -\binom{v-k-1}{k-1}$, so

$$|S| \leq \binom{v-1}{k-1}.$$

# To do

- Starting from the matrix of eigenvalues - are we using (2.6) to compute the multiplicities? Why is the multiplicity of $k$ one? Define $m_\theta, m\tau$.
- Computing $Q$ from $P$ - formalize better.
- Computing $p_{i,j}(k), q_{i,j}(k)$ - are we showing how to compute them?
- Should we divide this chapter into sections? Some things are a bit out of context.
- Proof of Theorem 3.1 is messy and confusing.

# Chapter 4

# Kronecker product

The *Kronecker product* of two matrices, $A$ and $B$, is defined by

$$A \otimes B = (A_{ij}B)_{i,j}.$$

The Kronecker product has the following properties:

- $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$,
- $(A \otimes B)(C \otimes D) = AC \otimes BD$,
- $A \otimes B = (A \otimes I)(I \otimes B)$.

We define

$$\mathrm{vec}(M) = \begin{pmatrix} Me_1 \\ \vdots \\ Me_n \end{pmatrix}.$$

Then

$$\mathrm{vec}(AMB^T) = (B \otimes A)\,\mathrm{vec}(M),$$

and this can be used to reduce some problems to solving a system of linear equations. For example, a matrix $M$ commutes with $A$ if and only if $MA - AM = 0$ and, since

$$\mathrm{vec}(MA - AM) = ((A^T \otimes I) - (I \otimes A))\,\mathrm{vec}(M)$$

finding the matrices that commute with $A$ is reduced to finding the kernel of

$$(A^T \otimes I) - (I \otimes A)$$

If $A$ and $B$ are square, $A \circ B$ is a principal submatrix of $A \otimes B$. This fact leads to a proof of Schur's theorem that the product of positive semidefinite matrices is positive semidefinite.

## To do

- This chapter is awkwardly short. Can we make it longer or move it into another chapter?

# Chapter 5

# A central identity

Define

$$\mu(A \otimes B) = AB, \quad \text{and} \quad \sigma(A \otimes B) = A \circ B.$$

The maps $\mu$ and $\sigma$ have adjoints $\mu^*$ and $\sigma^*$. We have

$$\begin{aligned} \langle C, AB \rangle &= \langle C, \mu(A \otimes B) \rangle = \langle \mu^*(C), A \otimes B \rangle, \\ \langle C, A \circ B \rangle &= \langle C, \sigma(A \otimes B) \rangle = \langle \sigma^*(C), A \otimes B \rangle. \end{aligned}$$

We will use $\tau$ to denote the transpose map on a space of matrices.

**Claim.** $* \otimes I$ is self-adjoint. We have

$$\begin{aligned} \langle \mu^*(I), A^* \otimes B \rangle &= \langle I, \mu(A^* \otimes B) \rangle \\ &= \langle I, A^* B \rangle \\ &= \operatorname{tr}(A^* B) \\ &= \langle A, B \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \sigma^*(J), \bar{A} \otimes B \rangle &= \langle J, \bar{A} \circ B \rangle \\ &= \operatorname{sum}(\bar{A} \circ B) \\ &= \langle A, B \rangle. \end{aligned}$$

We conclude that $(* \otimes I)\mu^*(I) = (- \otimes I)\sigma^*(J)$. Let's assume $\mathcal{A}$ is real and symmetric. Then $\mu^*(I) = \sigma^*(J)$.

$$\mu^*(I) = \sum_{i,j} \gamma_{ij} E_i \otimes E_j$$

$$\langle \mu^*(I), A \otimes B \rangle = \langle I, AB \rangle = \operatorname{tr}(AB)$$

$$\langle \mu^*(I), E_i \otimes E_j \rangle = \delta_{ij} m_j$$

and so

$$\mu^*(I) = \sum_j \frac{1}{m_j} E_j \otimes E_j.$$

Similarly,

$$\sigma^*(J) = \sum_i \frac{1}{vv_i} A_i \otimes A_i.$$

We have thus proved the following identity, due to Koppinen [**?**].

**Theorem 5.1.** *For any association scheme,*

$$\sum_i \frac{1}{vv_i} A_i \otimes A_i^T = \sum_j \frac{1}{m_j} E_j \otimes E_j.$$

If we apply $I \otimes *$ to this we get

$$\sum_i \frac{1}{vv_i} A_i \otimes A_i = \sum_j \frac{1}{m_j} E_j \otimes \bar{E}_j;$$

we can produce a number of such variants.

## 5.1 Clique coclique bound

Define $T : C \otimes D \to \operatorname{tr}(DX)C$. Then

$$\begin{aligned} T(A_i \otimes A_i) &= \operatorname{tr}(A_i)A_i \\ T(E_j \otimes E_j) &= \operatorname{tr}(XE_j)A_j \end{aligned}$$

Let $C$ be a clique and $S$ be a coclique with character vectors $y$ and $z$ respactively. Set $x = y \otimes z$. Then

$$\begin{aligned} x^T(A_i \otimes A_i)x &= (y^T \otimes z^T)(A_i \otimes A_i)(y \otimes z) \\ &= y^T A_i y \, z^T A_i z \\ &= y^T A_0 y \, z^T A_0 z \\ &= |C||S|. \end{aligned}$$

Also,

$$\begin{aligned} \frac{|C||S|}{v} &= x^T \sigma^*(J) x \\ &= x^T \mu^*(I) x \\ &= \sum_j \frac{1}{m_j} y^T E_j y \, z^T E_j z \\ &\geq \frac{|C|^2|S|^2}{v^2} \end{aligned}$$

and so $|C||S| \leq v$. If equality holds, $y^T E_j y\, z^T E_j z = 0$, for $j = 1, \ldots, d$. Apply $T$ to both sides of K's identity,

$$\sum_i \frac{\langle X, A_i \rangle}{v v_i} A_i = \sum_j \frac{\langle X, E_j \rangle}{m_j} E_j.$$

1. A scheme is *pseudocyclic* if $v_1 = \cdots = v_d$ and $m_1 = \cdots = m_d$, e.g. prime cycles, Paley graphs.
2. A connected, regular graph is strongly regular if and only if it has exactly three distinct eigenvalues.

Assume $\mathcal{A}$ is pseudocyclic. Then

$$\mu^*(I) = \sum_i \frac{A_i \otimes A_i}{v v_i} = \frac{1}{v} I + \frac{1}{v_1} \sum_i A_i^{\otimes 2}$$

and

$$\sigma^*(J) = E_0^{\otimes 2} + \frac{1}{m_1} \sum_j E_j \otimes E_j.$$

The matrices $A_i \otimes A_j$ form an association scheme. We conclude that $\sum_i A_i^{\otimes 2}$ is strongly regular.

# Chapter 6

# Quotients and subschemes

Let $M$ be a matrix. Let $\rho$ be a partition of the rows of $M$ and let $R$ be the characteristic matrix of $\rho$. Let $\rho^*$ be the partition given by the relation 'equals' on the columns of $R^T M$. Let $N$ be the matrix with the distinct columns of $R^T M$ as its columns. If $S$ is a characteristic matrix of $\rho^*$, then $R^T M = NS^T$. We call $\rho^*$ the *induced partition*.

**Theorem 6.1.** *If the rows of $M$ are linearly independent, then $|\rho^*| \geq |\rho|$.*

*Proof.* If $0 = x^T R^T M$, then $x^T R^T = 0$ and so $x^T = 0$. Therefore the rows of $NS^T$ are linearly independent and so $|\rho^*|$, the number of columns of $NS^T$ is at least as large as the number of rows, which is $|\rho|$. □

## 6.1 Equitable partitions

Suppose $A$ is a linear map on $V$. Then $U$ is $A$-invariant if and only if there is a matrix $B$ such that if $u_1, \ldots, u_m$ are a basis for $U$, then

$$A\begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix} B.$$

A partition, $\pi$, of the vertices of a scheme is *equitable* if the space of functions constant on the cells of $\pi$ is $\mathbb{C}[\mathcal{A}]$-invariant. If $P$ is the characteristic matrix of a partition $\pi$ of $V(X)$, then $\pi$ is equitable if and only if $\operatorname{col}(P)$ is $A$-invariant.

**Lemma 6.2.** $U = \operatorname{col}(P)$ *for some partition if*

*(a)* $\mathbb{1} \in U$, *and*

*(b)* $U$ *is Schur-closed.*

**Example.** Choose $u \in V(\mathcal{A})$ and set

$$C_r := \{x \in V : (A_r)_{ux} \neq 0\}.$$

This gives us a partition. The characteristic vector of $C_r$ is $A_r e_u$. The characteristic matrix is then

$$N = \begin{pmatrix} A_0 e_u & A_1 e_u & \cdots & A_d e_u \end{pmatrix}$$

and its column space is $\mathcal{A}$-invariant. The non-zero vectors $E_j e_u$ form an orthogonal basis for the column space of $N$.

There are matrices, $B_0, \dots, B_d$ (of order $(d+1) \times (d+1)$) such that $A_r N = N B_r$. Note that $N^T N$ is invertible and so

$$B_r = (N^T N)^{-1} N^T A N$$

(thus it is determined by $A_r$). The map $A_r \mapsto B_r$ extends to a homomorphism from $\mathbb{C}[\mathcal{A}]$ to the algebra generated by $B_0, \dots, B_d$.

There is a second way of getting the matrices $B_r$: the matrices $A_i$ act on $\mathbb{C}[\mathcal{A}]$ by left multiplication. Then $B_r$ represents $A_r$ in this action. Now,

$$A_r A_i = \sum_j p_{r,i}(j) A_j$$

and so

$$(B_r)_{i,j} = p_{r,j}(i).$$

(Exercise.)

## 6.2 More invariant subspaces

Suppose $C \subseteq V(\mathcal{A})$ with characteristic vector $x$. We form the matrix

$$N = \begin{pmatrix} A_0 x & A_1 x & \cdots & A_d x \end{pmatrix}.$$

Then $\mathrm{col}(N)$ is $A$-invariant and the non-zero vectors, $E_j x$, form an orthogonal basis. We view $\mathrm{col}(N)$ as the $\mathbb{C}[\mathcal{A}]$-module generated by $x$ and denote it by $\langle x \rangle_d$ and $\langle x \rangle$.

Let $\mathcal{A}$ be a scheme with matrix of eigenvalues $P$. We define a map $T$ on $\{0, \dots, d\}$ by $A_{i^T} = (A_i)^T$. Then if $C \subseteq \{0, \dots, d\}$, $C^T = \{i^T : i \in C\}$. Any subscheme determines a partition of the idempotents $A_0, \dots, A_d$ and hence a partition $\pi$ of $\{0, 1, \dots, d\}$:

(a) $\{0\} \in \pi$

(b) If $C_i$ is a cell then either $C_i^T = C_i$ or $C_i^T \cap C_i = \emptyset$

The problem is to characterize the partitions that give us a subscheme. Set $B := \sum_{j \in C_i} A_j$. Then $\sum B_i = J$, $B_0 = I$ and $B_i^T \in \{B_0, \dots, B_e\}$ for each $i$. The algebra generated by $\mathcal{B} = \{B_0, \dots, B_e\}$ is commutative and $\mathcal{B}$ is an association scheme if and only if

$$\dim(\langle \mathcal{B} \rangle) = e + 1.$$

Let $N$ be the characteristic matrix of the partition $\pi$. The columns of $PN$ give us the eigenvalues of the matrices $B_i$. The dimension of $\langle \mathcal{B} \rangle$ is the number of distinct rows of $PN$. Thus the number of eigenvalues is $|\pi^*|$. It follows that $\mathcal{B}$ is a scheme if and only if $|\pi^*| = |\pi|$. (We have $|\pi| \leq |\pi^*|$ because $P$ is invertible.)

If $R$ is the characteristic matrix of $\pi^*$ then $PN = RP_1$ and $P_1$ is the matrix of eigenvalues of $\mathcal{B}$.

# Chapter 7

# Duality

## 7.1 Group schemes

Let $G$ be a group. If $g \in G$, let $P_g$ be the permutation matrix representing $G$ its right regular representation. Then

$$\begin{aligned} P_e &= I, \\ \sum_g P_g &= J, \\ (P_g)^T &= P_{g^{-1}}, \\ P_g P_h &= P_{gh}. \end{aligned}$$

This is a representation of the group algebra. If $G$ is abelian we get a so called *group scheme*.

**Remark.** We have that $v_i = 1$ for all $g \in G$. This characterizes group schemes.

A *translation scheme* is a subscheme of a group scheme. Each Schur idempotent in a translation scheme is a Cayley graph for $G$. The Hamming scheme and the forms schemes are translation schemes.

Suppose $\mathcal{A}$ is a group scheme. Assume $v = |G|$. There exist matrix idempotents $E_j$ for $j = 0, \dots, v-1$. We have

$$m_j = \text{rank}(E_j) = 1 \quad \text{for all } j.$$

Next, the Schur product of rank one matrices has rank one, so

$$E_i \circ E_j = E_l \quad \text{for some } l.$$

The matrix of eigenvalues of $\mathcal{A}$ is flat and is unitary. Hence $E_j = x_j x_j^*$ where $x_j$ is flat and has norm one. So

$$E_j \circ \bar{E}_j = J.$$

It follows that the matrix idempotents form an abelian group $G^*$ under Schur multiplication. It is called the character group of $G$.

*Exercise.* We have $G \simeq G^*$.

If $\mathcal{A}$ and $B$ are the schemes for groups $G$ and $H$ respectively, then the matrices

$$A_r \otimes B_s, \quad A_r \in \mathcal{A},\ B_s \in \mathcal{B}$$

form the group scheme for $G \times H$.

## 7.2 Subschemes

We can view $G^*$ as a group on the vectors $x$ such that $xx^*$ is one of the matrix idempotents. The matrix of eigenvalues for $G^*$ is $Q$. If we order things so that $P$ is symmetric (this is possible) then $P = \bar{Q}$.

If $\mathcal{A}$ is a translation scheme for a group $G$ corresponding to a partition $\pi$ of $G$ (i.e. of $\{0, 1, \ldots, v-1\}$), the induced partition $\pi^*$ gives a partition of $G^*$. If $\pi$ is a "good" partition of $G$, then it determines a subscheme if and only if $|\pi| = |\pi^*|$. Since $P$ is invertible, $(\pi^*)^* = \pi$ (Exercise).

It follows that $\pi^*$ gives us an association scheme on $G^*$ thus translation schemes come in dual pairs.

