So Few Moore Graphs

C. D. Godsil [1]

Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1

ABSTRACT

These notes provide an introduction to some of the basic theory of Moore graphs.

1. The Moore Graphs of Diameter Two

A *Moore graph* is a k-regular graph of diameter d with girth $2d+1$. It has been proved (by Damerell, and independently by Bannai and Ito) that a Moore graph has diameter at most two. The Moore graphs of diameter one are the complete graphs, about which no more need be said. A Moore graph of diameter two can be characterised as a graph with diameter two, maximum valency k and k^2+1 vertices. (The latter is the maximum possible number of vertices for a graph with the first two properties.)

Following Hoffman and Singleton, we determine the possible valencies for a Moore graph of diameter two. Such a graph must have girth five, and any two non-adjacent vertices have a unique common neighbour. Let A be the adjacency matrix of such a graph G. Then we have:

$$A^2 + A = (k-1)I + J.$$

Since G is regular the vector j is an eigenvector of A with eigenvalue k. Since A is symmetric, all other eigenvectors of A can be assumed to be

[1] Support from a National Sciences and Engineering Council of Canada operating grant is gratefully acknowledged.

orthogonal to j. If z is an eigenvector and $Jz = 0$ then

$$(A^2 + A)z = ((k-1)I + J)z = (k-1)z.$$

Consequently any eigenvalue of A belonging to an eigenvector orthogonal to j must be a zero of the polynomial $x^2 + x - (k-1)$. Hence A can have at most two such eigenvalues.

If there is only one such eigenvalue then it follows that any vector orthogonal to j is an eigenvector for A. In particular, if e_i is the i-th standard basis vector then $e_i - e_j$ is an eigenvector for A. From this it is not too difficult to verify that G must be complete. Since complete graphs do not have diameter two, we may thus assume that both zeros of $x^2 + x - (k-1)$ are eigenvalues of A. Denote them by θ and τ. As their product is equal to $1-k$, one is positive and one is negative. We assume that $\theta > \tau$. Consider the multiplicities of θ and τ, which we denote by a and b respectively. Since A has k^2+1 eigenvalues, and since their sum is $\mathrm{tr}(A)$, which is zero, we have

$$a + b + 1 = k^2 + 1, \qquad a\theta + b\tau + k = 0.$$

Thus we find that

$$a = \frac{k^2\tau + k}{\tau - \theta}, \qquad b = \frac{k^2\theta + k}{\theta - \tau}. \tag{1}$$

We must distinguish two cases. If the quadratic $x^2 + x - (k-1)$ is irreducible over the rationals then θ and τ are algebraically conjugate, and hence the same holds for a and b. Since a and b must be integers, this implies that $a = b$. From (1) we now deduce that $k^2\theta + k = -k^2\tau - k$. Therefore $k^2(\theta+\tau) + 2k = 0$. But $\theta + \tau$ is the coefficient of the linear term in $x^2 + x - (k-1)$, and so is equal to -1. We deduce finally that $k = 2$.

We now suppose that $x^2 + x - (k-1)$ factors over the rationals. The zeros θ and τ are then integers. Since $\theta + \tau = -1$, from (1) we find that

$$b - a = \frac{k^2(\theta+\tau) + 2k}{\theta - \tau} = -\frac{k^2 - 2k}{2\theta + 1}. \tag{2}$$

Now

$$\begin{aligned}
16(k^2 - 2k) &= 4k(4k - 8) \\
&= (4\theta^2 + 4\theta + 4)(4\theta^2 + 4\theta - 4) \\
&= (2\theta(2\theta+1) + (2\theta+1) + 3))(2\theta(2\theta+1) + (2\theta+1) - 5))
\end{aligned}$$

Hence, if $b - a$ is an integer then $2\theta + 1$ must divide 15. This implies that $\theta \in \{1, 2, 7\}$ and thus $k \in \{3, 7, 57\}$.

Thus we have shown that if a Moore graph of valency k exists then $k \in \{2, 3, 7, 57\}$. The first three cases are realised uniquely and respectively by the pentagon, Petersen's graph and the Hoffman-Singleton graph. It is is not known whether there is a Moore graph of diameter two and valency 57. Following G. Higman we will show that, if this graph exists, it cannot be vertex transitive.

