The Crossing Number of K_{11} is 100

January 29, 2007

Shengjun Pan[1] and R. Bruce Richter[2]
University of Waterloo

Abstract

The crossing number of K_n is known for $n \leqslant 10$. We develop several simple counting properties that we shall exploit in showing by computer that $cr(K_{11}) = 100$, which implies that $cr(K_{12}) = 150$. We also determine the numbers of non-isomorphic optimal drawings of K_9 and K_{10}.

1 Introduction

Guy [4] conjectured that the crossing number $cr(K_n)$ of the complete graph K_n is equal to

$$Z(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor .$$

He proved this for $n \leqslant 10$ and also determined that, for $n = 4, 5, 6, 7, 8$, the number of optimal drawings of K_n is $1, 1, 1, 1, 5, 3$, respectively.

In general, we know that $.8594\, Z(n) \leq cr(K_n) \leq Z(n)$. The latter inequality follows from the existence of a drawing (see [8] for one example) and the former is proved in [2].

In this paper we use some simple counting properties to provide the basis of an algorithm which we programmed to show that $cr(K_{11}) = Z(11)$. In particular, we determine that K_9 and K_{10} have 3080 and 5679 optimal drawings, respectively. Along the way, we answer affirmatively an open question of Brodsky, Durocher and Gethner [1] by showing that every good drawing (to be defined in the next section) of K_n induces a 3-connected planar graph.

2 Theory

In this section, we provide the simple theoretical background required for our algorithm.

---

[1]Current address: University of California, San Diego

[2]NSERC support for the research of RBR is gratefully acknowledged.

A *drawing* of a graph G consists of a set of distinct points of the sphere, one for each vertex, and a simple curve for each edge, joining the points representing the ends of the edge, without any vertex-point in its interior. Two drawings are *isomorphic* if there is a homeomorphism of the sphere to itself mapping the image of one drawing to the image of the other such that vertex-points are mapped to vertex-points.

A *crossing* in a drawing D of a graph G is an ordered pair $(x, \{e_1, e_2\})$ consisting of a non-vertex-point x of the sphere and distinct edges e_1, e_2 of G whose representing curves both contain x. The *crossing number* $cr(D)$ is the number of crossings of D. Choosing the vertex-points as the corners of a convex polygon and using line-segments for the edges shows that G has a drawing with finitely many crossings. We shall only be concerned with drawings having finitely many crossings. The *crossing number* of a graph G, denoted by $cr(G)$, is the minimum $cr(D)$, taken over all drawings D of G. A drawing is *optimal* if its number of crossings achieves the minimum.

A drawing is *satisfactory* if two edges share at most one common point, including endpoints, and each non-vertex intersection between two edges is a transverse crossing. It is an easily proved folklore fact that every optimal drawing is satisfactory. A satisfactory drawing is *good* if no non-vertex point of the plane is in three edge-representing arcs. Likewise, it is an easy and well-known fact that if D is a satisfactory drawing of a graph G, then there is a good drawing D' of G having the same number of crossings. In particular, some optimal drawing of G is good.

The main theoretical result we need is the following.

Theorem 1.

1) *For $n \leqslant 8$, every optimal drawing of K_n contains an optimal drawing of K_{n-1}.*

2) *A good optimal drawing of K_9 contains a good drawing of K_8 with at most 20 crossings. Any good drawing of K_8 with at most 20 crossings contains an optimal drawing of K_7.*

3) *A good drawing of K_{11} with fewer than 100 crossings contains a good drawing of K_{10} with at most 62 crossings. Any good drawing of K_{10} with at most 62 crossings contains an optimal drawing of K_9.*

We need the following facts to prove Theorem 1; the first is due to Kleitman [5], while the other two are standard counting results.

Lemma 2. *If n is odd, then the number of crossings in any good drawing of K_n has the same parity as $Z(n)$.*

Deleting a vertex of a good drawing of K_n produces a drawing of K_{n-1}. Any crossing occurs in $n-4$ of these n subdrawings. That is,

Lemma 3. *For $n \geqslant 5$,*

$$cr(K_n) \geqslant \left\lceil \frac{n}{n-4} \cdot cr(K_{n-1}) \right\rceil .$$

The *responsibility* of a vertex v in a drawing is the total number of edge crossings of edges incident with v. Notice that the sum of all the responsibilities counts every crossing four times. That is,

Lemma 4. *Let G be a graph with n vertices and let D be a good drawing of G with $cr(D)$ crossings. Then there is a vertex v of G with responsibility at least $\lceil 4cr(D)/n \rceil$.*

Proof of Theorem 1. In every case, we compute $\lceil 4cr(D)/n \rceil$. For (1), it is easy to verify that, for $n \leqslant 8$, $\lceil 4cr(K_n)/n \rceil = Z(n) - Z(n-1)$, while $\lceil 4cr(K_9)/9 \rceil = 12$ yields the first part of (2). For the second part of (2), $\lceil 4 \cdot 20/8 \rceil = 10$ shows that every good drawing of K_8 with at most 20 crossings has a drawing of K_7 with at most 10 crossings. Now Lemma 2 implies any such drawing has in fact at most 9 crossings.