**Example.** Consider the cyclic group, $G = \langle g : g^6 = 1\rangle \simeq \mathbb{Z}_6$. Let $\pi = \{C_0, C_1, C_2, C_3\}$ be the following partition of $G$:

$$C_0 = \{1\}, \quad C_1 = \{g^2\}, \quad C_2 = \{g^4\}, \quad C_3 = \{g, g^3, g^5\},$$

(so $C_1^{-1} = C_2$ and $C_3^{-1} = C_3$) and let $\mathcal{A} = \{A_0, A_1, A_2, A_3\}$ be the corresponding translation scheme, i.e.

$$A_0 = I, \quad A_1 = A_2^T = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0 \end{pmatrix}.$$

Let $\zeta := e^{\frac{2\pi i}{6}}$ and note that $\zeta^3 = -1$. The character table of $G$ is

| | $1$ | $g$ | $g^2$ | $g^3$ | $g^4$ | $g^5$ |
|---|---|---|---|---|---|---|
| $1_{G^*}$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\chi_1$ | $1$ | $\zeta$ | $\zeta^2$ | $-1$ | $-\zeta$ | $-\zeta^2$ |
| $\chi_2$ | $1$ | $\zeta^2$ | $-\zeta$ | $1$ | $\zeta^2$ | $-\zeta$ |
| $\chi_3$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $-1$ |
| $\chi_4$ | $1$ | $-\zeta$ | $\zeta^2$ | $1$ | $-\zeta$ | $\zeta^2$ |
| $\chi_5$ | $1$ | $-\zeta^2$ | $-\zeta$ | $-1$ | $\zeta^2$ | $\zeta$ |

Denote the character table by $M$ and let $R$ be the characteristic matrix for the partition $\pi$, thus

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & -1 & -\zeta & -\zeta^2 \\ 1 & \zeta^2 & -\zeta & 1 & \zeta^2 & -\zeta \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\zeta & \zeta^2 & 1 & -\zeta & \zeta^2 \\ 1 & -\zeta^2 & -\zeta & -1 & \zeta^2 & \zeta \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now since $\zeta - \zeta^2 = 1$, we get

$$R^T M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta^2 & -\zeta & 1 & \zeta^2 & -\zeta \\ 1 & -\zeta & \zeta^2 & 1 & -\zeta & \zeta^2 \\ 3 & 0 & 0 & -3 & 0 & 0 \end{pmatrix}$$

The first four columns of $R^T M$ are all distinct, column five is the same as column two, and column six is the same as column three, thus, the dual partition, $\pi^*$ of $G^*$ is

$$D_0 = \{1_{G^*}\}, \quad D_1 = \{\chi_1, \chi_4\}, \quad D_2 = \{\chi_2, \chi_5\}, \quad D_3 = \{\chi_3\}.$$

Note that for all $x \in G$ we have

$$\chi_1(x)^{-1} = \chi_5(x), \quad \chi_2(x)^{-1} = \chi_4(x) \quad \text{and} \quad \chi_3(x)^{-1} = \chi_3(x),$$

i.e. $D_1^{-1} = D_2$ and $D_3^{-1} = D_3$. Clearly, this partition defines a translation scheme that is not isomorphic to $\mathcal{A}$, thus $\mathcal{A}$ is not self-dual.

## 7.3 Duality

First, an example on $\mathbb{Z}_2^d$. The maps

$$\psi_a : x \mapsto (-1)^{\langle a,x \rangle}$$

are eigenvectors of the group scheme. Then

$$P = \left((-1)^{\langle a,b \rangle}\right)_{a,b} \quad \text{and} \quad E_a = \frac{1}{v}\psi_a \psi_a^*.$$

The duality, $\Theta$, is defined by

$$\Theta(A_u) = \sum_j p_u(j) A_j.$$

This maps $\mathbb{C}[\mathcal{A}]$ to itself. It is linear and represented by $P$. Since $p_i(j) = \overline{q_i(j)}$, $(P = \bar{Q})$, we have

$$\begin{aligned} \Theta(A_u) &= v\bar{E}_u \\ \Theta(I) &= J \\ \Theta(J) &= vI \end{aligned}$$

Also

$$\Theta(A_u) \circ \Theta(A_v) = \Theta(A_u A_v).$$

In general, if $\mathcal{A}$ is a scheme and $P = \bar{Q}$ we define

$$\Theta(A_i) = v\bar{E}_i.$$

Then $\Theta(I) = J$ and $\Theta(J) = vI$. We have

$$\Theta(A_i)\Theta(A_j) = v^2 \bar{E}_i \bar{E}_j = v\,\Theta(A_i \circ A_j).$$

By linearity,

$$\Theta(M)\Theta(N) = v\,\Theta(M \circ N).$$

Also

$$\begin{aligned}\Theta(v\bar{E}_i) &= \Theta\left(\sum_j \overline{q_i(j)} A_j\right)\\ &= v\sum_j q_i(\bar{j})\bar{E}_j\\ &= v\sum_{j,r} q_i(j)\overline{q_j(r)}A_r\\ &= vA^T.\end{aligned}$$

**Corollary 7.1.** $\Theta^2(M) = vM^T$.

Let $\mathcal{A}$ be an association scheme with $P = \bar{Q}$. We have

$$\Theta : \mathbb{C}[\mathcal{A}] \to \mathbb{C}[\mathcal{A}], \quad \Theta(A_i) = vE_i,$$

then $\Theta(I) = J$ and $\Theta(J) = vI$. Further,

$$\begin{aligned}\Theta(M \circ N) &= \frac{1}{v}\Theta(M)\Theta(N)\\ \Theta(v\bar{E}_i) &= vA_i^T.\end{aligned}$$

Hence $\Theta^2(M) = vM^T$ for $M \in \mathbb{C}[\mathcal{A}]$ and in consequence,

$$\Theta(MN) = \Theta(M) \circ \Theta(N).$$

In general $\Theta^4(M) = v^2 I$. Define

$$\hat{\Theta}(M) = \frac{1}{\sqrt{v}}\Theta(M).$$

Then $\hat{\Theta}^T(M) = M^T$. We see that the symmetric matrices in $\mathbb{C}[\mathcal{A}]$ are an eigenspace for $\hat{\Theta}^2$ with eigenvalue 1, the skew symmetric matrices are an eigenspace with eigenvalue $-1$. So if $M = \bar{M}$ then $(\hat{\Theta}^2 - I)(M) = 0$ and thus

$$(\hat{\Theta} - I)(\hat{\Theta} + I)(M) = 0.$$

Hence $(\hat{\Theta} + I)(M)$ is an eigenvector for $\hat{\Theta}$ with eigenvalue 1 and similarly, $(\hat{\Theta} - I)(M)$ is an eigenvector with eigenvalue $-1$.

Similarly if $M^T = -M$ then $(\hat{\Theta}^2 + I)(M) = 0$ and the eigenvectors are of the form $\hat{\Theta}(M) \pm iM$ with eigenvalues $\pm i$.

# Questions and remarks

1. How do we know if a scheme is a translation scheme?
2. If $\mathcal{A}$ is a translation scheme, then the matrices $M + M^T$ for $M \in \mathbb{C}[\mathcal{A}]$ form the Bose-Mesner algebra of a symmetric scheme.
3. If $\mathcal{A}$ and $\mathcal{B}$ are schemes, then we have the product scheme $\mathcal{A} \otimes \mathcal{B}$. If $P_{\mathcal{B}} = \bar{Q}_{\mathcal{A}}$ then $\mathcal{A} \otimes \mathcal{B}$ is a self dual scheme. (It is true that $P_{\mathcal{A}\otimes\mathcal{B}} = P_{\mathcal{A}} \otimes P_{\mathcal{B}}$.)

# To do

- Add example of non-selfdual translation scheme.
- Duality section could be better organized.

# Chapter 8

# Type-II matrices

If $W$ is a matrix with all entries non-zero, then we denote by $W^{(-)}$ the Schur inverse of $W$. We say that $W$ is a *type-II matrix* if $W$ is $v \times v$ and $WW^{(-)T} = vI$.

**Example.**

1. Hadamard matrices.
2. The matrix of eigenvalues of a group scheme (character table of an abelian group).
3. $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & t & -t \\ 1 & -1 & -t & t \end{pmatrix}, \quad t \neq 0.$$

Note that examples 1. and 2. are flat, but 3. is not. If $W_1$ and $W_2$ are type-II, so is $W_1 \otimes W_2$. The following operations take type-II to type-II:

(a) Transpose.

(b) Rescaling rows and/or columns.

(c) Permuting rows and/or columns.

We say that two type-II matrices are *equivalent* if one can be obtained from the other using any of the above operations.

**Theorem 8.1.** *Any two of the following imply the third:*

*(a) $W$ is type-II,*

*(b) $W$ is flat,*

*(c) $W$ is unitary.*

Unitary matrices $M$ and $N$ are *unbiased* if $M^*N$ is flat.

**Example** (Potts model)**.** If $W = (t-1)I + J$, then $W^{(-)} = (t^{-1}-1)I + J$ and

$$WW^{(-)} = (2 - t - t^{-1})I + (t + t^{-1} - 2 + v)J.$$

Then $W$ is type-II if $t + (v-2) + t^{-1} = 0$.

## 8.1 Nomura algebras

Let W be a Schur-invertible matrix. Define

$$W_{a/b} = (We_a) \circ (We_b)^{(-)}.$$

The Nomura algebra, $\mathcal{N}_W$ of all $W$ is

$$\{M : W_{a/b} \text{ is an eigenvector for all } a, b\}.$$

Then $\mathcal{N}_W$ is a matrix algebra.

**Lemma 8.2.** *If $W$ is square and $W^{(-)}$ exists then $J \in \mathcal{N}_W$ if and only if $W$ is type-II.*

If $W$ is invertible, the vectors $W_{a/b}$ span $\mathbb{C}^v$ and $\mathcal{N}_W$ is diagonalizable and commutative. If $W_1$ and $W_2$ are equivalent, then $\mathcal{N}_{W_1} \simeq \mathcal{N}_{W_2}$. If $W$ is the character table of an abelian group $G$, then $\mathcal{N}_W$ is $\mathbb{C}[G]$ (the group ring of $G$).

If $M \in \mathcal{N}_W$, let $\Theta(M)$ be the $v \times v$ matrix such that $\Theta(M)_{a,b}$ is the eigenvalue of $M$ on $W_{a/b}$. Clearly,

$$\Theta(I) = J.$$

Also,

$$\Theta(J) = vI.$$

The set $\Theta(\mathcal{N}_W)$ is Schur-closed (because $\Theta(MN) = \Theta(M) \circ \Theta(N)$ and contains $J$). The goal now is to prove that $\mathcal{N}_W$ is Schur-closed. Choose a vertex $a$ and define

$$F_i = \frac{1}{v} W_{a/i}(W_{i/a})^T.$$

This is a rank one matrix. We have

$$F_iF_j = \frac{1}{v^2} W_{a/i}(W_{i/a})^T W_{a/j}(W_{j/a})^T.$$

Here,

$$\begin{aligned}(W_{i/a})^T W_{a/j} &= \left(We_i \circ We_a^{(-)}\right)^T \left(We_a \circ We_j^{(-)}\right)\\ &= D_{W_a}^{-1}(We_i)^T(We_j)^{(-)}D_{W_a}\\ &= v\delta_{i,j}.\end{aligned}$$

It follows that $F_iF_j = \delta_{i,j}F_i$. Since $\operatorname{tr}(F_i) = 1$, we have $\operatorname{tr}\left(\sum F_i\right) = n$, and so

$$\left(\sum F_i\right)^2 = \sum F_i.$$

Hence

$$\sum_i F_i = I.$$

**Lemma 8.3.** *If $M \in \mathcal{N}_W$ then*

$$M = \sum_i \Theta(M)_{a,i} F_i.$$

**Lemma 8.4** (Nomura)**.** *We have*

$$\Theta_W(M)(W^T)_{r/s} = vM_{s,r}(W^T)_{r/s}.$$

This implies that $\Theta_W(M) \in \mathcal{N}_{W^T}$ and $\Theta_W(\Theta_{W^T}(M)) = vM^T$.

# Chapter 9

# Dual bases

If $x_1, \ldots, x_2 \in \mathbb{C}^d$ and $y_1, \ldots, y_d \in \mathbb{C}^d$ and

$$\langle y_i, x_i \rangle = \delta_{i,j}$$

then $\{y_j\}$ is a *dual basis* for $\{x_i\}$. In this case the $x_i y_j^*$ are idempotents and the sum to the identity. Also distinct idempotents are orthogonal.

**Example.** A Schur invertible matrix $W$ is type-II if and only if the columns of $W^{(-)}$ are a dual basis to the columns of $W$.

Suppose $W$ is type-II of order $v \times v$. We define rank-1 matrices

$$\mathcal{F}_{i,j} = \frac{1}{v} W_{i/j} (W_{j/i})^T.$$

Then $\mathcal{F}_{i,i} = \frac{1}{v} J$. These matrices are idempotents and

$$\mathcal{F}_{i,r} \mathcal{F}_{i,s} = 0, \quad \mathcal{F}_{r,i} \mathcal{F}_{s,j} = 0.$$

Also

$$\sum_r \mathcal{F}_{i,r} = I = \sum_s \mathcal{F}_{s,j}.$$

Note that

$$\mathcal{F}_{i,j}^T = \mathcal{F}_{j,i}$$
$$\mathcal{F}_{i,j}^{(-)} = v^2 \mathcal{F}_{j,i}$$

**Theorem 9.1** (Nomura)**.** *If $M \in \mathcal{N}_W$ then $\Theta_{W^T}(\Theta_W(M)) = vM^T$.*

*Proof.* Assume $M \in \mathcal{N}$. Then

$$M \mathcal{F}_{i,j} = \Theta(M)_{i,j} \mathcal{F}_{i,j}$$

and, summing this over $j$ we have

$$M = \sum_j \Theta(M)_{i,j} \mathcal{F}_{i,j}.$$

Then

$$\begin{aligned} M_{r,s} &= \frac{1}{v} \sum_j \Theta(M)_{i,j} \frac{W_{r,i}}{W_{r,j}} \frac{W_{s,j}}{W_{s,i}} \\ &= \frac{W_{r,i}}{W_{s,i}} \sum_j \Theta(M)_{i,j} \frac{W_{s,j}}{W_{r,j}} \\ vM_{r,s} \frac{W_{s,i}}{W_{r,i}} &= \sum_j \Theta(M)_{i,j} \frac{W_{s,j}}{W_r, j}. \end{aligned}$$

□

**Consequences**

- $\Theta_W$ and $\Theta_{W^T}$ are invertible.
- $\mathcal{N}_W$ is closed under transpose.
- Since $\operatorname{im}(\Theta_{W^T})$ is Schur closed, it follows that $\mathcal{N}_W$ is Schur closed. Hence $\mathcal{N}_W$ is the Bose-Mesner algebra of a scheme, and so is $\mathcal{N}_{W^T}$.

# To do

- The last equations are disturbing to look at. Try to write them more neatly.

# Chapter 10

# Magic unitary matrices

A magic unitary matrix is a $n \times n$ matrix whose entries are $d \times d$ projections such that each row and column sums to $I_d$. The projections in a row or column are orthogonal.

**Example.** Any permutation matrix.

**Lemma 10.1.** *If $P$ is a magic unitary matrix, it is unitary.*

*Proof.* Exercise. $\square$

We will also be concerned with the case where we use idempotents in place of projections. Graphs $X$ and $Y$ with $V(X) = V(Y) = v$ are *quantum isomorphic* if there is a magic unitary $P$ such that

$$(A(X) \otimes I)P = P(A(Y) \otimes I).$$

Since $P$ is invertible, $A(X) \otimes I$ and $A(Y) \otimes I$ are similar and so $X$ and $Y$ are cospectral.

**Lemma 10.2.** *If $P$ is a magic unitary with index $d$, then $P$ commutes with $J \otimes I_d$.*

*Proof.* Exercise. $\square$

**Corollary 10.3.** *If $X$ and $Y$ are quantum isomorphic, they are cospectral with cospectral complements.*

**Lemma 10.4.** *If $M \otimes I$ anf $N \otimes I$ commute with $P$ then so does $M \circ N$.*

*Proof.* The $(i, j)$-block of $(M \otimes I)P$ is

$$\sum_r M_{i,r} P_{r,j}$$

and by hypothesis this is equal to the $(i,j)$-block of $P(M \otimes I)$:

$$\sum_s M_{s,j} P_{i,s}.$$

We have corresponding expressions for $N \otimes I$ and

$$\sum_r M_{i,r} P_{r,j} \sum_s N_{i,s} P_{s,j} = \sum_r (M)_{i,r} M_{i,r}) P_{r,j}.$$

Similarly

$$\sum_r M_{r,j} P_{i,r} \sum_r N_{r,j} P_{i,r} = \sum M_{r,j} N_{r,j} P_{j,r}.$$

$\square$

A *coherent algebra* is an algebra of matrices that is closed under transpose, contains $J$ and is Schur-closed.