2. A Feasibility Condition for Automorphisms

Let G be a graph with adjacency matrix A. Since A is symmetric, there is an orthogonal matrix L and a diagonal matrix D such that $A = LDL^T$. Hence we have

$$A = \sum_\theta LI_\theta L^T, \tag{1}$$

where θ runs over the distinct eigenvalues of A and I_θ is a diagonal matrix with all non-zero entries equal to 1. (Thus $D = \sum_\theta \theta I_\theta$.) Assume $E_\theta := LI_\theta L^T$. Then

(a) $E_\theta^2 = E_\theta$ and $E_\theta^T = E_\theta$,
(b) $E_\theta E_\tau = 0$ is θ and τ distinct,
(c) $\sum_t hE_\theta = I$,
(d) $AE_\theta = \theta E_\theta$.

Note that (a) implies that each matrix is an orthogonal projection, and from (d) it follows that it is the orthogonal projection onto the eigenspace of A associated to θ. The trace of E_θ is the multiplicity of the eigenvalue θ.

From (1) we find that

$$A = \sum_\theta \theta E_\theta,$$

which is the so-called *spectral decomposition* of the symmetric matrix A. If $p(x)$ is a polynomial then it follows that

$$p(A) = \sum_\theta p(\theta) E_\theta,$$

whence it follows that E_θ is a polynomial in A. We recall that the automorphism group of X can be identified with group of permutation matrices P

such that $PA = AP$. If $P \in \mathrm{Aut}(X)$, then P must commute with each projection E_θ and this implies in turn that P leaves invariant the eigenspace associated to θ.

If a matrix P leaves a subspace U of $\mathbb{R}^n$ invariant, then it determines a linear mapping from U. We call the trace of this linear mapping the *trace of P restricted to U*, and denote it by $\mathrm{tr}_U(P)$. We note that tr_U is a sum of eigenvalues of P.

2.1 Lemma. *Let X be a graph, let P be a permutation matrix in $\mathrm{Aut}(X)$ and let U be an eigenspace of X. If E is the orthogonal projection on U then $\mathrm{tr}_U(P) = \mathrm{tr}(PE)$ is an algebraic integer.*

Proof. As $\mathrm{Aut}(X)$ is finite, there is a least positive integer m such that $P^m = I$. Therefore the eigenvalues of P are zeros of the monic polynomial $x^m - 1$ and $\mathrm{tr}_U(P)$ is an algebraic integer.

Let $u_1, \dots, u_n$ be an orthonormal basis for $\mathbb{R}^n$ such that $u_1, \dots, u_m$ is an orthonormal basis for U. Then

$$\mathrm{tr}_U(P) := \sum_{i=1}^{m} \langle u_i, Pu_i \rangle = \sum_{i=1}^{m} \langle Eu_i, PEu_i \rangle = \sum_{i=1}^{m} \langle u_i, EPEu_i \rangle$$

Next, $Eu_j = 0$ if $j > m$ and so the last sum is equal to

$$\sum_{i=1}^{n} \langle u_i, EPEu_i \rangle.$$

This equals the trace of EPE. Since $E^2 = E$ we have

$$\mathrm{tr}(EPE) = \mathrm{tr}(PEE) = \mathrm{tr}(PE),$$

whence the lemma follows. □

In general it is not at all convenient to compute $\mathrm{tr}_U(P)$. For strongly regular graphs it is easier though.

2.2 Lemma. *Let X be a strongly regular graph with valency k on v vertices. Let λ be an eigenvalue of X distinct from k with multiplicity m. Then*

$$E_\lambda = \frac{m}{v}\left(I + \frac{\lambda}{k}A - \frac{\lambda+1}{v-1-k}\overline{A}\right). \qquad \square$$

2.3 Corollary. *If $P \in \mathrm{Aut}(X)$ then*

$$\frac{m}{v}\left(\mathrm{tr}(P) + \frac{\lambda}{k}\,\mathrm{tr}(PA) - \frac{\lambda+1}{v-1-k}\,\mathrm{tr}(P\overline{A})\right)$$

is an algebraic integer. □

By way of example, consider the Petersen graph. Suppose P maps each vertex to one of its neighbours. Then

$$\mathrm{tr}(P) = \mathrm{tr}(P\overline{A}) = 0, \quad \mathrm{tr}(PA) = 10.$$

Now 1 is an eigenvalue of the Petersen graph with multiplicity five, so the corollary asserts that

$$\frac{5}{10}\frac{1}{3}10 = \frac{5}{3}$$

is an algebraic integer, which is false. Therefore no such automorphism exists. If P mapped each vertex to a distinct non-adjacent vertex then

$$\frac{5}{10}\frac{2}{6}10 = \frac{5}{3}$$

would have to be an algebraic integer. It follows that the Petersen graph is not a circulant.

