Partial Spreads

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1. Spreads

Let Z be a complete graph on n^2 vertices. A *parallel class* in Z is a spanning subgraph isomorphic to nK_n. We say two parallel classes S_1 and S_2 are *orthogonal* if they have no edges in common. If A_i denotes the adjacency matrix of A_i, then S_1 and S_2 are orthogonal if and only if

$$(A_1 + I)(A_2 + I) = J.$$

It is also not difficult to verify that

$$(A_i + I)^2 = n(A_i + I).$$

A *partial spread* is a set of pairwise orthogonal parallel classes.

Now suppose that $S_1, \ldots, S_r$ is a partial spread of size r. The graph X formed by the union of (the edges in) the parallel classes is a regular graph with valency $r(n-1)$; we show that it is strongly regular. Let A be the adjacency matrix of X. Then

$$\begin{aligned}(A + rI)^2 = \left(\sum_{i=1}^{r}(A_i + I)\right)^2 &= \sum_{i=1}^{r}(A_i + I)^2 + \sum_{i \neq j}(A_i + I)(A_j + I) \\ &= n\sum_{i=1}^{r}(A_i + I) + r(r-1)J \\ &= nA + nrI + r(r-1)J\end{aligned}$$

and therefore

$$A^2 - (n - 2r)A - (nr - r^2)I = r(r-1)J.$$

This shows that A is strongly regular, with parameters

$$\big(n^2,\ r(n-1);\ r(r-3)+n,\ r(r-1)\big).$$

Its eigenvalues are $-r$, $n-r$ and $r(n-1)$. If $r=1$ then $X=nK_n$, which is a trivial strongly regular graph and iff $r=2$ then X is $L(K_{n,n})$. When $r=3$, the graph X is best known as a *Latin square graph.*

The multiplicities of the eigenvalues of the graph of a partial spread are

$$(n-1)(n+1-r),\ r(n-1),\ 1.$$

Now set $r=-s$ and $n=-m$. Then, if $m \le s(s+1)$, there could be a strongly regular graph with parameters

$$\big(m^2,\ s(m+1);\ s(s+3)-m,\ s(s+1)\big).$$

Its eigenvalues would be s, $s-m$ and $s(m+1)$, with respective multiplicities

$$(m+1)(m-1+s),\ s(m+1),\ 1.$$

In fact, strongly regular graphs with these parameters do exist in some cases, and are said to be of *negative Latin square* type.

Two especially interesting cases occur when $m=s(s+1)$ and $s=1$ or 2. The corresponding parameter vectors are

$$(16,\ 5;\ 0,\ 2), \qquad (100,\ 22;\ 0,\ 6).$$

The first is associated to the *Clebsch* graph, the second to the *Higman-Sims* graph. The vertices at distance two from a given vertex in the Clebsch graph form a triangle-free graph on 10 vertices with valency $5-2=3$. Given this hint, it is not hard to construct the Clebsch graph from the Petersen graph.

The vertices at distance two from a given vertex in the Higman-Sims graph form a triangle-free on 77 vertices that is regular with valency 16. It can be shown that this graph too is strongly regular. (This is a difficult, but not impossible, exercise.)