**Example.** The commutant of a set of permutation matrices.

A coherent algebra has a unique basis of 01-matrices. If $I$ is an element of this basis, the algebra is *homogeneous*. Any matrix generates a coherent algebra. If $X$ and $Y$ are quantum isomorphic, the coherent algebras they generate are isomorphic.

# Chapter 11

# Type-II matrices and magic unitaries

The $v^2 \times v^2$ matrix $\mathcal{F}$ with $(i,j)$-entries equal to $\mathcal{F}_{i,j}$ is the *matrix of idempotents*. Since $\mathcal{F}_{i,j}^T = \mathcal{F}_{j,i}$ we see that $\mathcal{F}$ is symmetric. If $\tau$ is partial transpose, then

$$\mathcal{F}^\tau = \frac{1}{v^2}\mathcal{F}^{(-)}.$$

Let $S$ act on $V \otimes V$ by $S(u \otimes v) = v \otimes u$. Then $\mathcal{F}_{W^T} = S\mathcal{F}_W S$. (Exercise.)

**Theorem 11.1.** *If $W$ is type-II, then $\mathcal{F}_W$ is type-II. If in addition, $W$ is flat, then $\mathcal{F}$ is flat and is a magic unitary.*

*Proof.* Exercise. □

For matrices $A$ and $B$ let $[A,B] := AB - BA$. Note that $[A,B] = 0$ if and only if $A$ and $B$ commute.

**Theorem 11.2.** *Assume $W$ is type-II with matrix of idempotents $\mathcal{F}$. Then the set of matrices $M$ such that $[I \otimes M, \mathcal{F}_W] = 0$ is $\mathcal{N}_W$ and the set of matrices $N$ such that $[N \otimes I, \mathcal{F}_W] = 0$ is $\mathcal{N}_{W^T}$.*

*Proof.* The $(i,j)$-block of $(I \otimes M)\mathcal{F}$ is $M\mathcal{F}_{i,j}$; the $(i,j)$-block of $\mathcal{F}(I \otimes M)$ is $\mathcal{F}_{i,j}M$. So $I \otimes M$ and $\mathcal{F}$ commute if and only if $M$ commutes with all idempotents $\mathcal{F}_{i,j}$. So $[(I \otimes M), \mathcal{F}] = 0$ if and only if $M \in \mathcal{N}_W$. □

We have $\mathcal{F}_{i,j} = W_{i/j}(W_{j/i})^T$ and $\mathcal{F} = (\mathcal{F}_{i,j})$. We denote by $\delta_i(M)$ the diagonal matrix fromed from the $i$-th column of $M$. Then $W_{i/j} = \delta_i(W)We_j^{(-)}$ and

$$\mathcal{F}_{i,j} = \delta_i(W)We_j^{(-)}(We_j)^T\delta_i(W)^{-1}.$$

**Remark.** $(\mathcal{F}_{i,j})_{r,s} = (\mathcal{F}_{r,s})_{i,j}$

## Remarks and Questions

1. What is the commutant of $\mathcal{F}_W$? It contains $\mathcal{N}_W \otimes \mathcal{N}_{W^T}$. Could it be equal to this?
2. Could we have $\mathcal{F}_W = S(W \otimes W^T)$?
3. Characterize the magic unitaries equal to $\mathcal{F}_W$ for some flat type-II $W$.
4. Is there a nice relation between $\mathcal{F}_W$ and $\mathcal{F}_{W^T}$?
5. How does $\Theta$ play with $\mathcal{F}_W$? Can we prove Nomura's identity,
$$(\Theta_{W^T}(\Theta_W(M))) = vM^T\,?$$
6. If $W$ is a real Hadamard, then $\mathcal{F}_W$ is a Hadamard matrix of Bush type.
7. If $\mathcal{F}_{i,j} \in \mathcal{N}_W$ then $\Theta(\mathcal{F}_{i,j})$ is a permutation matrix, so they are dual permutations.

# Chapter 12

# Spin models

## 12.1 Knots and braids

We will not give a formal definition of a *braid*, but define them by illustations. Figure 12.1 shows a braid on eight strands.

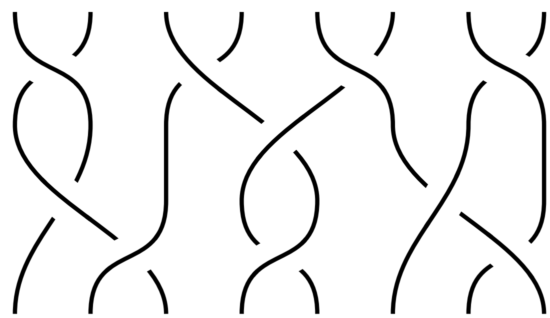


Figure 12.1: A braid on eight strands

There is only one braid on one strand but infinitely many on two (or more) strands (see Figure 12.2).

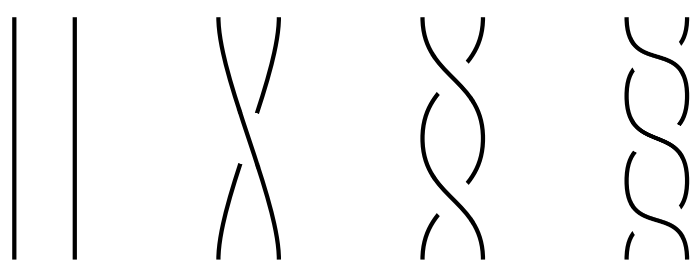


Figure 12.2: Four distinct braids on two strands

The set of braids on $n$ strands forms an infinite group, $B_n$. The identity of this group is obvious, and it is not too hard to see what the multiplication looks like. To see that every element has an inverse, we note that the braid group, $B_n$ is generated by $n-1$ elements, $\sigma_1, \ldots, \sigma_{n-1}$, shown in Figure 12.3. Their inverses are depeicted in Figure 12.4

In Figure 12.2, we have the identity of $B_2$ to the far left, then $\sigma_1$, $\sigma_1^2$ and $\sigma_1^3$ respectively. Note that $\sigma_1$ has infinite degree.

The generators for a general braid group is depicted in Figure 12.3.

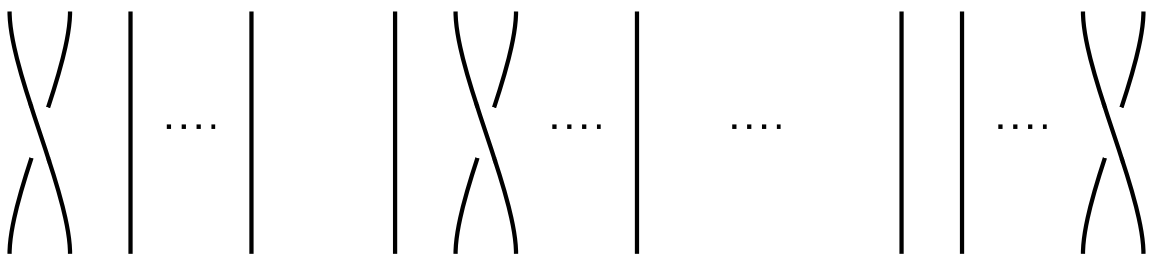


Figure 12.3: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$

Now it is not too hard to see what the inverses of these elements are (see Figure 12.4).

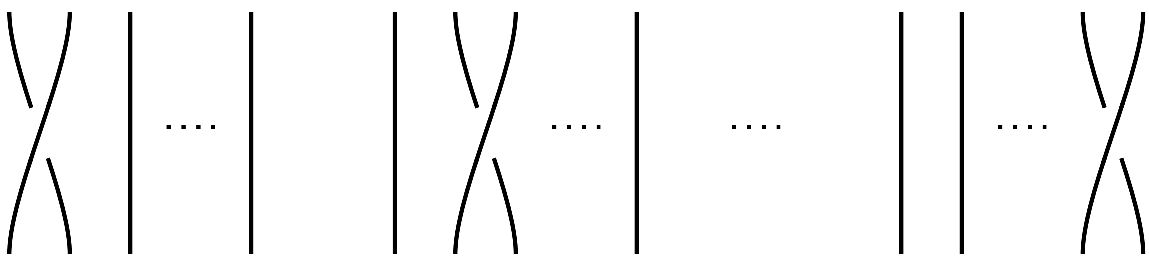


Figure 12.4: $\sigma_1^{-1}, \sigma_2^{-1} \ldots, \sigma_n^{-1}$

Given an arbitrary braid, $\beta$, the corresponding *link* is shown in Figure 12.5

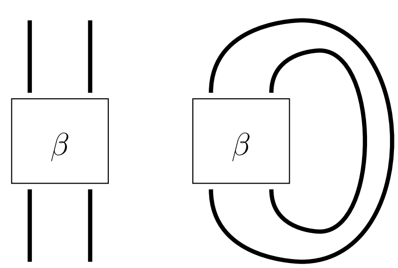



Figure 12.5: Link of a braid

**Question.** When do two braids give the same link?

Let $\beta$ be an arbitrary braid. The following two operations preserve its link (here, $\gamma$ is any arbitrary braid):

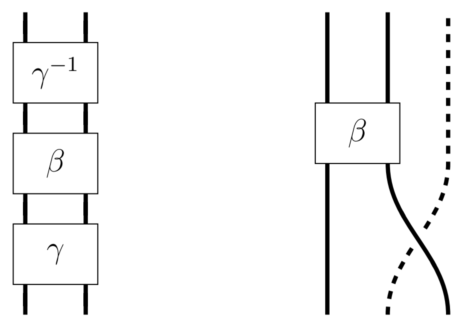



Figure 12.6: Link-preserving operations

**Theorem 12.1** (Markov)**.** *Two braids have the same closure if and only if they are related by a sequence of these operation.*

If we have a finite dimensional matrix representation, $\Phi$, of $B_n$ then the map

$$\beta \mapsto \mathrm{tr}(\Phi(\beta))$$

will give a link invariant if it behaves nicely under the second Markov move.

## 12.2 Braid relations

There are two sets of relations on the generators of $B_n$:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$.

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

We call this second identity *the braid relation.*

If $aba = bab$ then

$$(aba)^2 = abaaba = ababab = (ab)^3.$$

If we have $\varphi(a) = (12)$ and $\varphi(b) = 23$, then

$$(12)(23)(12) = (23)(12)(23).$$

## 12.3 A generalization of the Nomura algebra

Let $A$ and $B$ be two $n \times n$ matrices. Then

$$\mathcal{N}_{A,B} = \{M : Ae_i \circ (Be_j)^{(-)} \text{ is an eigenvector for all } i, j\}$$

(so if $A^{(-)}$ exists, then $\mathcal{N}_A = \mathcal{N}_{A,A^{(-)}}$). If $M \in \mathcal{N}_{A,B}$ then $\Theta_{A,B}(M)$ is the matrix of eigenvalues of $M$. We have

$$\Theta_{A,B}(MN) = \Theta_{A,B}(M) \circ \Theta_{A,B}(N).$$

Again, $\mathcal{N}_{A,B}$ is a matrix algebra and $\Theta_{A,B}(\mathcal{N}_{A,B})$ is Schur-closed. As before, $\Theta_{A,B}(I) = J$.

**Lemma 12.2.** *If $A^{-1}$ and $B^{(-)}$ exist then $\mathcal{N}_{A,B}$ is a commutative matrix algebra.*

*Proof.* Exercise. □

**Lemma 12.3.** *$W$ is type-II if and only if $J \in \mathcal{N}_{W,W^{(-)}}$.*

## 12.4 Braid relation

Let $A, B, C$ by $n \times n$ matrices and define linear maps on $\mathrm{Mat}_{n\times n}(\mathbb{C})$ by

$$X_A(M) = AM, \quad \Delta_B(M) = B \circ M, \quad Y_C(M) = MC^T.$$

Using the isomorphism $\mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}^n \otimes \mathbb{C}^n$ we see that

$$\begin{aligned} &X_A = I \otimes A \\ &Y_C = C \otimes I \\ &\Delta_B \text{ - diagonal operator.} \end{aligned}$$

If $A \in \mathcal{N}_{A,B}$, $A^{-1}$ exists, $\Theta_{A,B}(A) = B$ and

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B,$$

then the map

$$\sigma_1 \mapsto X_A, \quad \sigma_2 \mapsto \Delta_B$$

is a braid representation.

**Lemma 12.4.** *Suppose $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Then $R \in \mathcal{N}_{A,B}$ and $S = \Theta_{A,B}(R)$ if and only if*

$$X_R \Delta_B X_A = \Delta_B X_A \Delta_S.$$

*Proof.* Apply each side to $e_i \otimes e_j$ (note that $\Delta_M(e_i \otimes e_j) = M_{i,j}(e_i \otimes e_j)$). Then

$$\begin{aligned} X_A e_i e_j^T &= A e_i e_j^T \\ \Delta_B X_A e_i e_j^T &= B \circ (A e_i e_j^T) \\ &= (B e_j \circ A e_i) e_j^T \\ X_R \Delta_B X_A e_i e_j^T &= R(B e_j \circ A e_i) e_j^T. \end{aligned} \tag{12.1}$$

On the other hand,

$$\begin{aligned} \Delta_S e_i e_j^T &= S_{i,j} e_i e_j^T \\ X_A \Delta_S e_i e_j^T &= S_{i,j} A e_i e_j^T \\ \Delta_B X_A \Delta_S e_i e_j^T &= S_{i,j}(B e_j \circ A e_i) e_j^T. \end{aligned} \tag{12.2}$$

Comparing (12.1) and (12.2), yields the result. □

Relative to the trace inner product,

$$(X_A)^T = X_{A^T}, \quad (\Delta_B)^T = \Delta_B$$

**Theorem 12.5** (Exchange relation)**.** *Let $A, B, C, Q, R, S$ be $v \times v$ matrices. Then*

$$X_A \Delta_B X_C = \Delta_Q X_R \Delta_S$$

*if and only if*

$$X_A \Delta_C X_B = \Delta_R X_Q \Delta_{S^T}.$$

*Proof.* Apply each expression to $e_i e_j^T$. □

**Theorem 12.6.** *Let $A$ be a $v \times v$ type-II matrix. THen $\Theta_A$ is a bijection from $\mathcal{N}_A$ to $\mathcal{N}_{A^T}$ and if $R \in \mathcal{N}_A$ then $\Theta_{A^T}(Q_A(R)) = vR^T$.*

*Proof.* Suppose $R \in \mathcal{N}_A$ and $\Theta_A(R) = S$. Then

$$X_R \Delta_{A^{(-)}} X_A = \Delta_{A^{(-)}} X_A \Delta_S$$

and the transpose of this is

$$X_{A^T} \Delta_{A^{(-)}} X_{R^T} = \Delta_S X_{A^T} \Delta_{A^{(-)}}.$$

If we apply the exchange identity to this, we get

$$X_{A^T} \Delta_{R^T} X_{A^{(-)}} = \Delta_{A^T} X_S \Delta_{A^{(-)T}}$$

Multiply this on the left by $\Delta_{A^{(-)T}}$ and on the right by $X_{A^{(-)-1}}$ to get

$$X_S \Delta_{A^{(-)T}} X_{A^{(-)-1}} = \Delta_{A^{(-)T}} A_{A^T} \Delta_{R^T}.$$

As $A^{(-)-1} = \frac{1}{v} A^T$, this yields

$$X_S \Delta_{A^{(-)T}} X_{A^T} = \Delta_{A^{(-)T}} X_{A^T} \Delta_{vR^T}.$$

Hence $S \in \mathcal{N}_{A^{(-)T} A^T} = \mathcal{N}_{A^T}$ and $\Theta_{A^T}(S) = vR^T$. □

## 12.5 Duality

We say that $W$ is a *spin model* if $W \in \mathcal{N}_W$. If $W \in \mathcal{N}_W$ then $W^T \in \mathcal{N}_W$ and $\Theta_W(W^T) = W^{(-)T}$. Hence

$$X_{W^T}\Delta_{W^{(-)}}X_W = \Delta_{W^{(-)}}X_W\Delta_{W^{(-)T}}.$$

Denote either side of this by $\Lambda$

**Theorem 12.7.** *If $R \in \mathcal{N}_W$ and $S = \Theta_W(R)$ then*

$$\Lambda^{-1}X_R\Lambda = \Delta_S, \quad \textit{and} \quad \Lambda^{-1}\Delta_{S^T}\Lambda = X_R.$$

**Remark.** If $X_A\Delta_B X_A = \Delta_B X_A \Delta_B$ then

$$(X_A\Delta_B)^3 = X_A\Delta_B X_A\Delta_B X_A\Delta_B = (X_A\Delta_B X_A)^2$$

from which it follows that $(X_A\Delta_B X_A)^2$ commutes with $X_A\Delta_B X_A$ and $X_A\Delta_B$, and hence it commutes with $X_A$ and $\Delta_B$.