For (3), Lemma 3 implies that any good drawing D of K_{11} has at least 95 crossings. Lemma 2 implies $cr(D)$ is even. So $cr(D) < 100$ implies $cr(D) \in \{96, 98\}$. If $cr(D) = 96$, then $\lceil 4 \cdot 96/11 \rceil = 35$, so some K_{10} is drawn in D with at most $96 - 35 = 61$ crossings. If $cr(D) = 98$, then $\lceil 4 \cdot 98/11 \rceil = 36$, so some K_{10} is drawn in D with at most $98 - 36 = 62$ crossings. Finally, if D is a drawing of K_{10} with $cr(D) \in \{60, 61, 62\}$, then $cr(D) - \lceil 4cr(D)/10 \rceil$ is either $60 - 24 = 36$ or $61 - 25 = 36$ or $62 - 25 = 37$. Thus, some subdrawing of K_9 has at most 37 crossings; by Lemma 2 it has at most 36. □

3 Algorithm

In this section, we describe our algorithm for showing $cr(K_{11}) \geqslant 100$. It is based on extending good drawings of K_{n-1} to good drawing of K_n.

A *face* of a drawing D of a graph G in the sphere $\mathbb{S}$ is a component of $\mathbb{S} \setminus D$. If $cr(D) < \infty$ and G is connected, then each face is homeomorphic to an open disc. The *induced planar graph* of D, denoted as G_D, is the graph with vertices $V(G_D) = V(G) \cup \{\text{crossings}\}$, where the edges are the components of $D \setminus V(G_D)$ and v is incident to e if and only if v is in the closure of e. Any vertex in $V(G)$ is a *non-crossing (vertex)* of G_D, and any crossing of D is a *crossing (vertex)* of G_D. We define the *dual graph* of a good drawing D to be the dual graph of the induced planar graph G_D.

Let $\mathcal{D}$ be a set of good drawings. In this section we shall use the following notation:

$cr(\mathcal{D})$:	the minimum number of crossings over all the drawings in $\mathcal{D}$;
D_F^+ :	the set of all good drawings obtained by inserting a new vertex v in a face F of a good drawing D, and drawing new edges from v to all vertices of D;
$\mathcal{D}^+$:	the set of all good drawings so that deleting some vertex and its incident edges leaves a drawing in $\mathcal{D}$;
$\mathcal{K}_n^c$:	the set of all good drawings of K_n with c crossings;
$\mathcal{K}_n^{\leqslant c}$:	the set of all good drawings of K_n with at most c crossings.

Obviously, $\mathcal{D}^+ = \bigcup_{D\in\mathcal{D}} \left(\bigcup_{F\in\mathcal{F}(D)} D_F^+ \right)$, where $\mathcal{F}(D)$ denotes the set of faces of a drawing D.

3.1 Idea

Given a good drawing D of K_n with vertices v_i, $i = 1, 2, \cdots, n$, and a face $F \in \mathcal{F}(D)$, let $d(F, v_i)$ be the minimum distance in the dual graph from F to the faces incident to v_i. Then $cr(D) + \sum_{i=1}^n d(F, v_i)$ is a lower bound $Lb(D, F)$ for $cr(D_F^+)$. Therefore, letting $Lb[\mathcal{D}] = \min\{Lb(D, F) \mid D \in \mathcal{D}, F \in \mathcal{F}(D)\}$,

$$cr(\mathcal{D}^+) \geqslant Lb[\mathcal{D}]. \tag{1}$$

Here is how we will show $cr(K_{11}) \geqslant 100$. Theorem 1(3) says $\mathcal{K}_{11}^{\leqslant 99} \subseteq \left(\mathcal{K}_{10}^{\leqslant 62}\right)^+$, so if $\mathcal{K}_{11}^{\leqslant 99} \neq \varnothing$, then Inequality (1) implies that

$$99 \geqslant cr(\mathcal{K}_{11}^{\leqslant 99}) \geqslant Lb\left[\mathcal{K}_{10}^{\leqslant 62}\right].$$

The algorithm will show that $Lb\left[\mathcal{K}_{10}^{\leqslant 62}\right] \geqslant 100$, giving a contradiction. Thus $\mathcal{K}_{11}^{\leqslant 99} = \varnothing$, i.e., $cr(K_{11}) \geqslant 100$.

We give an algorithm for generating all the drawings in $\mathcal{K}_{10}^{\leqslant 62}$ as follows.

