Notes on Kasami Graphs

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Abstract

We demonstrate that the Kasami graphs form an infinite family of counterexamples to the now-disproved rank colouring conjecture.

1 Introduction

The rank of the adjacency matrix of a graph over the real field is a natural parameter associated with the graph, and there has been substantial interest in relating this to other graph theoretic parameters. In particular the relationship between the rank and chromatic number of a graph, and the rank and the maximum number of vertices of a graph has been extensively studied. This was at least in part motivated by a conjecture of Van Nuffelen [3] that

$$\chi(X) \leq \mathrm{rk}(X)$$

where $\chi(X)$ is the chromatic number of the graph. This conjecture was proved incorrect by Alon & Seymour [1] who found a counterexample with rank 29, and chromatic number 32.

Their counterexample turns out to be the complement of the folded 7-cube. The folded 7-cube Υ_7 is a distance-regular graph with intersection array

$$\{7, 6, 5; 1, 2, 3\}$$

and eigenvalues

$$7^1, 3^{21}, -1^{35}, -5^7.$$

This graph is 7-regular, and because $b_1 = 6$, it follows that there are no edges in the neighbourhood of any vertex, and hence that Υ_7 is triangle free.

If X is a k-regular graph on n vertices with eigenvalues

$$k = \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$$

then its complement $\overline{X}$ has eigenvalues $n - k - 1$ and

$$-1 - \theta_2 \leq \cdots \leq -1 - \theta_n$$

Therefore the graph $\overline{\Upsilon_7}$ has eigenvalues

$$56^1, -4^{21}, 0^{35}, 4^7$$

and so has rank 29.

As Υ_7 is triangle-free, the size of any independent set in $\overline{\Upsilon_7}$ is at most 2, and so $\chi(\overline{\Upsilon_7})) \geq 32$. It is clear that Υ_7 has a perfect matching with 32 edges and so

$$\chi(\overline{\Upsilon_7}) = 32.$$

2 Kasami graphs

It is natural to attempt to generalize this example, but the obvious generalization to the folded n-cubes for $n \neq 7$ does not provide any further examples.

However Dom de Caen suggested that the folded 7-cube is also a member of the family of graphs related to the *Kasami codes.* There are several families of distance-regular graphs related to these codes (page 358 of Brouwer, Cohen & Neumaier [2]). If q is a power of 2, then there is a distance regular graph $K_{q,j}$ on q^{4j+2} vertices with intersection array

$$\{q^{2j+1} - 1,\ q^{2j+1} - q,\ q^{2j}(q-1) + 1;\ 1,\ q,\ q^{2j} - 1\}.$$

This graph has four distinct eigenvalues, which are eigenvalues of the 4×4 matrix

$$\begin{pmatrix} a_0 & b_0 & 0 & 0 \\ c_1 & a_1 & b_1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{pmatrix}$$

For the Kasami graphs with the above parameters, this matrix is

$$\begin{pmatrix} 0 & q^{2j+1}-1 & 0 & 0 \\ 1 & q-2 & q^{2j+1}-q & 0 \\ 0 & q & q^{2j}-q-2 & q^{2j}(q-1)+1 \\ 0 & 0 & q^{2j}-1 & q^{2j}(q-1) \end{pmatrix}$$

It is straightforward (with the aid of Maple) to verify that the characteristic polynomial of this matrix is

$$(x+1)(x-(q^{2j+1}-1))(x-(q^{j+1}-1))(x+(q^{j+1}+1))$$

and therefore the eigenvalues of the Kasami graph are

$$q^{2j+1}-1, q^{j+1}-1, -1, -(q^{j+1}+1)$$

The valency $q^{2j+1}-1$ has multiplicity one, but we need to calculate the multiplicities of the other 3 eigenvalues. Suppose that the eigenvalues $q^{j+1}-1$, -1 and $-(q^{j+1}+1)$ have multiplicities a, b and c respectively. The total number of eigenvalues is equal to the number of vertices of $K_{q,j}$ and so we have

$$a+b+c = q^{4j+2}-1.$$

Setting A to be the adjacency matrix of the graph, we have the the sum of the eigenvalues is $\mathrm{tr}A$, which is equal to 0, and so

$$a(q^{j+1}-1) - b - c(q^{j+1}+1) = 1 - q^{2j+1}.$$

Finally, the sum of the squares of the eigenvalues is $\mathrm{tr}A^2$, which is equal to the sum of the valencies of the vertices, and so

$$a(q^{j+1}-1)^2 + b + c(q^{j+1}+1)^2 = q^{4j+2}(q^{2j+1}-1) - (q^{2j+1}-1)^2$$

This system of linear equations can be solved, and has the following (unique) solution.

$$\begin{aligned} a &= (q^{4j+1} + q^{3j+1} - q^{2j} - q^j)/2 \\ b &= q^{4j+2} - q^{4j+1} + q^{2j} - 1 \\ c &= (q^{4j+1} - q^{3j+1} - q^{2j} + q^j)/2 \end{aligned}$$

Each of these expressions is divisible by the valency $q^{2j+1} - 1$ and so we finally get

$$\begin{aligned} a &= (q^{2j+1} - 1)q^j(q^j + 1)/2 \\ b &= (q^{2j+1} - 1)(q^{2j+1} - q^{2j} + 1) \\ c &= (q^{2j+1} - 1)q^j(q^j - 1)/2 \end{aligned}$$

3 Rank and Chromatic Number

Following the example of the folded 7-cube, we now consider the complements of the Kasami graphs. Each of the eigenvalues equal to -1 of the graph $K_{q,j}$ becomes a zero eigenvalue of its complement. Hence

$$\begin{aligned} \mathrm{rk}(\overline{K_{q,j}}) &= q^{4j+2} - (q^{4j+2} - q^{4j+1} + q^{2j} - 1) \\ &= q^{4j+1} - q^{2j} + 1 \end{aligned}$$

The chromatic number of the complement of a Kasami graph is only easy to determine when $q = 2$. In that situation $K_{2,j}$ is triangle free and so the same argument as before shows that the chromatic number of its complement is q^{4j+1}.

Therefore, for $q = 2$, and any $j > 0$, we have a family of graphs $\overline{K_{2,j}}$ such that

$$\begin{aligned} \mathrm{rk}(\overline{K_{2,j}}) &= 2^{4j+1} - 2^{2j} + 1 \\ \chi(\overline{K_{2,j}}) &= 2^{4j+1} \end{aligned}$$

and hence the chromatic number exceeds the rank by $2^{2j} - 1$

Question (Open?): Can the chromatic number of the graphs $\overline{K(4,j)}$ be determined?

These graphs certainly give additional examples of graphs with chromatic number exceeding rank, but by how much is not entirely clear.

References

[1] Alon, N.; Seymour, P. D. A counterexample to the rank-coloring conjecture. *J. Graph Theory* **13** (1989), no. 4, 523–525.

[2] Brouwer, A.; Cohen, A and Neumaier, A. Distance-Regular Graphs, Springer-Verlag, 1989.

[3] Van Nuffelen, Cyriel. The rank of the adjacency matrix of a graph. *Bull. Soc. Math. Belg. Sér. B* **35** (1983), no. 2, 219–225.