*Proof.* Since $R$ and $W^T$ commute,

$$\begin{aligned}\Lambda^{-1}X_R\Lambda &= X_{W^{-1}}\Delta_W X_{W^{-T}}X_R X_{W^T}\Delta_{W^{(-)}}X_W\\ &= X_{W^{-1}}\Delta_W X_R\Delta_{W^{(-)}}X_W\\ &= \Delta_S.\end{aligned}$$

Similarly, $\Lambda^{-1}\Delta_{S^T}\Lambda = X_R$. $\square$

If $\Lambda^{-1}X_R\Lambda = \Delta_S$ and $\Lambda^{-1}\Delta_{S^T}\Lambda = X_S$ then

$$\Lambda^{-2}\Delta_{S^T}\Lambda^2 = \Delta_S.$$

Similarly, $\Lambda^{-2}X_R\Lambda^2 = X_{R^T}$. In consequence, $\Lambda^4$ commutes with $X_R$ and $\Delta_S$.

If $\Lambda^{-1}X_R\Lambda = \Delta_S$ then

$$S = \Delta_S(J) = \Lambda^{-1}X_R\Lambda(J)$$

and here

$$\Lambda(J) = \Delta_{W^{(-)}}X_W\Delta_{W^{(-)T}}(J).$$

Since $W \in \mathcal{N}_W$, its diagonal is constant and so

$$\begin{aligned}\Delta_{W^{(-)}}X_W\Delta_{W^{(-)T}}(J) &= \Delta_{W^{(-)}}X_W(W^{(-)T})\\ &= \Delta_{W^{(-)}}(WW^{(-)T})\\ &= v\Delta_{W^{(-)}}(I)\\ &= v\delta I\end{aligned}$$

for some $\delta$. Therefore $S = \Lambda^{-1}(R)$.

# To do

- Try to illustrate links.
- Define links (maybe pictures are enough).
- There are two sections with the name Braid relation.

# Chapter 13

# Galois theory

## 13.1 Bose-Mesner automorphisms

A linear map $\psi : \mathbb{C}[\mathcal{A}] \to \mathbb{C}[\mathcal{A}]$ is a *Bose-Mesner automorphism* if

(a) $(MN)^\psi = M^\psi N^\psi$

(b) $(M \circ N)^\psi = M^\psi \circ N^\psi$.

It follows that

(c) $\psi$ is invertible.

Since $J \circ J = J$, we have

$$J^\psi \circ J^\psi = J^\psi$$

and so $J^\psi$ is a 01-matrix. As $J^2 = vJ$,

$$vJ^\psi = (J^\psi)^2$$

and thus $J^\psi = J$.

Next, $(A_r)^\psi$ is a 01-matrix and as $J = \sum A_r$,

$$J^\psi = \sum_r A_r^\psi$$

and therefore $\psi$ must permute the minimal Schur idempotents. This means that $\psi$ maps a basis to basis, whence it is invertible (so (c) holds).

Claim: $(A_i A_j) \circ I \neq 0$ if and only if $A_j = A_i^*$. (Exercise)

It follows that

(d) $(M^*)^\psi = (M^\psi)^*$ for all $M \in \mathbb{C}[\mathcal{A}]$.

**Example.** The transpose map.

Let $\mathcal{A}$ be an association scheme, let $L$ be the splitting field generated by the eigenvalues and let $K$ be the field generated by the Krein parameters. We define $\Gamma := \mathrm{Gal}(L/\mathbb{Q})$ and $H := \mathrm{Gal}(L/K)$. Then $H \leq \Gamma$.

If $\sigma \in \Gamma$ and $M, N \in L[\mathcal{A}]$ then $(MN)^\sigma = M\sigma N\sigma$ and $(M \circ N)^\sigma = M^\sigma \circ N^\sigma$ but $\sigma$ is not a Bose-Mesner automorphism - it is not linear.

If $\tau \in \Gamma$, define a map $\hat{\tau}$ on $L[\mathcal{A}]$ as follows: if

$$M = \sum \mu_i E_i$$

then

$$M^{\hat{\tau}} = \sum \mu_i E_i^\tau.$$

This is linear over $L$.

**Example.** If $\tau$ is complex conjugation, then $\hat{\tau}$ is the transpose.

**Theorem 13.1.** *Let $\mathcal{A}$ be an association scheme with splitting field $L$. If $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ then $\hat{\tau}$ is an algebra automorphism if and only if $\tau$ fixes $K$ (i.e. $\tau \in H$).*

*Proof.* There are a number of steps.

(1) If $M \in L[\mathcal{A}]$ and $M = \sum \mu_j E_j$ then since $E_j^* = E_j$ we have

$$(M^*)^\tau = \sum_j \mu_j^* E_j^\tau = (M^{\hat{\tau}})^*$$

(2) As noted above,

$$(MN)^\sigma = M^\sigma N^\sigma,$$
$$(M \circ N)^\sigma = M^\sigma \circ N^\sigma$$

Since $(E_i)^\tau (E_j)^\tau = (E_i E_j)^\tau$, we have

$$(MN)^{\hat{\tau}} = M^{\hat{\tau}} N^{\hat{\tau}}.$$

(3) We show that $\hat{\tau}$ commutes with Schur multiplication if and only if $\tau \in H$. On one hand,

$$\begin{aligned}(E_i \circ E_j)^{\hat{\tau}} &= \frac{1}{v} \sum_r q_{i,j}(r) E_r^{\hat{\tau}} \\ &= \frac{1}{v} \sum_r q_{i,j}(r) E_r^\tau\end{aligned}$$

while on the other hand,

$$\begin{aligned} E_i^{\hat{\tau}} \circ E_j^{\hat{\tau}} &= E_i^{\tau} \circ E_j^{\tau} \\ &= (E_i \circ E_j)^{\tau} \\ &= \frac{1}{v} \sum_r q_{i,j}(r)^{\tau} E_r^{\tau}. \end{aligned}$$

So $\hat{\tau}$ commutes with Shur multiplication if and only if $q_{i,j}(r)^{\tau} = q_{i,j}$ for all $i, j, r$.

□

**Lemma 13.2.** $H \le Z(\Gamma)$.

*Proof.* Assume $\sigma \in \Gamma$ and $\tau \in H$. Then

$$E_j^{\sigma\hat{\tau}} = E_j^{\tau\sigma} = \frac{1}{v} \sum_i q_j(i)^{\sigma\tau} A_i$$

and similarly,

$$E_j^{\hat{\tau}\sigma} = E_j^{\tau\sigma} = \frac{1}{v} \sum_i q_j(i)^{\tau\sigma} A_i.$$

Next,

$$\begin{aligned} \left( \sum_i q_j(i) A_i \right)^{\sigma\hat{\tau}} &= \sum_i q_j(i)^{\sigma} A_i^{\hat{\tau}} \\ &= \left( \sum_i q_j(i) A_i \right)^{\hat{\tau}\sigma}. \end{aligned}$$

Here the first term is $E_j^{\sigma\hat{\tau}}$ and the second is $E_j^{\hat{\tau}\sigma}$ hence $q_j(i)^{\sigma\tau} = q_j(i)^{\tau\sigma}$ for all $i, j$. □

**Theorem 13.3.** *Let $\mathcal{A}$ be an association scheme with splitting field $L$ and let $H =$ $\mathrm{Gal}(L/K)$. Let $F$ be a subfield of $L$ that contains $K$ Then the matrices in $L[\mathcal{A}]$ with eigenvalues and entries in $F$ are the subscheme fixed by elements of* $\mathrm{Gal}(L/F)$.

*Proof.* Let $\mathcal{F}$ be the set of matrices in $L[\mathcal{A}]$ with eigenvalues and entries in $F$. If $M \in L[\mathcal{A}]$ and $M = \sum \mu_i E_i$ then

$$M^{\hat{\tau}\tau^{-1}} = \sum_i \mu_i^{\tau^{-1}} E_i.$$

It follows that $\hat{\tau}$ fixes $M$ if and only if $\tau$ fixes the eigenvalues of $M$. □

# Chapter 14

# Distance regular graphs

A graph with diameter $d$ is *distance regular* if its distance matrices form an association scheme.

**Example.**

- Strongly regular graphs.
- Cycles.
- Johnson graph, $J(v, k)$.
- Hamming schemes, $H(n, q)$.
- Bilinear forms.
- Grassmann

**Lemma 14.1.** *If $\mathcal{A}$ is an association scheme with $d$ classes and $Y$ is a graph in the scheme, then* $\operatorname{diam}(Y) \leq d$. *If equality holds, then $Y$ is distance regular.*

A scheme with $d$ classes is *metric (P-polynomial)* if some graph has diameter $d$.

## 14.1 Three-term recurrence

Assume $\mathcal{A}$ is metric relative to $X_1$. Then there are polynomials, $p_0, \dots, p_d$ such that

$$A_r = p_r(A_1)$$

and $\deg(p_r) = r$. The product, $A_1 A_r$ is a linear combination of $A_{r-1}, A_r$ and $A_{r+1}$. Define scalars, $a_i, b_i, c_i$ such that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1},$$

or equivalently,

$$A_{i+1} = \frac{1}{c_{i+1}} \left( (A_1 - a_i I) A_i - b_{i-1} A_{i-1} \right).$$

## 14.2 Distance partitions

There is a module isomorphism from $\mathbb{C}[\mathcal{A}] \to \mathbb{C}^{d+1}$ given by

$$M \mapsto M e_u \quad u \in V(\mathcal{A}).$$

(Think of $u$ as the first vertex, so $Me_u$ is the first column of $M$.)

The vectors $A_i e_u$ for $i = 0, \dots, d$ are linearly independent. Set

$$N = [A_0 e_u, \dots, A_d e_u].$$

(Delsarte called this the "outer distribution matrix".) The column space of $N$ is $A_1$-invariant, hence $\mathcal{A}$-invariant. But $N$ is a characteristic matrix of the *distance partition* relative to $u$. Thus this partition is equitable. If $\partial_u$ denotes this partition, then

(a) $\partial_u$ is equitable for each $u \in V$.

(b) The quotient $X/\partial_u$ is independent of $u$. ($V(X/\partial_u) = \{0, 1, \dots, d\}$.)

Claim: If (a) and (b) hold for some graph, then $X$ is distance regular. Since $\partial_u$ is a distance partition, $X/\partial_u$ is a weighted path.

Since $\partial_u$ is equitable, there are $(d+1) \times (d+1)$ matrices

$$B_0, \dots, B_d$$

such that

$$A_i N = N B_i.$$

This is a homomorphism from $\mathbb{C}[\mathcal{A}]$ to $\mathrm{Mat}_{(d+1)\times(d+1)}(\mathbb{C})$; in fact an isomorphism.

We calculate

$$\begin{aligned} A_1 A_i e_0 &= (b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}) e_0 \\ &= b_{i-1} A_{i-1} e_0 + a_i A_i e_0 + c_{i+1} A_{i+1} e_0 \end{aligned}$$

which implies

$$B_1 = \begin{pmatrix} 0 & b_1 & 0 & \cdots \\ 1 & a_1 & b_2 & \cdots \\ 0 & c_2 & a_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} = A(X/\partial_u).$$

**Lemma 14.2.** *Let $X$ be distance regular with diameter $d$. Then*

*(a)* $b_0 \geq b_1 \geq \cdots \geq b_{d-1}$

*(b)* $c_1 \leq c_2 \leq \cdots \leq c_d$

*(c) The sequence $v_0, \ldots, v_d$ is unimodal.*

*Proof.* Exercise. □

## 14.3 Completely regular subsets

Suppose $\mathcal{A}$ is a scheme and $C \subseteq V(\mathcal{A})$ with characteristic vector $x$. Let $C_i$ denote the set of vertices of $\mathcal{A}$ that are $i$-related to a vertex in $C$ (so $C_0 = C$). We say that $C$ is a *completely regular subset* of $V(\mathcal{A})$ if the partition $\{C_0, \ldots, C_d\}$ is $\mathcal{A}$-equitable.

**Example.**

1. $C$ is a vertex in a distance regular graph.
2. $X$ is distance regular and $C$ is the set of vertices at distance $d$ from $u$.
3. $C$ is a coclique in a strongly regular graph $X$ and $|C|$ meets the ration bound, then $C$ is completely regular.

# To do

- We use $\partial$ for the distance partition. I think I used $\delta$ for the same later. Find it and change it.
- We haven't defined $i$-related.
- In the definition of a completely regular subset, is $\{C_0, \ldots, C_d\}$ necessarily a partition? Should $C_i$ be the set of vertices at distance $i$ from $C$?

# Chapter 15

# Imprimitivity

An association scheme is *primitive* if $X_1, \dots, X_d$ are connected, otherwise it is imprimitive. (Our main interest will be metric schemes.)

**Example.** If $\mathcal{A}$ is metric and $X$ is bipartite, then $\mathcal{A}$ is imprimitive.

Suppose $\mathcal{A}$ is a scheme and $X_r$ is not connected ($r \neq 0$).

**Lemma 15.1.** *A connected graph $X$ is regular if and only if there is a polynomial, $p$ such that $p(A) = J$.*

Given this, there is a polynomial, $q$ such that $q(A_r)$ is block diagonal with each block equal to $J$. Since $q(A_r)$ and $J$ commute, it follows that all components of $X_r$ have the same size. An immediate consequence is that the partition by $X_r$-components is $\mathcal{A}$-equitable. Let $K = q(A_r)$ be the block-diagonal matrix of $J$'s.

Claim: The matrices $A_i$ such that $A_i \circ K \neq 0$ sum to $K$, and generate a Schur-closed subalgebra of $\mathbb{C}[\mathcal{A}]$. (Exercise.)

The components of $X_r$ are completely regular subsets.

## 15.1 Imprimitivity in distance regular graphs

**Theorem 15.2.** *If $X$ is distance regular, imprimitive and not a cycle, then either $X$ is bipartite or antipodal.*

**Remark.** In the second case $X_d$ is not connected, it is a disjoint union of complete graphs.

*Proof.* If $X_2$ is not connected, then either $X$ is bipartite or complete multipartite (exercise). Now assume $X$ is imprimitive, $X_1$ and $X_2$ are connected and let $r$ be the smallest number such that $X_r$ is not connected. We show that if $r = d$ then $X$ is a cycle. We say $X_1$ has $(i,j,k)$ triangle if $p_{i,j}(k) > 0$.