Input: a set $\mathcal{D}$ of good drawings of K_n ($n \geqslant 4$), and an integer $\delta \in \{0, 1, 2\}$.

Output: all the drawings in $\mathcal{D}^+$ having at most $cr(K_{n+1}) + \delta$ crossings.

Applying the following inputs $(\mathcal{D}, \delta)$, one after the other, to the algorithm:

$$(\mathcal{K}_4^0, 0),\ (\mathcal{K}_5^1, 0),\ (\mathcal{K}_6^3, 1),\ (\mathcal{K}_7^9, 2),\ (\mathcal{K}_8^{\leqslant 20}, 0)\ , (\mathcal{K}_9^{36}, 2),$$

by Theorem 1, we shall get the sequence of sets of drawings:

$$\mathcal{K}_5^1,\ \mathcal{K}_6^3,\ \mathcal{K}_7^9,\ \mathcal{K}_8^{\leqslant 20},\ \mathcal{K}_9^{36},\ \mathcal{K}_{10}^{\leqslant 62}.$$

3.2 Generating Drawings

For each D in the input $(\mathcal{D},\delta)$ and each face F in $\mathcal{F}(D)$, a good drawing in D_F^+ consists of D plus the new vertex and curves joining the new vertex to the vertices in D. In general, these curves correspond to walks in the dual graph. Claim 1 below shows that, in our context, it suffices to consider paths.

The algorithm first adds a new vertex v in F, then searches for all the paths in the dual graph from v to a face incident to v_i with length at most $d(F,v_i)+2$. Denote such a set of paths by S_i. The algorithm then checks each combination $(P_1,P_2,\cdots,P_n)$, where $P_i \in S_i$. For each combination $(P_1,P_2,\cdots,P_n)$, if

1) any two new edges can been drawn without crossing each other,

2) any new edge can be drawn by crossing each edge in D at most once, and

3) the total length is no more than $cr(K_{n+1})+\delta-cr(D)$,

then $(P_1,P_2,\cdots,P_n)$ determines a way to add new edges to D so that the new drawing of K_{n+1} is valid for output.

The following claim shows that it is sufficient to search for paths in the dual graph.

Claim 1. *Let $\mathcal{D}$ be a set of good drawings of K_n, let $\delta \in \{0,1,2\}$, let $D \in (\mathcal{D},\delta)$, and let $F \in \mathcal{F}(D)$. Let $D' \in D_F^+$, let v be the new vertex and let $(W_1,W_2,\cdots,W_n)$ be the dual walks in D corresponding to the curves incident with v. Then each W_i is a path.*

We first prove a lemma, which answers affirmatively an open question in [1].

Lemma 5. *For $n \geq 4$, the induced planar graph G_D of a good drawing D of K_n is 3-connected.*

Proof. For $n=4$, there are only two good drawings of K_4; one is optimal and the other has a unique crossing. The corresponding induced planar graphs are K_4 and a 4-wheel, which are both 3-connected.

For $n \geqslant 5$ suppose otherwise there is a separating set $S \subseteq V(G_D)$ with size at most 2. Then there is a partition C_1,C_2 of $V(G_D)\setminus S$ into nonempty sets so that there is no edge of G_D between C_1 and C_2.

Let m_i be the number of non-crossings in C_i, $i=1,2$, and m_0 be the number of non-crossings in S. Then $m_1+m_2+m_0=n$. First we prove that $m_1>0$ and $m_2>0$. Suppose $m_1=0$. Let v be a crossing in C_1. Then in G_D there are four internally disjoint paths from v to four non-crossings $u_j \in S\cup C_2, j=1,2,3,4$. Hence each of these paths goes through a vertex in S. However, $|S| \leqslant 2$, a contradiction. So $m_1>0$. Similarly $m_2>0$.

Let $u_1,u_2,\cdots,u_{m_1}$ be non-crossings in C_1, and $v_1,v_2,\cdots,v_{m_2}$ be non-crossings in C_2. Then, for each $j \in \{1,2,\cdots,m_1\}$ and $k \in \{1,2,\cdots,m_2\}$, there is $\{u_j,v_k\}$-path P_{jk} in G_D going through only crossings. Since D is a good drawing, $P_{j1},P_{j2},\cdots,P_{jm_2}$ are internally disjoint for any fixed j. So

$m_2 \leqslant |S| - m_0 \leqslant 2 - m_0$. Similarly $m_1 \leqslant 2 - m_0$. Thus $n = m_1 + m_2 + m_0 \leqslant 4 - m_0 \leqslant 4$, which contradicts the assumption that $n \geqslant 5$. □

Proof of Claim 1. For the drawing D of K_n in the input $(\mathcal{D}, \delta)$, let D' be the good drawing of K_{n+1} determined by the combination of walks $(W_1, \cdots, W_n)$. Since D is a good drawing, G_D is a simple graph and, since D' is good, no W_i can have one face F both immediately preceded and succeeded by the same face F'. By Lemma 5, G_D is 3-connected. It is well known that the dual graph of any simple and 3-connected graph is simple and 3-connected (for example, see Theorem 2.6.7 in [7]), so the dual graph of G_D, i.e., the dual graph of D, is also simple and 3-connected.