We now apply this theory to a possible Moore graph of valency 57. This is a $(3250, 57; 0, 1)$ strongly regular graph, with eigenvalues -8, 7, and 57. We calculate that the multiplicity of 7 is 1729, whence for any element P of $\mathrm{Aut}(X)$ the trace of P on this eigenspace is

$$\frac{1729}{3250}\left(\mathrm{tr}(P) + \frac{7}{57}\,\mathrm{tr}(PA) - \frac{8}{3192}\,\mathrm{tr}(P\overline{A})\right) \tag{2}$$

must be an integer.

3. The Moore Graphs of Valency 57

Let G be the Moore graph of diameter two and valency 57 and let γ be an automorphism of it, with order two. Let P be the corresponding permutation matrix. Our aim is to prove that γ has exactly 56 fixed points, i.e., that $\mathrm{tr}(P) = 56$.

If γ fixes two points at distance two in G then it must also fix their unique common neighbour. Consider the subgraph F induced by $\mathrm{fix}(\gamma)$. If there is a vertex in F adjacent to all the other vertices, we will call F a *star*. If F is not a star then it must be a connected graph of diameter two and girth five; hence it is a Moore graph with 5, 10 or 50 vertices. As a star in G has at most 58 vertices, it follows that $|\mathrm{fix}(\gamma)| \leq 58$.

Suppose now that there is an edge uv in G with its ends swapped by γ. Then γ must swap the 56 neigbours of u distinct from v with the 56 neighbours of v distinct from u. This gives us a set of 56 paths of length three, each fixed by γ. Each path of length three lies in a unique 5-cycle, and thus determines a unique vertex at distance two from both u and v. Hence we have a set S of 56 vertices fixed by γ. Any vertex at distance two from u and v and not in S determines a path of length three on uv not fixed by γ. Therefore γ has exactly 56 fixed points, as required.

Assume then that there is no pair of adjacent vertices swapped by γ. (Thus $\mathrm{tr}(PA) = 0$.) If $x\gamma \neq x$, then γ must fix the unique common neighbour, z say, of x and $x\gamma$. For each fixed point z there at most 56 such points x. Thus we have

$$\mathrm{tr}(P\overline{A}) \leq 56\,\mathrm{tr}(P).$$

As $\mathrm{tr}(PA) = 0$, we also have

$$\mathrm{tr}(P) + \mathrm{tr}(P\overline{A}) = 3250.$$

Together these two equations imply that $\mathrm{tr}(P) \geq 58$, and accordingly $\mathrm{tr}(P) = 58$ and $\mathrm{tr}(P\overline{A}) = 3192$. From Equation (2) in the previous section we find now that the trace of P on the eigenspace with eigenvalue 7 is

$$\frac{1729}{3250}(58 - 8) = \frac{1729}{65} = \frac{133}{5}.$$

This is certainly not an integer, whence we conclude that any involution must swap the vertices in some edge, and hence have exactly 56 fixed points.

We can now show that $\mathrm{Aut}(G)$ is not transitive. Note first that any involution has exactly $3194/2 = 1597$ 2-cycles, and hence is an odd permutation. This implies that $|\mathrm{Aut}(G)|$ is not divisible by four. For suppose that this is not true, and let P be a Sylow 2-subgroup of G. Then $Z(P)$ is not the identity, hence we can find two involutions α and β in P. Then $\alpha\beta$ is again an involution. Now α and β are both odd, and therefore their product is even. Thus we are forced to conclude that $|\mathrm{Aut}(G)|$ is not divisible by four.

We now see that if $u \in V(G)$ then G_v contains a Sylow 2-subgroup of G (this is true even if Aut(G) has odd order) and thus that $|G : G_v|$ is odd. Since $|V(G)|$ is even, $\mathrm{Aut}(G)$ cannot be transitive.

Remark: The exposition in the last three sections is loosely based on the relevant part of the article by Peter Cameron in *Selected Topics in Graph Theory 2*, eds. Beineke and Wilson, Academic Press, London, 1983.