We consider three cases:

(a) $a_r = 0$. Suppose not. Then there is a $(1,r,r)$ triangle and so $p_{r,r}(1) \neq 0$. Let $u$ be a vertex and let $C$ be the component of $X_r$ containing $u$. We can assume $u$ is adjacent to $v$ in $X_1$. It follows that each neighbour of $u$ in $X_1$ lies in $C$. Since $X_1$ is connected, this is impossible.

(b) $b_{r-1} = 1$. If $b_{r-1} > 1$ then there is a $(2,r,r)$ triangle in $X_1$. Since $X_2$ is connected this is impossible.

(c) $c_{r+1} = 1$. If $c_{r+1} > 1$, there is a $(2,r,r)$ triangle and again this is impossible.

Now,

$$1 = c_{r+1} \geq c_r, \quad 1 = b_{r-1} \geq b_r$$

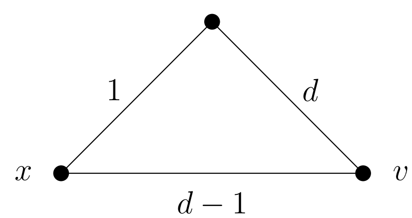
and so

$$b_r = c_r = 1, \quad a_r = 0$$

and therefore the valency of $X$ is 2.

Suppose $X_d$ is not connected, and $d \geq 3$. Then each vertex is adjacent to at most one vertex in any $X_d$-component that does not contain it. If $u$ is adjacent to $v$ in a second component, then it is at distance $d-1$ from the remaining vertices in that component.

Assume $u \sim v$ and $x$ is in the $X_d$-component of $X_1$ that contains $u$. Then we have a $(d, 1, d-1)$ triangle. Hence there must be a triangle




and so if two components are joined by an edge, they are joined by a matching. $\square$

We will refer to the components of $X_d$, $d \geq 3$ as *fibres*.

**Lemma 15.3.** *Each fibre is completely regular - the distance partition relative to a fibre is equitable.*

If $C \subseteq V(X)$ for some $X$ then the *covering radius* of $C$ is

$$\min\{\operatorname{dist}(x, C) : x \in V(X)\}.$$

The *packing radius* is the maximum value of $r$ such that balls of radius about distinct vertices in $C$ are disjoint. A code is *perfect* if its packing radius is equal to its covering radius.

Claim. A fibre in an antipodal distance regular graph is a perfect code (Exercise).

# To do

- Should the definitions of covering / packing radius and perfect code be moved to chapter 19 - Codes? (We haven't defined codes yet.)

# Chapter 16

# Orthogonal polynomials

A distance regular graph can be viewed as a combinatorial realization of a family of orthogonal polynomials. We use inner products on the vector space of real polynomials.

**Example.**

1. $\langle p, q\rangle = \displaystyle\int_0^\infty p(t)g(t)e^{-t}\,dt$
2. $\langle p, q\rangle = \displaystyle\sum_i w_i p(\theta_i)q(\theta_i)$, where $w_i \geq 0$, $\sum w_i = 1$
3. $\langle p, q\rangle = \mathrm{tr}(p(A)q(A))$, for a symmetric matrix, $A$.
4. $\langle p, q\rangle = \mathrm{tr}(p(A)q(A)M)$, for a symmetric matrix, $A$ and a positive semidefinite matrix, $M$.

The inner product must satisfy

(a) $\langle tp, q\rangle = \langle p, tq\rangle$

(b) If $f \geq 0$ then $\langle 1, f\rangle \geq 0$ and if equality holds, $f = 0$.

Note that (a) says that multiplication by $t$ is self-adjoint. Given an inner product we can apply Gram-Schmidt to the polynomials $1, t, t^2, \dots$ and obtain a sequence of orthogonal polynomials $p_0, p_1, \dots$. We can normalize the latter sequence; three typical ways are

1. $\langle p_n, p_n\rangle = 1$
2. $p_n(a) = 1$ for some specified $a$.
3. $p_n$ is monic for all $n$.

## 16.1 Real simple

Suppose $(p_r)_{r\geq 0}$ is a sequence of orthogonal polynomials. Then for each $n$ the zeros of $p_n$ are real and simple.

**Lemma 16.1.** *Let $(p_r)_{r\geq 0}$ be a sequence of orthogonal polynomials. If $f(t) \mid p_n(t)$ and $f(t) \geq 0$ then $f$ is constant.*

*Proof.* Note that $\langle p_n, t^m\rangle = 0$ if $m < n$ by construction. Suppose $p_n = fq$. If $\deg(q) < n$, then

$$0 = \langle p_n, q\rangle = \langle fq, q\rangle = \langle 1, fq^2\rangle.$$

But $fq^2 \geq 0$, hence $fq^2 = 0$. □

## 16.2 Three-term recurrence

Assume $(p_n)_{n\geq 0}$ is a sequence of monic polynomials. Then

$$tp_n = a_0 p_{n+1} + a_1 p_n + \cdots + a_{n+1} p_0.$$

Since $p_n$ and $p_{n+1}$ are monic, $a_0 = 1$. Next,

$$\langle tp_n, p_m\rangle = \langle p_n, tp_m\rangle$$

and hence $\langle tp_n, p_m\rangle = 0$ if $m \leq n-2$. So one expression for $tp_n$ can be written as

$$tp_n = p_{n+1} + a_n p_n + b_n p_{n-1}.$$

Equivalently,

$$p_{n+1} = (t - a_n)p_n - b_n p_{n-1}.$$

The matrix representing multiplication by $t$ is

$$\begin{pmatrix} a_0 & 1 & 0 & 0 & \cdots \\ b_1 & a_1 & 1 & 0 & \cdots \\ 0 & b_2 & a_2 & 1 & \cdots \\ 0 & 0 & b_3 & a_2 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

We can express $a_n$ and $b_n$ in terms of $\langle p_n, p_n\rangle$ and $\langle tp_n, p_n\rangle$. Assume $(p_n)_{n\geq 0}$ satisfies

$$p_{n+1} = (t - a_n)p_n - b_n p_{n-1}.$$

Taking inner products with $p_n$ yields

$$\langle tp_n, p_n\rangle - a_n\langle p_n, p_n\rangle = 0$$

whence $a_n = \langle tp_n, p_n\rangle / \langle p_n, p_n\rangle$. Next, take the inner product with $p_{n-1}$,

$$0 = \langle tp_n, p_{n-1}\rangle - b_n \langle p_{n-1}, p_{n-1}.$$

Here

$$\langle tp_n, p_{n-1}\rangle = \langle p_n, tp_{n-1}\rangle = \langle p_n, p_n\rangle,$$

and so

$$b_n = \frac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle}.$$

## 16.3 Tridiagonal matrices

If $T$ is given and $q_r$ is the characteristic polynomial of the leading $r \times r$ submatrix of $T$ then

$$q_{r+1} = (t - a_r)q_r - b_r q_{r-1}$$

and hence $q_r = p_r$.

**Lemma 16.2.** *Suppose $D$ is diagonal with positive diagonal entries and $S = D^{-1}TD$. Then*

$$S_{i+1,i}\, S_{i,i+1} = T_{i+1,i}\, T_{i,i+1} \quad \text{and} \quad S_{i,i} = T_{i,i}.$$

This implies that tridiagonal matrices give sequences of polynomials satisfying a 3-term recurrence. If $X$ is distance regular, $u \in V(X)$ and $\delta_u$ is the distance partition relative to $u$, then $X/\delta_u$ is a weighted path and $A(X/\delta_u)$ is tridiagonal.

It is always possible to choose $D$ so that $D^{-1}TD$ is symmetric. In fact, if $B_{0,0} = 1$, and

$$B_{r,r} = \prod_{j=1}^{r} b_j$$

then

$$B^{-1/2}TB^{1/2} = \begin{pmatrix} a_0 & \sqrt{b_1} & 0 & 0 & \cdots \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & 0 & \cdots \\ 0 & \sqrt{b_2} & a_2 & \sqrt{b_3} & \cdots \\ 0 & 0 & \sqrt{b_3} & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

One consequence of this is that the zeros of $p_{n-1}$ interlace the zeros of $p_n$ and that $\gcd(p_{n-1}, p_n) = 1$.

## 16.4 Eigenvectors

We have the equation

$$T\begin{pmatrix} p_0 \\ \vdots \\ p_{n-1} \end{pmatrix} = t\begin{pmatrix} p_0 \\ \vdots \\ p_{d-1} \end{pmatrix} - p_d(t)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and thus if $p_d(\theta) = 0$, then

$$\begin{pmatrix} p_0(\theta) \\ \vdots \\ p_{n-1}(\theta) \end{pmatrix}$$

is an eigenvector for $T$ with eigenvalue $\theta$. Let $\hat{T}$ be the symmetric form of $T$; assume

$$\hat{T} = B^{-1/2}TB^{1/2};$$

so if $Tz = \lambda z$ then

$$\hat{T}B^{-1/2}z = B^{-1/2}z.$$

Set

$$\tilde{q} := \begin{pmatrix} q_0(t) \\ \vdots \\ q_{n-1}(t) \end{pmatrix} = B^{-1/2}\begin{pmatrix} p_0(t) \\ \vdots \\ p_{n-1}(t) \end{pmatrix},$$

If $\theta, \tau$ are distinct eigenvalues for $T$ then the vectors $\tilde{q}(\theta)$ and $\tilde{q}(\tau)$ are orthogonal. It follows that

$$\sum_r \frac{p_r(\theta)p_r(\tau)}{B_{r,r}} = 0.$$

In other words, the matrix, $\begin{pmatrix}\tilde{q}(\theta_1) & \cdots & \tilde{q}(\theta_n)\end{pmatrix}$ is orthogonal, hence its rows are orthogonal, i.e.

$$\sum_\theta p_r(\theta)q_s(\theta) = 0, \qquad (r \neq s).$$

This translates to an orthogonality relation on the polynomials $p_r$,

$$\sum_\theta w_\theta p_r(\theta)p_s(\theta) = 0, \qquad (r \neq s).$$

# Chapter 17

# Locally distance regular graphs

A graph $X$ is *locally distance regular* if the distance partition $\delta_u$ relative to the vertex $u$ is equitable for all $u$.

**Theorem 17.1.** *Let $T$ be an $n \times n$ tridiagonal matrix with polynomial sequence $p_0, \ldots, p_n$ (so $p_n = \det(tI - T)$). Then these polynomials are orthogonal relative to*

$$\langle p, q \rangle = \operatorname{tr}\left(p(T)q(T)e_0 e_0^T\right).$$

*Proof.* Exercise. □

If we have an inner product on then we call

$$\langle 1, t^n \rangle, \quad n \geq 0$$

the *moment sequence*. Note that if

$$\langle p, g \rangle = \int p(t)q(t)w(t)\, dt$$

then $\langle 1, t^n \rangle = \int t^n w(t)\, dt$. The moment sequence determines the polynomials. Now suppose $X$ is a graph, $u \in V(X)$ and the distance partition is equitable. Let $T = A(X/\delta_u)$. Then $T$ is tridiagonal.

**Lemma 17.2.** $W_{u,u}(X, t) = W_{u,u}(X/\delta_u, t)$.

*"Proof".* [Tikz] □

Note that the number of closed walks at $u$ of length $m$ in $X/\delta_u$ is

$$e_u^T T^m e_u = \operatorname{tr}(T^m e_u e_u^T).$$

So the coefficients of $W_{u,u}$ are moments.

**Theorem 17.3.** *If $X$ is locally distance regular then either*

*(a) $X$ is distance regular, or*

*(b) $X$ is bipartite and semiregular and its halved graphs are distance regular.*

Denote by $k_u$ the degree of $u$.

**Lemma 17.4.** *Suppose $X$ has diameter $d$, $u \in V(X)$ such that $\delta_u$ is equitable and $v \sim u$. Then*

$$\langle e_v, A^r e_u \rangle = \frac{1}{k_u} \langle e_u, A^{r+1} e_u \rangle.$$

*Proof.* Let $U$ be the cyclic module generated by $e_u$ — i.e., the span of the vector $A^r e_u$, $r \geq 0$. Let $z_i$ be the characteristic vector of the $i$-th cell of $\delta_u$. Then the vectors $z_i$ are an orthogonal basis for $U$. If $u \in \mathbb{R}^n$, the projection of $w$ onto $U$ is

$$\hat{w} = \sum_i \frac{\langle w, z_i \rangle}{\langle z_i, z_i \rangle} z_i.$$

If $v \in V(X)$ and $\operatorname{dist}(u, v) = j$ then

$$\hat{e}_v = \frac{\langle e_v, z_j \rangle}{\langle z_j, z_j \rangle} z_j = \frac{1}{\langle z_j, z_j} z_j$$

and if $v \sim u$ then

$$\hat{e}_v = \frac{1}{k_u} A e_u.$$

Hence

$$\langle e_v, A^r e_u \rangle = \langle \hat{e}_v, A^r e_u \rangle = \frac{1}{k_u} \langle A e_u, A^r e_u \rangle = \frac{\langle e_u, A^{r+1} e_u \rangle}{k_u}$$

□

If $\delta_v$ is equitable, it follows that

$$\frac{\langle e_u, A^{r+1} e_u \rangle}{k_u} = \frac{\langle e_v, A^{r+1} e_v \rangle}{k_v}, \quad r \geq 0.$$

If $u$ and $v$ have the same valency then

$$\langle e_u, A^r e_u \rangle = \langle e_v, A^r e_v \rangle$$

for all $r \geq 0$. This implies that if $X$ is locally distance regular and regular then the quotients $X/\delta_u$ ($u \in V(X)$) are equal. Hence $X$ is distance regular.

Suppose $v, w \sim u$ and $u$ and $v$ have exactly $a$ neighbours in common [Tikz]. Since the neighbourhood of $u$ is regular, $w$ has exactly $a$ neighbours in common with $u$, and as $\delta_u$ is equitable, $w$ has $k_v - 1 - a$ neighbours at distance two from $u$. Therefore, $\deg(w) = \deg(v)$ and so the vertices at distance two have the same valency ($X_2$ is regular).

# Chapter 18

# Spectral excess

This theory is based on work by a bunch of people, centered on Fiol and Garriga []. Suppose $X$ has $n$ vertices and eigenvalues $\theta_0, \ldots, \theta_d$. Then there are orthogonal polynomials $p_r$ relative to the inner product

$$\langle p, q \rangle = \operatorname{tr}(p(A)q(A)).$$

It will be convenient sometimes to view $p$ and $q$ as polynomials on $\{\theta_0, \ldots, \theta_d\}$. We scale our orthogonal polynomials so that

$$\operatorname{tr}\left(p_i(A)^2\right) = np_i(\theta_0).$$

For this to work, we need

$$p_i(\theta_0) \neq 0 \quad \text{for} \quad i = 0, 1, \ldots, d$$

but if $p_i(\theta_0) = 0$ then $(\theta_0 - t)$ is a non-negative factor of the orthogonal polynomial $p_i$. Hence, $p_i(\theta_0)$ is not zero. We note that $p_0$ and $p_1$ are multiples of 1 and $t$. Further, if $j > i$ then $p_i(A) \circ A_j = 0$. Hence $A_j = A(X_j)$; we are not assuming that $X$ is distance regular.