Suppose some walk W_i is not a path. Then there is a subsequence $(F_j, F_{j+1}, \cdots, F_{j+k})$ in W_i such that $F_j = F_{j+k}$. Removing the subsequence $(F_{j+1}, \cdots, F_{j+k})$ from W_i gives a new walk W_i'. Replacing W_i with W_i' gives a different new drawing D'' of K_{n+1}. By our earlier remark, $k \geqslant 3$. If D'' remains a good drawing, $D'' \in \mathcal{D}^+$. Then

$$cr(D'') \leqslant cr(D) - k \leqslant cr(\mathcal{D}_n^+) + \delta - 3 < cr(\mathcal{D}_n^+),$$

which contradicts that $D'' \in \mathcal{D}^+$. In D'' there may be crossings between new edges. These are easily eliminated to give a good drawing $D''' \in \mathcal{D}^+$, which also leads to a contradiction by a similar argument. □

3.3 Checking Isomorphism

To determine the number of non-isomorphic drawings, we need to tell a computer how to check drawing isomorphism. For any two good drawings D^1, D^2 of K_n, $n \geqslant 5$, by Lemma 5, their induced planar graphs G_{D^1}, G_{D^2} are 3-connected, and obviously simple. According to Whitney's Theorem (e.g., see Theorem 4.3.2 in [3], page 96), a planar graph has a unique drawing, up to isomorphism, if it is simple and 3-connected. Then the problem is reduced to checking graph isomorphism.

In our code we use **nauty** (**n**o **aut**omorphisms, **y**es?), created by McKay [6], to determine graph isomorphism. **nauty** is a set of very efficient programming procedures for calculating the automorphism group of a vertex-colored graph. It can be used to test graph isomorphism.

3.4 Results

The results from our code show that, for $n = 5, 6, 7, 8$, the number of non-isomorphic drawings of K_n is indeed $1, 1, 5, 3$ respectively. Our results also show that there are 3080 optimal drawings of K_9 and 5679 optimal drawings of K_{10}, up to isomorphism.

With all the drawings in $\mathcal{D}_{10}^{\leqslant 62}$, $Lb\left[\mathcal{D}_{10}^{\leqslant 62}\right]$ can be calculated straightforwardly by applying any shortest path algorithm. The value is 100 from our results. Therefore, $cr(K_{11}) \geqslant 100$, and so $cr(K_{11}) = 100$.

Moreover, by Lemma 3, it is easy to show that, for any odd n, $cr(K_n) = Z(n)$ implies $cr(K_{n+1}) = Z(n+1)$. Therefore $cr(K_{12}) = Z(12) = 150$.

We also found that there are many optimal drawings of K_9 which generate no optimal drawing of K_{10}. It was known that not every optimal drawing of K_n contains some optimal drawing of K_{n-1} (see [4] for an example). Now we also know that not every optimal drawing of K_n is contained in some optimal drawing of K_{n+1}.

Our code can be downloaded at

www.math.uwaterloo.ca/~brichter/pubs/publications.html.

References

[1] A. Brodsky, S. Durocher, and E. Gethner. The rectilinear crossing number of K_{10} is 62. *eprint arXiv:cs.DM/0009023/*, September 2000.

[2] E. de Klerk, D.V. Pasechnik, A. Schrijver, Reduction of symmetric semidefinite programs using the regular *-representation, Mathematical Programming Series B, to appear.

[3] R. Diestel, *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin · Heidelberg, 3rd edition, 2005.

[4] R. K. Guy. Crossing numbers of graphs. In *Graph Theory and Applications*, volume 303 of *Lecture Notes in Mathematics*, pages 111–124. Springer-Verlag, Berlin · Heidelberg · New York, May 1972.

[5] D. J. Kleitman. A note on the parity of the numbers of crossings of a graph. *J. Combin. Theory, Ser. B*, 21:88–89, 1976.

[6] B. D. McKay. The nauty page. http://cs.anu.edu.au/~bdm/nauty/.

[7] B. Mohar and C. Thomassen. *Graphs on Surfaces*. The Johns Hopkins University Press, Baltimore and London, 2001.

[8] R. B. Richter and C. Thomassen. Relations between crossing numbers of complete and complete bipartite graphs. *Amer. Math. Monthly*, 104(2):131–137, 1997.