4. The Hoffman-Singleton Graph

There are at least three distinct constructions of the Hoffman-Singleton graph. One is based on the existence of a distance-regular 5-fold cover of $K_{5,5}$, which we will not mention again. A second is based on the existence of a distance-regular 6-fold cover of K_7; an advantage of this approach is that it also provides a uniqueness proof. This approach is due to Graham Higman, and is presented in Chapter 8 of *Graphs, Codes and Designs* by Cameron and Van Lint, London Math. Soc. Lecture Notes 43. The third approach is based on the assumption that the Hoffman-Singleton graph contains an independent set of size 15. For this, see Chapter 5 of *Algebraic Graph Theory* by Godsil and Royle.

5. The Moore Graphs of Diameter Greater Than Two

6. Why the Last Section is Empty

A Moore graph, we may recall, is a regular graph of girth g and diameter d such that $g = 2d + 1$. Suppose that G is a Moore graph with valency k and diameter d. Then G is distance-regular, and the quotient matrix of the distance partition with respect to any vertex is

$$B_{d+1} = \begin{pmatrix} 0 & k & & & \\ 1 & 0 & k-1 & & \\ & & \ddots & & \\ & & 1 & 0 & k-1 \\ & & & 1 & k-1 \end{pmatrix}$$

We assume henceforth that $k \geq 3$ and $d \geq 2$. Our goal will be to determine the multiplicities of the eigenvalues of $A(G)$ from B_{d+1}. The formulas we derive for these will be valid whenever both $k \geq 3$ and $d \geq 2$.

Define polynomials p_i for $i = -1, 0, \ldots$ by setting

$$p_{-1}(x) = p_0(x) = 1$$

and, if $m > 1$, taking $p_m(x)$ to be the characteristic polynomial of the matrix formed by the entries in the last m rows and columns of B_{d+1}. (Since d can be arbitrarily large, p_m is defined for all m.) Let $\varphi_{d+1}(x)$ be the characteristic polynomial of B_{d+1}. If θ is a zero of B_{d+1} then it is a zero of $A(G)$ with multiplicity

$$n \frac{p_d(\theta)}{\varphi'_{d+1}(\theta)}. \tag{1}$$

We intend to prove that if $d > 2$ then the expression in (1) cannot be an integer.

Our first step will be to determine the polynomials p_m. For all non-negative m,

$$p_{m+1}(x) = xp_m(x) - (k-1)p_{m-1}(x) \tag{2}$$

and

$$\begin{aligned} \varphi_{d+1}(x) &= xp_d(x) - kp_{d-1}(x) \\ &= p_{d+1}(x) - p_{d-1}(x). \end{aligned} \tag{3}$$

Let $P(x,t)$ be the generating function

$$\sum_{m \geq 0} p_m(x) t^m.$$

Then

$$\begin{aligned} P(x,t) - 1 &= \sum_{m \geq 0} p_{m+1}(x) t^m \\ &= \sum_{m \geq 0} (xp_m(x) - (k-1)p_{m-1}(x)) t^m \\ &= xtP(x,t) - (k-1)t^2(P(x,t) + t^{-1}) \end{aligned}$$

which yields

$$P(x,t) = \frac{1-(k-1)t}{1 - xt + (k-1)t^2}.$$

We have

$$\sum_{m \geq 0} \frac{\sin(m+1)\alpha}{\sin \alpha} t^m = (1 - 2t\cos\alpha + t^2)^{-1}. \tag{4}$$

(To prove this, multiply both sides by $1 - xt + (k-1)t^2$ and then compare coefficients.) If we set $\beta = \sqrt{k-1}$, $t = s/\beta$ and $x = 2\beta\cos\alpha$ then

$$P(2\beta\cos\alpha, s) = \frac{1-\beta s}{1-2s\cos\alpha + s^2}.$$

Comparing this with (4), we finally obtain

$$\frac{p_m(2\beta\cos\alpha)}{\beta^m} = \frac{\sin(m+1)\alpha - \beta\sin m\alpha}{\sin\alpha} \tag{5}$$

which, taken with (2) implies

$$\varphi_{d+1}(2\beta\cos\alpha) = \frac{\beta^{d-1}}{\sin\alpha}[\beta^2\sin(d+2)\alpha - \beta^3\sin(d+1)\alpha - \sin d\alpha + \beta\sin(d-1)\alpha].$$