**Lemma 18.1.** *If $X$ is distance regular, then $p_i(A_1) = A_i$.*

**Lemma 18.2.** *If $X$ is regular and has girth $g$, and $2i < g$, then $p_i(A) = A_i$.*

*Proof.* Exercise. □

**Theorem 18.3.** *Assume $X$ has and exactly $d + 1$ distinct eigenvalues. If the spectral idempotents are $E_0, \ldots E_d$ then*

$$\sum_{i=0}^{d} p_i(A) = nE_0.$$

*Proof.* Let $m_j$ be the multiplicity of $\theta_j$. Then $p_i(A)E_j = p_i(\theta_j)E_j$ which implies that $\mathrm{tr}(p_i(A)E_j) = m_j p_i(\theta_j)$. Consequently,

$$E_j = \sum_{i=0}^{d} \frac{\langle E_j, p_i(A)\rangle}{\langle p_i(A), p_i(A)\rangle} \, p_i(A) = \sum_i \frac{m_j p_i(\theta_j)}{n p_i(\theta_0)} \, p_i(A).$$

In particular, $m_0 = 1$, thus

$$E_0 = \frac{1}{n} \sum_i p_i(A).$$

□

If $i < d$ we have $A_d \circ p_i(A) = 0$, so this implies that

$$A_d \circ E_0 = \frac{1}{n} A_d \circ p_d(A).$$

Assume $X$ is regular, then $\sum p_i(A) = J$. Now assume that the number of distinct eigenvalues $D$ is equal to the diameter, $d$. Then

$$p_i(A) \circ A_j = 0, \quad \text{if } j > i.$$

It follows that $A_d \circ p_d(A) = A_d$, hence we can write

$$p_d(A) = A_d + S$$

where $A_d \circ S = 0$.

**Theorem 18.4** (Fiol & Garriga)**.** *Suppose $X$ is $k$-regular, has diameter $d$ and $d + 1$ distinct eigenvalues. Then $p_d(k) \geq \frac{1}{n} \operatorname{sum}(A_d)$ and if equality holds then $p_d(A) = A_d$.*

*Proof.* Let $\hat{k}_d = \frac{1}{n} \operatorname{sum}(A_d)$. Recall that $\langle p_i(A), p_i(A)\rangle = n p_i(k)$. Then

$$\begin{aligned} n p_d(k) &= \langle p_d(A), p_d(A)\rangle \\ &= \langle A_d + S, A_d + S\rangle \\ &= \operatorname{sum}((A_d + S) \circ (A_d + S)) \\ &= \operatorname{sum}(A_d^{\circ 2} + A_d \circ S + S \circ A_d + S^{\circ 2}) \\ &= \operatorname{sum}(A_d) + \operatorname{sum}(S^{\circ 2}). \end{aligned}$$

□

**Corollary 18.5.** *If $X$ is $k$-regular with diameter $d$ and $p_d(k) = \hat{k}_d$, then $X$ is distance regular.*

*Proof.* The polynomials, $p_0, \dots, p_d$ satisfy a three-term recurrence. There are polynomials, $p_0^*, \dots, p_d^*$ such that

$$p_{d-s}(A) = p_s^*(A)p_d(A) = p_s^*(A)(A_d).$$

If $r < d - s$, then

$$\langle A_r, p_{d-s}(A) \rangle = 0.$$

Also, if $r > d - s$, then $\langle A_r, p_{d-s}(A) \rangle = 0$. Therefore

$$A_{d-s} = p_s^*(A)A_d = p_{d-s}(A).$$

□

**Lemma 18.6.** *Suppose $X$ is connected, $k$-regular with $D + 1$ distinct eigenvalues and girth $g$. If $g \geq 2D - 1$ then $X$ is distance regular.*

*Proof.* If $r < D$, then $p_r(D) = A_r$ (induction). Also, there is a polynomial $q$ of degree $D$ such that $q(A) = J$. It follows that $A_d$ is a polynomial in $A_1$ of degree at most $d$. □

## 18.1 Twisted Grassmann graphs

The Grassmann graph $J_q(d, e)$ has the $e$-dimensional subspaces of $V(d, q)$ as its vertices and two subspaces are adjacent if their intersection has dimension $e - 1$. In particular,

$$J_q(d, 1) = K_{[d]}, \quad \text{where} \quad [d] = \frac{q^d - 1}{q - 1}.$$

The Grassmann graphs are distance transitive. They are $q$-analogs of the Johnson graphs $J(d, e)$. We define

$$[n] = \frac{q^n - 1}{q - 1}; \quad [0] = 1$$

$$[n]! = [n][n-1]! \quad n \geq 1$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

The last is the *q-binomial coefficient* and $[n]!$ is the $q$-factorial. The number of $k$-dimensional subspaces of $V(n, q)$ is $\begin{bmatrix} n \\ k \end{bmatrix}$.

**Remark.**

(a) $\begin{bmatrix} n \\ 1 \end{bmatrix} = [n]$; this is the number of 1-dimensional subspaces of $V(n, q)$.

(b) $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$.

(c) $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k}\begin{bmatrix} n-1 \\ n-k \end{bmatrix}$ and $\begin{bmatrix} n \\ k \end{bmatrix} = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ n-k \end{bmatrix}$. **check!**

From these recurrences, we can prove the claims about subspaces.

We define a graph related to $J_q(2e+1, e)$. Let $V = V(2e+1, q)$ and let $H$ be a hyperplane in $V$. Then vertices of our new graph are

- the $(e+1)$-dimensional subspaces of $V$ not in $H$, and
- the $(e-1)$-dimensional subspaces of $H$.

The edges are as follows,

- If $\alpha, \beta$ are not in $H$ then $\alpha \sim \beta$ is $\dim(\alpha \cap \beta) = e$.
- If $\alpha$ is not in $H$ and $\beta \leq H$, then $\alpha \sim \beta$ if $\beta \leq \alpha$.
- If $\alpha, \beta \leq H$ then $\alpha \sim \beta$ if $\dim(\alpha \cap \beta) = e-2$.

We call this the *twisted Grassmann graph* and denote it by $\tilde{G}$.

**Theorem 18.7** (Koolen, Van Dam)**.** *The twisted Grassmann graph is distance regular with the same parameters as* $J_q(2e+1, e+1)$.

We have
$$|V(\tilde{G})| = \begin{bmatrix} d-1 \\ e-1 \end{bmatrix} + \begin{bmatrix} d \\ e+1 \end{bmatrix} - \begin{bmatrix} d-1 \\ e+1 \end{bmatrix}, \qquad (d = 2e+1).$$
Here,
$$\begin{bmatrix} d \\ e+1 \end{bmatrix} - \begin{bmatrix} d-1 \\ e+1 \end{bmatrix} = q^{d-e-1}\begin{bmatrix} d-1 \\ e \end{bmatrix}$$
by the first recurrence. Now,
$$\begin{bmatrix} d-1 \\ e-1 \end{bmatrix} + q^{d-e-1}\begin{bmatrix} d-1 \\ e \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix}.$$

We construct a partial linear space. Let $V$ and $H$ be as above. The points are the $e$-dimensional subspaces of $V$. There are two sorts of lines,

(a) the $(e+1)$-dimensional subspaces not in $H$,

(b) the $(e-1)$-dimensional subspaces in $H$.

The lines in (a) are incident with with their own subspaces and the lines in (b) are incident with the $e$-dimensional subspaces of $H$ that contain them. Note that the number of points is equal to the number of lines.

**Claim.** If $\alpha$ and $\beta$ are points, they are collinear if and only if $\dim(\alpha \cap \beta) = e - 1$. (So the distance-two graph on the points is $J_q(d, e)$.)

**Claim.** The distance-two graph on the lines is $\tilde{G}$.

We have $J_q(d, e) \simeq J_q(d, d - e)$. We can use a non-degenerate bilinear form to produce an explicit isomorphism.

# To do

- Citation missing

# Chapter 19

# Codes

A code is *perfect* if its packing radius is equal to its covering radius. An *e-code* is a code with packing radius $e$. If the code $\mathcal{C}$ has a packing radius $e$, then the minimum distance $\delta$ of $\mathcal{C}$ is at least $2e + 1$.

By way of example, a perfect 1-code in a $k$-regular graph is a subset of $\mathcal{C}$ of the vertices such that any two vertices in $\mathcal{C}$ are at distance at least 3, and the "balls" of radius partition the vertices of the graph.

**Example.** A fibre in a drackn.

If $\mathcal{C}$ is a perfect 1-code, then the partition $(\mathcal{C}, V(X)\backslash\mathcal{C})$ is equitable with quotient

$$\begin{pmatrix} 0 & k \\ 1 & k-1 \end{pmatrix}.$$

Here, $-1$ is an eigenvalue.

**Lemma 19.1.** *If $X$ is regular and contains a perfect 1-code, then $-1$ is an eigenvalue. If the valency of $X$ is $k$, the size of the code is* $\frac{|V(X)|}{k+1}$

**Delsarte's temptation:** there are no perfect 1-codes in $J(v, k)$. (True if $v \leq 2^{250}$.)

The Hamming graphs contain perfect 1-codes in some cases. Assume $X = H(n, 2)$ (binary codes), and suppose there is a perfect 1-code. Then

(a) $-1$ is an eigenvalue of $H(n, 2)$ if and only if $n$ is odd.

(b) If a perfect 1-code exists, then $n + 1 \mid 2^n$.

If (b) holds, a perfect 1-code exists, in fact a linear code. Let $G$ be the $n \times (2^n - 1)$ matrix with distinct non-zero binary vectors as its columns. Let $\mathcal{C} = \ker(G)$. Note that $\operatorname{rank}(G) = n$ and so $|\mathcal{C}| = 2^{2^n - 1} - n$.

**Claim.** The minimum distance of $G$ is three.

We have equality in the sphere-packing bound, so $\mathcal{C}$ is perfect. There are (many) perfect 1-codes that are not linear. If $\mathcal{C}$ is a perfect 1-code, then the words of weight three in $\mathcal{C}$ are the characteristic vectors of the blocks of a Steiner triple system.

## 19.1 Completely regular codes

A code in a distance regular graph is *completely regular* if its distance partition is equitable. Perfect codes are completely regular.

**Theorem 19.2.** *Suppose $\mathcal{C}$ is a completely regular linear code in $H(n,q)$. Then if the minimum distance is at least three, the quotient of $H(n,q)$ over the cosets is a distance regular graph.*

*Proof.* The key is that each vertex in the quotient is completely regular. Since the quotient is regular, it is distance regular. □

## 19.2 Outer distribution matrix

Let $C$ be a subset of $V(X)$ with characteristic vector $x_C$. Assume we have an association scheme with $d$ classes on $V(X)$. The *outer distribution matrix* $N$ is

$$\begin{pmatrix} A_0 x_C & \cdots & A_d x_C \end{pmatrix}.$$

The column space of $N$ is $\mathcal{A}$-invariant — it is the cyclic module generated by $x_C$.

The vectors $E_j x_C$, $(j = 1, \dots, d)$ that are non-zero form an orthogonal basis for $\operatorname{col}(N)$. The *dual degree* $s^*$ of $\mathcal{C}$ is $\operatorname{rank}(N) - 1$. (If $\mathcal{C}$ is linear, $s^*$ is the degree of the dual code, $\mathcal{C}^\perp$.)

**Lemma 19.3.** *Let $\mathcal{C}$ be a code in a distance regular graph with covering radius $t$ and outer distribution matrix $N$. Then $N$ has at least $t+1$ distinct rows, and equality holds if and only if $\mathcal{C}$ is completely regular.*

*Proof.* The supports of the vectors $(A_1 + I)^r x$ for $r = 0, 1, \dots, t$ are a strictly increasing sequence of subsets of $V$. Thus $N$ has at least $t+1$ distinct rows, and these rows are linearly independent. Therefore $t + 1 \le s^* + 1$, and so $t \le s^*$.

If the distance partition of $\mathcal{C}$ is equitable, relative to $X_1$ then it is an equitable partition of $X_r$ for $r = 1, \dots, d$. Because the scheme is metric relative to $X_1$, it follows that the vector

$$(S_j(n) \cap C_0)_{j=0}^d$$

is determined by $\operatorname{dist}(u, C_0)$. But the vector is the $u$-row of $N$. Hence if $\mathcal{C}$ is completely regular then $t = s^*$ and $N$ has exactly $t+1$ distinct rows. □

**Lemma 19.4.** *The column space of $N$ is spanned by the first $s^*+1$ columns or by the first $s^*$ columns and $\mathbb{1}$.*

*Proof.* If $i \leq d$ with $A_i x \in \operatorname{span}\{x, A_1 x, \ldots, A_{i-1}x\}$, then $\operatorname{rank}(N) = i$ so $i = s^*+1$. For the second part, suppose

$$\sum_{i=0}^{j} \alpha_i A_i x = \mathbb{1}.$$

Then $q(A_1)x = \mathbb{1}$, where $\deg(q) \leq j$ and if $l \neq 0$,

$$0 = E_l \mathbb{1} = q(\theta_l) E_l x.$$

There are $s^*+1$ values of $l$ such that $E_l x \neq 0$. Here, $q$ vanishes on $s^*+1$ eigenvalues so $\deg(q) \geq s^*+1$. □

**Theorem 19.5.** *Let $\mathcal{C}$ be a code in a distance regular graph with minimum distance $\delta$ and dual degree $s^*$. If $\delta \geq 2s^*-1$, then $\mathcal{C}$ is completely regular.*

**Remark.** (a) Since $r \leq s^*$, we have $\delta \leq 2s^*+1$

(b) If $\delta = 2s^*+1$, then $\mathcal{C}$ is perfect.

*Proof.* We first prove that $s^* = r$. We have $2r+1 \geq \delta \geq 2s^*-1$ and so $s^* \geq r \geq s^*-1$. But if $r = s^*-1$, then $2s^*-1 = \delta = 2r+1$ and so $\mathcal{C}$ is perfect, implying $r = s^*$, a contradiction. Thus $r = s^*$.

Now, since the $\delta \geq 2r-1$, the balls of radius $r-1$ about the vertices in $\mathcal{C}$ are pairwise disjoint. If $i \leq r-1$ and $u, v \in \mathcal{C}$, it follows that $e_u^T N = e_v^T N$.

If $\operatorname{dist}(x, \mathcal{C}) > r-1$ then $\operatorname{dist}(x, \mathcal{C}) = r$, so $Nx_i = 0$ for $i = 0, \ldots, r-1$. It follows that $N$ has exactly r+1 distinct rows. Consequently, $\mathcal{C}$ is completely regular. □

## 19.3 Perfect codes in $H(n,2)$

If there is a perfect code in $H(n,2)$, then

1. $|\mathcal{C}| \mid 2^n$,

2. $\varphi(X/\partial(\mathcal{C}), t) \mid \varphi(X, t)$

Assume that $e = 2$.

1. $|\mathcal{C}| \cdot \left(1 + n + \binom{n}{2}\right) = 2^n$

2. (add tikz?)

$$A(X/\partial) = \begin{pmatrix} 0 & n & 0 \\ 1 & 0 & n-1 \\ 0 & 2 & n-2 \end{pmatrix}$$

$$\begin{pmatrix} t & -n & 0 \\ -1 & t & -(n-1) \\ 0 & -2 & t-(n-2) \end{pmatrix} \sim \begin{pmatrix} t-n & -n & 0 \\ t-n & t & -(n-1) \\ t-n & -2 & t-(n-2) \end{pmatrix}$$

$$\det \begin{pmatrix} t+n & -(n-1) \\ n-2 & t-(n-2) \end{pmatrix}$$

The zeros of this are $\frac{1}{2}(-2 \pm \sqrt{4n-4}) = -1 \pm \sqrt{n-1}$. Since these must be the eigenvalues of $H(n,2)$, we deduce that $n-1$ is a perfect square.

