Map Graphs

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Let X and F be graphs. We say that a map f from $V(X)$ to $V(F)$ is *conical* if, for each vertex u in X there is a vertex v in F such that $f(N_X(u)) \subseteq N_F(v)$. A conical map f from X to F is *locally onto* if for each vertex u in X there is a vertex v in F such that $f(N_X(u)) = N_F(v)$. We will only consider locally-onto maps when F is reduced, in which case the vertex v is unique.

We note that any homomorphism from X to F is conical. Any automorphism of F is locally onto. If X is uniquely n-colourable, then any homomorphism $f : X \to K_n$ is locally onto. The graphs $X \times K_2$ and $Y \times K_2$ are isomorphic if and only if there is a locally-onto bijection from $V(X)$ to $V(Y)$.

1.1 Lemma. *Let X and F be graphs. The neighbourhood of a vertex f in F^X is non-empty if and only if f is conical.* $\square$

1.2 Lemma. *Let X and F be graphs. If F is reduced and the vertex f in F^X is conical and locally onto, it has a unique neighbour in F^X.* $\square$

Consider $K_3^{K_4}$, and identify the vertices of this graph with the ternary sequences of length four. Then 1122 is conical and locally onto. Its unique neighbour is the constant map 0000.

Let f be a homomorphism from C_m to C_n. Suppose that $V(C_m) = \{0, 1, \dots, m-1\}$ and i is adjacent to $i-1$ and $i+1$ (mod n), for each vertex i. We say that f *folds* C_m at i if $f(i+1) = f(i-1)$. (Thus f has no folds if and only if it locally onto.)

1.3 Lemma. *Suppose m and r are odd and $m > r \geq 3$. Any homomorphism from C_{2m} to C_r has an even number of folds.* □

Now, when m is odd, $C_{2m} \cong K_2 \times C_m$. Hence each homomorphism $f : C_{2m} \to C_r$ determines an arc (f_0, f_1) in $C_r^{C_m}$. Here f_0 and f_1 are the restrictions of φ to the two colour classes of C_{2m}. Define the *parity* of f_i $(i = 0, 1)$ to be the parity of the number of vertices in its colour class not folded by φ. Since the sum of the parities is even, f_0 and f_1 must have the same parity.

On the other hand, any conical map f from $V(X)$ to $V(F)$ lies in an edge of F^X. Consequently it has a well-defined parity, which is constant on the component of F^X that contains f. The constant maps have even parity. We will see that homomorphisms have odd parity.

Suppose X, Y and F are graphs and ψ is a homomorphism from $X \times Y$ to F. If $x \in V(X)$ then ψ determines a map, which we denote for now by $\psi(x, ?)$, that sends y in $V(Y)$ to $\psi(x, y)$ in F. Note that $\psi(x, ?) \in F^Y$, and that if x' in $V(X)$ is adjacent to x, then $\psi(x', ?)$ is adjacent to $\psi(x, ?)$ (in F^Y). Similarly, if $y \in V(X)$, then $\psi(?, y) \in F^X$ and adjacent vertices in Y determine adjacent elements of F^X. If we view ψ as a matrix with rows indexed by $V(X)$ and columns indexed by $V(Y)$, and with vertices of F as entries. Then the rows of this matrix describe vertices of F^Y and the columns vertices of F^X.

1.4 Lemma. *Suppose m, n and r are odd. If φ is a homomorphism from $C_m \times C_n$ to C_r, then the maps determined the rows of φ have the opposite parity to the maps determined by the columns.*

Proof. For now this is an exercise. □

To determine the parity of homomorphisms from C_m to C_r (when m and r are odd, let ψ be a homomorphism from C_m to C_r. Then the map Ψ that sends (i, j) in $V(C_m \times C_m)$ to $\psi(j)$ is a homomorphism into C_r.

Here each row of Ψ is equal to ψ, while each column is a constant map. Since constant maps have even parity, the lemma implies that ψ must have odd parity.

1.5 Lemma. *Let X be a graph that is not 3-colourable. If $f \in V(K_3^X)$ and f is not isolated, there is an odd circuit C in X such that the restriction of f to C is even.*

Proof. We remark that if C is a circuit in X then $C \to X$ and so $K_3^X \to K_3^C$. The restriction of f to C is the image of f under this homomorphism.

Let $Z(f)$ denote the set of vertices x in X such that there is y in $V(X)$ adjacent to X and $f(x) = f(y)$. Let g be a vertex in K_3^X adjacent to f. If $Z(f)$ does not contain an odd circuit, then it is bipartite, and we can partition its vertices into two sets Z_0 and Z_1. Define a map h from $V(X)$ to K_3 by

$$h(u) = \begin{cases} f(u), & u \notin Z_1; \\ g(u), & \text{otherwise.} \end{cases}$$

It is easy to verify that h is homomorphism, and so X is 3-colourable.

Since X is not 3-colourable, we deduce that $Z(f)$ is not bipartite. Therefore it contains an odd circuit, C say.

If $x \in V(C)$ then there is y in $V(X)$ adjacent to x such that $f(y) = f(x)$. Since $g \sim f$ and $x \sim y$, we that $g(x) \neq f(y)$. Therefore $g(x) \neq f(x)$, for any vertex x in C. It is an exercise to that this implies that f is even on C. □

1.6 Theorem. *If $X \times Y$ is 3-colourable, then X or Y is 3-colourable.*

Proof. We prove this by contradiction. We may assume there are connected graphs X and Y, neither 3-colourable, such that there is a homomorphism $\psi : X \times Y \to K_3$. If $y \in V(Y)$ then, since Y is connected, $\psi(?, y)$ is a vertex of F^X with positive valency. Hence there is an odd cycle C in X on which $\psi(?, y)$ is even. Choose a vertex x in C. Then similarly, there is a cycle D in F^Y on which $\psi(x, ?)$ is even.

Since Y is connected and $\psi(?, y)$ is even on C, each column of the induced homomorphism from $C \times Y$ to K_3 is even. Likewise, each row of the induced homomorphism from $X \times D$ to K_3 is even, and this forces us to the conclusion that both the rows and columns of the induced homomorphism from $C \times D$ to K_3 are even.

Let $h(X, Y)$ denote the number of homomorphisms from X to Y.

1.7 Lemma. $F^{K_2} \times K_2 \cong F \times F \times K_2$.

Proof. (Here F^2 denotes $F \times F$ and, later, $2Z$ denotes the disjoint union $Z \cup Z$.) We have

$$h(Z, F^{K_2} \times K_2) = h(Z, F^{K_2})\, h(Z, K_2)$$

and

$$h(Z, F^2 \times K_2) = h(Z, F)^2\, h(Z, K_2).$$

If Z is not bipartite, the right side of each of these expressions is zero. If Z is bipartite then $Z \times K_2 \cong 2Z$ and

$$h(Z, F^{K_2}) = h(Z \times K_2, F) = h(2Z, F) = h(Z, F)^2.$$

It follows that, for all graphs Z,

$$h(Z, F^{K_2} \times K_2) = h(Z, F^2 \times K_2). \qquad \square$$