Now

$$\begin{aligned}
&[\beta^2\sin(d+2)\alpha - \beta^3\sin(d+1)\alpha - \sin d\alpha + \beta\sin(d-1)\alpha] \\
&\quad = -\beta^2(\beta\sin(d+1)\alpha + \sin d\alpha) + (\beta^2(\sin d\alpha + \sin(d+2)\alpha) \\
&\qquad + \beta(\sin(d-1)\alpha + \sin(d+1)\alpha) - (\beta\sin(d+1)\alpha + \sin d\alpha) \\
&\quad = -(\beta^2+1)(\beta\sin(d+1)\alpha + \sin d\alpha) \\
&\qquad + 2\beta^2\sin(d+1)\alpha\cos\alpha - 2\beta\sin d\alpha\cos\alpha \\
&\quad = -(1-2\beta\cos\alpha+\beta^2)(\beta\sin(d+1)\alpha + \sin d\alpha)
\end{aligned}$$

whence we get

$$\varphi_{d+1}(2\beta\cos\alpha) = -\beta^{d-1}(1-2\beta\cos\alpha+\beta^2)\frac{\beta\sin(d+1)\alpha + \sin d\alpha}{\sin\alpha}. \tag{6}$$

6.1 Lemma. *Let G be a Moore graph of diameter d and valency k. Suppose that $\beta = \sqrt{k-1}$. Then $\theta = 2\beta\cos\alpha$ is an eigenvalue of $A(G)$ distinct from k if and only if*

$$\beta\sin(d+1)\alpha + \sin d\alpha = 0.$$

Proof. We have

$$(1-2\beta\cos\alpha+\beta^2) = k - 2\beta\cos\alpha$$

and so, from (6), any zero of φ_{d+1} distinct from k must be a zero of

$$(\beta\sin(d+1)\alpha + \sin d\alpha)/\sin\alpha,$$

viewed as a polynomial in $\cos\alpha$. □

6.2 Theorem. *Let G be a Moore graph on n vertices with diameter d and valency k. Suppose that g is the girth of G and that $\theta = 2\beta\cos\alpha$ is a zero of $\varphi_{d+1}(x)$ distinct from k. Then the multiplicity of θ as an eigenvalue of $A(G)$ is*

$$\frac{nk}{g}\frac{4(k-1)-\theta^2}{(k-\theta)\left(k+\frac{k-2}{g}+\theta\right)}.$$

Proof. Suppose $\theta = 2\beta\cos\alpha$. Differentiating (6) with respect to α and then choosing α so that θ is a zero of φ_{d+1} yields

$$\frac{2\beta\sin\alpha}{\beta^{d-1}}\varphi'_{d+1}(\theta) = (k-\theta)\frac{\beta(d+1)\cos(d+1)\alpha + d\cos d\alpha}{\sin\alpha}. \tag{7}$$

We also find that

$$p_d(\theta) = -k\beta^{d-1}\frac{\sin d\alpha}{\sin\alpha}$$

and therefore

$$\frac{p_d(\theta)}{\varphi'_{d+1}(\theta)} = -\frac{2k\beta\sin\alpha}{k-\theta}\frac{\sin d\alpha}{\beta(d+1)\cos(d+1)\alpha + d\cos d\alpha}. \tag{8}$$

Since $\beta\sin(d+1)\alpha + \sin d\alpha = 0$,

$$\beta(\sin d\alpha\cos\alpha + \cos d\alpha\sin\alpha) + \sin d\alpha = 0$$

and consequently

$$\tan d\alpha = -\frac{\beta\sin\alpha}{1+\beta\cos\alpha}. \tag{9}$$

Further

$$\cos(d+1)\alpha = \cos d\alpha\cos\alpha - \sin d\alpha\sin\alpha,$$

whence

$$\begin{aligned}\frac{\cos(d+1)\alpha}{\cos d\alpha} = \cos\alpha - \tan d\alpha\sin\alpha &= \cos\alpha + \frac{\beta\sin^2\alpha}{1+\beta\cos\alpha}\\ &= \frac{\beta+\cos\alpha}{1+\beta\cos\alpha}\end{aligned}$$

and so

$$\frac{\cos(d+1)\alpha}{\sin d\alpha} = \frac{\cos(d+1)\alpha}{\cos d\alpha}\frac{\cos d\alpha}{\sin d\alpha} = -\frac{\beta+\cos\alpha}{\beta\sin\alpha}. \tag{10}$$

Using (9) and (10) to simplify ??, we thus obtain

$$\frac{p_d(\theta)}{\varphi'_{d+1}(\theta)} = \frac{2k\beta^2 \sin^2\alpha}{k-\theta}[d(\beta^2+2\beta\cos\alpha+1)+\beta^2+\beta\cos\alpha]^{-1}.$$