The sphere-packing condition gives $n^2+n+2 \mid 2^{n+1}$

| $n$ | $n^2+n+2$ |
|---|---|
| 1 | 4 |
| 2 | 8 |
| 3 | 14 |
| 4 | 22 |
| 5 | 32 |
| 6 | 44 |
| 7 | 58 |

## 19.4 Linear Codes

If $\mathcal{C}$ is linear, $H(n,2)$ is a distance regular graph with diameter $e$. For $e=2$ we have a strongly regular graph.

If $n=5$ we get the Clebsch graph. If $n=10$ then $v=56$. This is the Gewirtz graph. (But there is no perfect code.)

# To do

- Define packing and covering radius.
- We are using both $r$ and $t$ as covering radius – maybe don't.
- Everything from Theorem 19.5, go over again.
- I don't know if I'm confusing $\delta$ and $\partial$ in some places.

# Chapter 20

# Representations

Let $X$ be a graph with an eigenvalue $\theta$ and spectral idempotent $E$. Let $U$ be the $n \times k$ matrix with columns forming an orthonormal basis of $\operatorname{col}(E) = \ker(A - \theta I)$, so $U^T U = I$ and

$$AU = \theta U, \quad E = UU^T.$$

Let $u_\theta(i) = e_i^T U$. The map on $V(X)$, defined by

$$i \mapsto u_\theta(i)$$

is a representation of $X$. Notice that

$$\theta u(i) = \sum_{j \sim i} u(j)$$

and

$$\langle u(i), u(j) \rangle = \langle e_i^T U, e_j^T U \rangle = e_i^T U U^T e_j = E_{i,j}.$$

We define

$$w_r = \frac{\langle u_\theta(i), u_\theta(j) \rangle}{\langle u_\theta(i), u_\theta(i) \rangle}$$

where $r = \operatorname{dist}(i, j)$, and call $w_0, \dots, w_d$ the *sequence of cosines* belonging to $\theta$.

**Remark.** $w_0 = 1$.

The vector $Uu_1(\theta)$ is an eigenvector for $A$ with eigenvalue $\theta$ which is constant on the cells of the distance partition relative to the vertex 1. If $P$ is the characteristic matrix of this partition and

$$w = \begin{pmatrix} w_0 \\ \vdots \\ w_d \end{pmatrix}$$

then $Uu_1(\theta) = Pw$ and so

$$\theta Pw = APw = PBw$$

whence $Bw = \theta w$, i.e $w$ is an eigenvector of $B$,

$$B = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ 1 & a_1 & b_2 & 0 & \cdots \\ 0 & c_2 & a_2 & b_3 & \cdots \\ 0 & 0 & c_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

**Theorem 20.1.** *Let $w_0, \ldots w_d$ be the sequence of cosines for the eigenvalue $\theta$ of the distance regular graph, $X$. If $\theta$ is the $i$-th largest eigenvalue of $A$, then the sequence has exactly $i$ sign changes; if $i \geq 2$ then the sequence of differences,*

$$\left(w_r - w_{r+1}\right)_{r=0}^{d-1}$$

*has exactly $i-2$ sign changes.*

**Remark.** The proof of this theorem includes Sturm sequences.

This implies that the sequence

$$w_0(\theta_1), \ldots, w_d(\theta_1)$$

is non-increasing.

## 20.1 Johnson graphs

Define incidence matrices, $W_{i,j}(v)$ indexed by $i$-subsets and $j$-subsets of $\{1, \ldots, v\}$, by

$$(W_{i,j})_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha \subseteq \beta \\ 0 & \text{otherwise.} \end{cases}$$

Define $\binom{v}{k} \times \binom{v}{k}$ matrices $C_r$ by

$$C_r = W_{r,k}^T W_{r,k}, \quad (r = 0, \ldots, k).$$

As a test case, $C_0 = J$. We need some properties of the matrices $W_{i,j}$.

1. We have

$$W_{i,j}W_{j,k} = \binom{v-i}{k-i} W_{i,k}.$$

   One consequence of this is that $\operatorname{row}(W_i, k) \leq \operatorname{row}(W_j, k)$.

2. We have
$$W_{s,k}W_{t,k}^T = \sum_i \binom{v-s-t}{v-i-k} W_{i,s}^T W_{i,t}.$$

*Proof of 2.* First,
$$(W_{s,k}W_{t,k}^T)_{\alpha,\beta} = \binom{v-|\alpha\cup\beta|}{k-|\alpha\cup\beta|}.$$
One the other hand,
$$(W_{i,s}^T W_{i,t})_{\alpha,\beta} = \binom{|\alpha\cap\beta|}{i}.$$
Now,
$$\sum \binom{v-s-t}{v-i-k}\binom{|\alpha\cap\beta|}{i} = \sum \binom{v-s-k+|\alpha\cap\beta|}{v-k}$$
and
$$|\alpha\cap\beta| = |\alpha|+|\beta|-|\alpha\cup\beta| = s+t-|\alpha\cup\beta|$$
and the result follows. □

A consequence of this is that
$$W_{t,k}W_{t,k}^T = \sum \binom{v-2t}{v-k-i} W_{i,t}^T W_{i,t}.$$
Note that $W_{t,t} = I$ and the right hand side is positive definite. Therefore $vk(W_{t,k}) = \binom{v}{t}$. We have
$$(C_r)_{\alpha,\beta} = (W_{r,k}^T W_{r,k})_{\alpha,\beta} = \binom{|\alpha\cap\beta|}{r},$$
hence
$$C_i = \sum_{j\geq i} \binom{j}{i} A_{k-j}$$
and thus $C_0,\dots,C_k$ lie in the Bose-Mesner algebra of $J(v,k)$. Since the change of basis matrix is backward triangular, we see that we can express $A_0,\dots,A_k$ in terms of $C_0,\dots C_k$. Therefore $C_0,\dots,C_k$ is a basis for the Bose-Mesner algebra. In fact,
$$A_{k-r} = \sum_j (-1)^{j-r}\binom{j}{r} C_j.$$
We can prove this using generating functions
$$\begin{aligned}\sum_i t^i C_i &= \sum_{i,j:j\geq i} t^i \binom{j}{i} A_{k-j}\\ &= \sum_j (1+t)^j A_{k-j}.\end{aligned}$$
From this, it is straightforward.

**Theorem 20.2.**

$$C_iC_j = \sum_{r=0}^{i\wedge j} \binom{v-i-j}{v-k-r}\binom{k-r}{i-r}\binom{k-r}{j-r} C_r.$$

*Proof.* We have

$$\begin{aligned} C_iC_j &= W_{i,k}^T W_{i,k} W_{j,k}^T W_{j,k} \\ &= \sum_r W_{i,k}^T \binom{v-i-j}{v-k-r} W_{r,i}^T W_{r,j} W_{j,k}. \end{aligned}$$

Since

$$W_{r,j}W_{j,k} = \binom{k-r}{j-r} W_{r,k}$$

and

$$W_{r,i} = \binom{k-r}{i-r} W_{r,k}$$

we get the stated result. $\square$

We have a chain of subspaces, $\mathrm{col}(W_{i,k}^T)$; these are invariant under $\mathbb{R}[\mathcal{A}]$. The subspaces

$$\mathrm{col}(W_{i-1,k}^T)^\perp \cap \mathrm{col}(W_{i,k}^T)$$

form an orthogonal decomposition of $\mathbb{R}^{\binom{v}{k}}$.

It follows from Theorem 20.2 that the matrix representing the action of $C_i$ (by multiplication) on $\mathbb{R}[\mathcal{A}] = \mathrm{span}\{C_r\}$ is lower triangular. We can read off the eigenvalues of $C_i$ from this and thus we can get the eigenvalues of $A_0, A_1, \ldots, A_k$. We derive these in another way.

Let $E_j$ be the orthogonal projection onto

$$\mathrm{col}(W_{j-1,k}^T)^\perp \cap \mathrm{col}(W_{j,k}^T).$$

Note that $E_jW_{i,k} = 0$ if $i < j$ and so $E_jC_i = 0$ if $i < j$. If we multiply both sides of Theorem 20.2 by $E_j$, we thus get

$$C_iC_jE_j = \binom{v-i-j}{v-k-j}\binom{k-j}{i-j} C_jE_j.$$

Therefore the columns of

$$C_jE_j = W_{j,k}^T W_{j,k} E_j$$

are eigenvectors for $E_j$.

## 20.2 $t$-designs

Let $x$ be the characteristic vector of a set, $\mathcal{D}$ of $k$-subsets of $\{1, \dots, v\}$. Then $\mathcal{D}$ is a $t$-design if and only if

$$W_{t,k} x = \lambda \mathbb{1}$$

for some positive integer, $\lambda$. Also,

$$W_{t,k} \mathbb{1} = \binom{v-t}{k-t} \mathbb{1}$$

and so $x$ is the characteristic vector of a $t$-design if and only if

$$W_{t,k} \left( \lambda^{-1} x - \binom{v-t}{k-t}^{-1} \mathbb{1} \right) = 0$$

and therefore, $x$ is a $t$-design if and only if

$$C_t \left( \lambda^{-1} x - \binom{v-t}{k-t}^{-1} \mathbb{1} \right) = 0.$$

**Corollary 20.3.** *$\mathcal{D}$ is a $t$-design if and only if*

$$E_i \left( \lambda^{-1} x - \binom{v-t}{k-t}^{-1} \mathbb{1} \right) = 0$$

*for $i = 1, \dots, t$.*

(No proof yet). So $\mathcal{D}$ is a $t$-design in $J(v,k)$ if and only if the eigenvalue support of $\mathcal{D}$ does not contain $\theta_1, \dots \theta_t$.

## 20.3 Eigenspaces

The eigenspace for $\theta_0$ in the Johnson graph $J(v,k)$ is spanned by $\mathbb{1}$. The column space of $W_{1,k}^T$ is a sum of the eigenspaces for $\theta_0$ and $\theta_1$. The $\theta_1$-eigenspace is $\mathbb{1}^T \cap \operatorname{col}(W_{1,k}^T)$. Hence the columns of

$$W_{1,k}^T - \frac{k}{v} J$$

span the $\theta_1$-eigenspace. We can view the rows of the above matrix as providing a representation of $J(v,k)$. If the columns of the matrix $M$ are linearly independent, define

$$E = M(M^T M)^{-1} M^T.$$

Then $E = E^T$ and $E^2 = E$ and $\operatorname{col}(E) = \operatorname{col}(M)$, i.e. $E$ represents an orthogonal projection onto $\operatorname{col}(M)$. So

$$F_j = W_{j,k}(W_{j,k}^T W_{j,k})^{-1} W_{j,k}^T$$

represents an orthogonal projection onto $\mathrm{col}(W_{j,k}^T)$. Consequently,

$$E_j = F_j - F_{j-1},$$

where $E_j$ is a projection on the $j$-th eigenspace.

## 20.4 Representations

Suppose $i \mapsto u_\theta(i)$ is a representation of a graph $X$ on an eigenspace with eigenvalue $\theta$. Then

$$\theta u(i) = \sum_{j \sim i} u(j)$$

and the inner products are given by entries of the idempotent $E = E_\theta$.

**Lemma 20.4.** *$X$ is walk-regular if and only if for each spectral idempotent $E$ we have $E \circ I = \gamma I$ (for some $\gamma$).*

**Definition.** $X$ is 1-*walk-regular* if it is walk regular and for each idempotent, there are constants $\gamma_\theta$ such that $E \circ A = \gamma_\theta A$.

Clearly any graph in an association scheme is 1-walk-regular, as is any arc-transitive graph.

In a 1-walk-regular graph we have

$$\begin{aligned}\theta \langle u_\theta(1), u_\theta(1) \rangle &= \sum_{i \sim 1} \langle u_\theta(i), u_\theta, (1) \rangle \\ &= k \langle u_\theta(i), u_\theta(1) \rangle, \quad i \sim 1.\end{aligned}$$

Therefore $w_1 = \theta / k$.

Suppose $C$ is a clique in $X$ and let $\tau$ be an eigenvalue of $X$. Then the submatrix of $E_i$ with rows and columns indexed by vertices in $C$ is a scalar multiple of

$$\begin{pmatrix} 1 & w_1 & \cdots & w_1 \\ w_1 & 1 & & w_1 \\ \vdots & & \ddots & \vdots \\ w_1 & w_1 & \cdots & 1 \end{pmatrix} = (1 - w_1) I + w_1 J \succcurlyeq 0.$$

Therefore the row sums of this matrix are non-negative;

$$1 + (|C| - 1) \frac{\tau}{k} \geq 0.$$

Assume $\tau < 0$. Then

$$1 \geq (|C| - 1) \frac{-\tau}{k}$$

and so

$$|C| \leq 1 - \frac{k}{\tau} \qquad \text{(ratio bound)}.$$

We can prove that

$$w_2 = \frac{1}{kb_1}(\theta^2 - a_1\theta - k),$$

from which it follows that

$$w_1 - w_2 = \frac{(k-\theta)(\theta+1)}{kb_1}$$

**Theorem 20.5.** *Let $\theta$ be an eigenvalue of the distance regular graph $X$. The corresponding representation is not injective if and only if one of the following holds:*

*(a)* $\theta = \theta_0$

*(b)* $\theta = \theta_d$ *and* $X$ *is bipartite*

*(c)* $\theta = \theta_r$, $r$ *is even and* $X$ *is antipodal.*

**Corollary 20.6.** *If $\theta \neq \theta_0, \theta_d$, then the representation is locally injective — images of vertices at distance two are distinct.*

# 20.5 Spherical designs

Let $\Omega$ be the unit sphere in $\mathbb{R}^d$. A subset, $\Phi$ of $\Omega$ is a *spherical $t$-design* if, for any polynomial $f$ of degree at most $t$,

$$\frac{1}{|\Phi|}\sum_{x\in\Phi} f(x) = \int_\Omega f\, d\mu$$

(average values of $f$ over $\Omega$).

The maximum value of $t$ for which this works is the *strength of the design*. We will use $\langle 1, f\rangle_\Phi$ to denote the left hand side and $\langle 1, f\rangle$ to denote the right hand side. Thus $\Phi$ is a 1-design if and only if

$$\sum_{x\in\Phi} x = 0.$$

**Lemma 20.7.** *A subset $\Phi$ is a 2-design if and only if*

$$\sum_{x\in\Phi} xx^T = \frac{|\Phi|}{d}\, I_d.$$

If $U$ is $n \times d$ and $U^TU = I$ then

$$U^TU = \sum u_i u_i^T, \quad u_i = (Ue_i)^T.$$

**Corollary 20.8.** *The image of an injective representation of a distance regular graph is a spherical design.*

If $\{x_1, \ldots, x_m\}$ is a 2-design, then $\{\pm x_1, \ldots, \pm x_m\}$ is a 3-design. The *degree* of a finite subset $\Phi$ of $\Omega$ is

$$|\{\langle x, y\rangle : x, y \in \Omega; x \neq y\}|.$$

## 20.6 $s$-distance sets

We want to get a good upper bound on the size of a subset $\Phi$ of $\Omega$ with degree $s$. For this we need information about polynomial functions on $\Omega$. Let $\mathrm{Pol}(\Omega)$ denote the space of polynomial functions on $\Omega$ and let $\mathrm{Pol}(\Omega, d)$ be the space of such functions with degree at most $d$.

**Theorem 20.9.** *If $\Phi$ is a subset of $\Omega$ with degree $s$, then*

$$|\Phi| \leq \dim(\mathrm{Pol}(\Omega, s)).$$

*Proof.* Let $\gamma_1, \ldots, \gamma_s$ be the inner products of distinct elements of $\Phi$. Define

$$f(t) = \prod_{i=1}^{s} \frac{t - \gamma_i}{1 - \gamma_i}.$$

Also, if $a \in \Omega$, define $f_a$ by

$$f_a(x) = f(\langle a, x\rangle).$$

Then $\{f_a : a \in \Phi\}$ is a linearly independent subset of $\mathrm{Pol}(\Omega, s)$. If $a, b \in \Phi$ then

$$f_a(b) = \delta_{a,b}.$$

□

We need to compute $\dim(\mathrm{Pol}(\Omega, s))$.