Now

$$\sin^2\alpha = 1 - \frac{\theta^2}{4k-4}$$

and

$$\begin{aligned} d(\beta^2+2\beta\cos\alpha+1)+\beta^2+\beta\cos\alpha &= dk+d\theta+k-1+\frac{\theta}{2} \\ &= (2d+1)\left(k+\frac{k-2}{2d+1}+\theta\right)/2. \end{aligned}$$

The theorem follows easily from this. □

We now show that there are no Moore graphs of diameter greater than two. First we need some information about the location of the zeros of $\varphi_{d+1}(x)$. All zeros distinct from k are of the form $2\beta\cos\alpha$, where α satisfies

$$\beta\frac{\sin(d+1)\alpha}{\sin\alpha} + \frac{\sin d\alpha}{\sin\alpha} = 0.$$

Let the solutions of this equation be $\alpha_1,\ldots,\alpha_d$, in decreasing order, and set θ_i equal to $2\beta\cos\alpha_i$. (Note that 0 is not a solution.)

6.3 Lemma. *If G has diameter d and $d\ge 3$ then $\theta_i+\theta_{d+1-i}<0$ and $-1<\theta_i+\theta_{d+1-i}<0$.*

Proof. We see that $\sin(d+1)\alpha/\sin\alpha$ has zeros at $\pi i/(d+1)$ for $i=1,\ldots d$, and $\sin d\alpha/\sin\alpha$ has $d-1$ solutions $\pi i/d$ for $i=1,\ldots,d-1$. It follows that α_i must lie in the open interval

$$\left(\frac{\pi i}{d+1},\frac{\pi i}{d}\right).$$

This implies that α_i cannot be $\pi/2$, and so the eigenvalues θ_i are all non-zero. Since $\cos\alpha$ is a strictly decreasing function on $[0,\pi]$, we have

$$\cos\alpha_i+\cos\alpha_{d+1-i} < \cos\frac{\pi i}{d+1}+\cos\frac{\pi(d+1-i)}{d+1} = 0.$$

To complete the proof we observe that $\operatorname{tr} B_{d+1}=k-1$. As k is an eigenvalue of B_{d+1}, it follows that $\sum_i\theta_i=-1$. As none of the θ_i are zero, we deduce that $-1<\theta_i+\theta_{d+1-i}$ if $d\ge 3$. □

The previous lemma implies that $\varphi_{d+1}(x)$ has non-integer zeros when $d > 2$. We derive a contradiction to this. Define $R(\theta)$ by

$$R(\theta) = \frac{4k-4-\theta^2}{(k-\theta)(f+\theta)}. \tag{11}$$

Thus $nkR(\theta)/g$ is the multiplicity of θ, and so $R(\theta)$ is rational. Suppose θ is irrational. From (11),

$$(R(\theta)-1)\theta^2 + R(\theta)(f-k)\theta + (4k-4-R(\theta)fk) = 0.$$

The polynomial on the left side here must be divisible by the minimal polynomial of θ, which is a monic polynomial. It follows that

$$\frac{R(\theta)(f-k)}{R(\theta)-1} \tag{12}$$

is the linear term in the minimal polynomial of θ, and it is therefore an integer. But

$$\frac{R(\theta)(f-k)}{R(\theta)-1} = \frac{4k-4-\theta^2}{\theta-(k+g(k-2))}.$$

Since $\theta \leq k$ this implies

$$\left|\frac{R(\theta)(f-k)}{R(\theta)-1}\right| \leq \frac{4k-4}{g(k-2)} < 1$$

provided that both $g \geq 9$ and $k \geq 3$. Hence (12) must be zero, whence $R(\theta) = 0$ and θ has multiplicity zero!

It only remains to dispose of the case when $g = 7$. We calculate that

$$\varphi_4(x) = (x-k)(x^3 + x^2 - 2(k-1)x - (k-1))$$

whence, for any zero $\theta \neq k$

$$k - 1 = \frac{\theta^2(\theta+1)}{2\theta+1}. \tag{13}$$

If θ is an integer then $2\theta+1$ must have non-trivial factors in common with θ or $\theta+1$, which is impossible. So all roots of (13) must be irrational, and therefore they must all have the same multiplicity. On the other hand, the equation $R(\theta) = \mu$ has at most two solutions and so there cannot be three distinct eigenvalues with the same multiplicity.