1. The monomials of degree $k$ (in $d$ variables) are linearly independent on $\mathbb{R}^d$ and there are $\binom{d+k-1}{k}$ of them.

2. The restrictions of these monomials to the unit are linearly independent.

3. If $f$ is a polynomial, $\deg(f(x_1^2 + \cdots + x_d^2)) = \deg(f) + 2$.

**Theorem 20.10** (Absolute bound)**.** *We have*

$$\dim(\mathrm{Pol}(\Omega, s)) = \binom{d+s-1}{s} + \binom{d+s-2}{s-1}.$$

For $s = 1$, we get
$$\binom{d}{1} + \binom{d-1}{0} = d + 1,$$
and for $s = 2$, the bound is
$$\binom{d+1}{2} + d = \frac{d(d+3)}{2}.$$
The image of a strongly regular graph under a representation is a sperical 2-design with degree two. So if the eigenvalue has multiplicity $m$, then
$$|V(X)| \leq \frac{m(m+3)}{2}.$$
There is a lower bound on the size of a $t$-design. Suppose $\Phi$ is a sperical $t$-design, define $m = \lfloor \frac{t}{2} \rfloor$ and let $\{g_1, \ldots, g_m\}$ be an orthonormal basis for $\mathrm{Pol}(\Omega, m)$. Note that $\deg(g_i g_j) \leq t$. Then
$$\begin{aligned}\delta_{i,j} &= \langle g_i, g_j \rangle \\ &= \langle 1, g_i g_j \rangle \\ &= \langle 1, g_i g_j \rangle_\Phi \\ &= \langle g_i, g_j \rangle_\Phi,\end{aligned}$$
so the restrictions $g_i|_\Phi$ are pairwise orthogonal and therefore form a linearly independent subset, $\mathbb{R}^\Phi$. Thus $m \leq |\Phi|$.

**Theorem 20.11.** *If $\Phi$ is a spherical $t$-design, then*
$$|\Phi| \geq \dim \mathrm{Pol}\left(\Omega, \left\lfloor \frac{t}{2} \right\rfloor\right).$$

**Remark.** If $\Phi$ has degree $s$ and strength $t$ then $t \leq 2s$. (Exercise.)

**Theorem 20.12.** *If a representation of a strongly regular graph $X$ is a 3-design, the neighbourhoods of a vertex in $X$ are strongly regular graphs.*

**Remark.** The image is a 3-design if and only if $a_{i,i}(i) = 0$. [Tikz?] Suppose $\deg(f) = 2$. Consider the average of $g_a$ on $\Phi$, where
$$g(t) = (t - \gamma_2) f,$$
and use this to show that the neighbourhoods are 2-designs.

**Theorem 20.13.** *Let $X$ be a distance regular graph with diameter $d$ and eigenvalue $\theta$ with multiplicity $m > 1$. If $d \geq 3m - 3$, then $X$ is a cycle.*

**Corollary 20.14.** *There are only finitely many distance regular graphs that are not cycles with an eigenvalue of multiplicity $m$.*

**Remark.**

1. If $\theta$ is an eigenvalue of $X$ and $\theta = \pm k$, then the $\theta$-representation is locally injetive — the images of vertices at distance two are distinct.

2. The dodehedron is distance regular with $d = 5$ and $\mathrm{mult}(\theta_1) = 3$. It follows that the bound in the theorem is tight.

*Proof of Theorem 20.13.* Assume $d \geq 3m - 3$. Let $u_0, u_1, \dots u_d$ be a geodesic path in $X$.

Claim: $b_{m-1} = 1$ (for proof, see [**?**]).

Recall that if $i + j \leq d$ then $c_j \leq b_j$ (ref: exercise). Also, the sequence $b_0, \dots, b_{d-1}$ is non-increasing and $c_1, \dots, c_d$ is non-decreasing. If $d \geq 3m - 3$, then $c_{2m-2} = 1$.

Claim: If $b_i = c_{i+r} = 1$ then $X$ has no odd cycle of length $2r + 1$. If $i = r = m - 1$, then we get $a_{m-1} = 0$. Hence

$$k = a_{m-1} + b_{m-1} + c_{m-1} = 2.$$

□

# To do

- Define walk-regular

# Chapter 21

# Cometric schemes

If $\mathcal{A}$ is an association scheme with $d$ classes and $X$ is a graph in the scheme, then $\operatorname{diam}(X) \leq d$. If equality holds, $X$ is distance regular and we say that $\mathcal{A}$ is *metric (P-polynomial)*.

The diameter of $X$ is the least integer $r$ such that all entries of $(A+I)^r$ are non-zero, i.e. it is the least integer $r$ such that $(A+I)^r$ is Schur invertible.

If $p(t) = p_0 t^m + \cdots + p_m$, then

$$p_0(E) = p_0 E^{\circ m} + \cdots + p_m J.$$

The *Schur diameter* of a matrix $M$ is the least integer $r$ such that there is a polynomial $p$ of degree $r$ and $p_0(M)$ is invertible.

If $M \in \mathbb{R}[\mathcal{A}]$ and the Schur diameter of $M$ is $r$, then $\mathcal{A}$ has at least $r$ classes. If $\mathcal{A}$ has $d$ classes and $E$ in $\mathbb{R}[\mathcal{A}]$ has Schur diameter $d$ then we say that $\mathcal{A}$ is *cometric (Q-polynomial)* relative to $E$.

## 21.1 Degree of functions

Each eigenvector of a graph is a function on its vertices. We can multiply functions, the corresponding operation will be Schur product. The $\theta_i$-eigenspace is $\operatorname{col}(E_i)$. We have

$$E_i \circ E_j = \frac{1}{v} \sum_k q_{i,j}(k) E_k.$$

Consider the Schur powers $(E_1+E_0)^{\circ r}$. If $(E_1+E_0)^{\circ s}$ is invertible, then $(E_1+E_0)^{\circ s} \cdot E_j \neq 0$ for $j = 0, \ldots, d$. The Schur diameter of $E_j$ is finite if and only if the $\theta_j$-representation is injective. If $\mathcal{A}$ is cometric relative to $\theta_1$, then there are polynomials $q_0, q_1, \ldots, q_d$ such that

$$E_j = q_j \circ (E_1), \qquad j = 0, 1, \ldots, d.$$

(This is the usual definition of cometric.)

Equivalently, the eigenvectors in the $\theta_j$-eigenspace are polynomials in $\theta_1$-eigenvectors with degree $j$.

**Example.**

(a) Cycles,

(b) metric translation schemes,

(c) $J(v,k)$, $H(n,q)$ — most metric schemes with names,

(d) any strongly regular graph,

(e) a drackn is cometric if and only if $r = 2$.

How de we prove that $J(v,k)$ is cometric?

The column space of $W_{i,k}^T$ is the sum of the first $i+1$ eigenspaces of $J(v,k)$. Let $F_i$ denote the orthogonal projection onto $W_{i,k}^T$ — thus

$$F_i = W_{i,k}^T (W_{i,k} W_{i,k}^T)^{-1} W_{i,k}.$$

Claim: $F_i$ is a Schur polynomial in $F_1$ of degree $i$.

Finally, $E_i = F_i - F_{i-1}$ and hence $E_i$ is a Schur polynomial of degree $i$ in $E_1$.

If the scheme $\mathcal{A}$ (on $d$ classes) is cometric with respect to $E_1$, then there are polynomials $p_0, \dots, p_d$ such that $\deg(p_j) = j$ and $E_j = p_j \circ E_1$. Relative to the inner product,

$$\langle f, g \rangle = \langle f \circ E_1, g \circ E_1 \rangle$$

these polynomials are orthogonal. If a scheme is metric anre cometric, it has two associated families of orthogonal polynomials. Leonard proved that the parameters of a scheme that is metric and cometric is determined by a set of only six parameters. **Reference? - Leonard 485 - 1992 section 8.1 BCN** The corresponding polynomials are members of the Askey-Wilson family.

If $\mathcal{A}$ is cometric and $S \subseteq V(\mathcal{A})$ with characteristic vector $x$, then we say that $S$ is a *$t$-design* if

$$E_j x = 0, \quad j = 1, \dots, t.$$

(For $J(v,k)$ and the unit sphere, this is equivalent to the usual definition.)

# Chapter 22

# Eigenvalues of the Johnson graph

Note that $J(v,1) = K_v$, with eigenvalues

$$v-1^{(1)}, \quad -1^{(v-1)}.$$

The valency of $J(v,k)$ is $k(v-k)$. To get the eigenvalues in general, we use two equations:

$$\begin{aligned} W_{k,k+1}W_{k,k+1}^T &= (v-k)I + A_1(k) \\ W_{k,k+1}^T W_{k,k+1} &= (k+1)I + A_1(k+1), \end{aligned}$$

where $A_1(i)$ is the adjacency matrix of $J(v,i)$. Since the matrices on the left have the same non-zero eigenvalues with the same multiplicities, we can compute the eigenvalues of $J(v,k+1)$ from those of $J(v,k)$.

| $k$ | $1$ | $v-1$ | $\binom{v}{2}-\binom{v}{1}$ | $\binom{v}{3}-\binom{v}{2}$ | $\cdots$ |
|---|---|---|---|---|---|
| 1 | $v-1$ | $-1$ | $\cdots$ | | |
| 2 | $2v-4$ | $v-4$ | $-2$ | $\cdots$ | |
| 3 | $3v-9$ | $2v-9$ | $v-7$ | $-3$ | $\cdots$ |

## 22.1 Delsarte cliques

We say that if $X$ is 1-walk regular with valency $k$ and least eigenvalue $\tau$, then

$$w(X) \leq 1 - \frac{k}{\tau}.$$

We investigate the case of equality in distance regular graphs. Assume $\mathcal{A}$ is a metric scheme with $d$ classes and let $C$ be a clique in $X_1$. Then the image of $\mathcal{C}$ under a representation gives a submatrix of $E_j$ of the form $\alpha I + \beta J$. Here,

$$\alpha + \beta = \frac{m(\theta_j)}{v}$$

and

$$\frac{\beta}{\alpha} = w_1 = \frac{\theta_j}{k}.$$

Since this $|C| \times |C|$ matrix is a Gram matrix, it is positive semidefinite. Its row sum is an eigenvalue, whence $1 + w_1(|C| - 1) \geq 0$. (This is useless if $\theta_j \geq 0$.) So

$$\begin{aligned} (-w_1)(|C| - 1) &\leq 1 \\ |C| - 1 &\leq \frac{-}{k}\theta_j \\ \Rightarrow |C| &\leq 1 - \frac{k}{\theta_j}. \end{aligned}$$

It follows that $w(X_1) \leq 1 - k/\theta_d$ (ratio bound for cliques). A clique of size $1 - \frac{k}{\theta_d}$ is a *Delsarte clique*.

**Lemma 22.1.** *A Delsarte clique in a distance regular graph is a completely regular subset.*

Recall that the covering radius $r$ of a subset is bounded above by the number of distinct rows of its outer distribution matrix,

$$N = \begin{pmatrix} A_0 x & A_1 x & \cdots & A_d x \end{pmatrix}.$$

If equality holds in the ratio bound the image of a Delsarte clique under the $\theta_d$– representation is a regular simplex centered at zero. In particular, the images of the vertices sum to zero.

Claim: If $y$ is at distance $r$ from the Delsarte clique, $C$, and there are exactly $\alpha_r$ vertices in $C$ at distance $r$ from $y$ then

$$\beta_r w_r + (|C| - \beta_r))w_{r+1} = 0.$$

This implies that the sequence $\beta_0, \ldots, \beta_d$ is determined by the cosine sequence, and conversely.

To complete the proof, we must first show that the covering radius of $C$ is $d - 1$.

## 22.2 Width and dual width

Let $X$ be a distance regular graph. The *width* of a subset $C$ of $V(X)$ is the maximum distance between two vertices in $C$. If $x$ is the characteristic vector of $C$, then the degree is

$$|\{x^T A_r x : r = 1, \ldots, d\}|.$$

We denote the width by $w$ and the degree by $s$. (Note that $w(C)$ is the maximum value of $r$ such that $x^T A_r x \neq 0$.)

Some of the motivation comes from the EKR problem: bound the size of $t$-intersecting family of $k$-subsets of a $v$-set and characterize the sets that achieve the bound. A $t$-intersecting family is a subset of vertices of the Johnson graph with width $k-t$.

There are dual concepts $s^*$ and $w^*$ to $s$ and $w$. The *dual degree*, $s^*$, is

$$|\{j : 1 \le j \le d, x^T E_j x \neq 0\}|.$$

The *dual width*, $w^*$, is the maximum value of $j$ such that $x^T E_j x \neq 0$.

**Remark.** Let $S$ be the set of $k$-subsets of $\{1,\dots,v\}$ that contain $\{1,\dots,t\}$ with characteristic vector $x$. Then the projection of $xx^T$ onto the Bose-Mesner algebra is

$$\binom{v}{t}^{-1} W_{t,k}^T W_{t,k}.$$

(Exercise.) It follows that $s^* = w^* = t$.

**Theorem 22.2.** *Let $\mathcal{A}$ be a metric scheme with $d$ classes. Let $C$ be a subset of its vertices with width $w$ and dual degree $s^*$. Then*

$$w + s^* \ge d.$$

**Remark.** Our canonical $t$-intersecting family $S$ satisfies this with equality.

*Proof.* Let $x$ be the characteristic vector of $C$. Then the projection of $xx^T$ onto the Bose-Mesner algebra is

$$\sum_{i=0}^{d} \frac{x^T A_i x}{v v_i} A_i = \sum_{j=0}^{d} \frac{x^T E_j x}{m_j} E_j.$$

Here, the left hand side is a polynomial in $A_1$, say $f(A_1)$ where $\deg(f) = w$. Since the eigenvalues of $f(A_1)$ are $f(\theta_0), \dots, f(\theta_d)$, we have

$$f(\theta_r) = \frac{x^T E_r x}{m_r}$$

and as $\deg(f) = w$, at most $w$ of the values $f(\theta_r)$can be zero. The numbers of non-zero terms on the right hand side is $s^* + 1$, hence $w + s^* \ge d$. $\square$

The dual of this theorem

**Theorem 22.3.** *Let $\mathcal{A}$ be a cometric scheme with $d$ classes. If $C \subseteq V(\mathcal{A})$ with degree $s$ and dual width $w^*$, then*

$$w^* + s \ge d.$$

*Proof.* Exercise. $\square$

Since we have the inequalities

$$s \leq w, \qquad s^* \leq w^*,$$

it follows that

$$w + w^* \geq d$$

and if equality holds, $w = s$ and $w^* = s^*$. If $w + w^* = d$ we say that $C$ is a *narrow subset*.

**Example.** A Delsarte clique is narrow. So is the canonical intersecting family.

## 22.3 Equality in the width bound

**Theorem 22.4.** *Suppose $\mathcal{A}$ is metric and $C \subseteq V(\mathcal{A})$ with width $w$ and dual degree $s^*$. If $w = d - s^*$ then $C$ is completely regular.*

*Proof.* Let $B$ be the outer distribution matrix of $C$. If $u \in V(\mathcal{A})$ with $\operatorname{dist}(u, C) = l$ and $y \in C$, then

$$l \leq \operatorname{dist}(u, y) \leq l + w.$$

Since $w + s^* = d$, we have $r = s^*$. (THINK!)

. . . □

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