Between Ends and Fibers

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Abstract

Let Γ be an infinite, locally finite, connected graph with distance function δ. Given a ray P in Γ and a constant $C \geq 1$, a vertex-sequence $\{x_n\}_{n=0}^{\infty} \subseteq VP$ is said to be *regulated by* C if, for all $n \in \mathbb{N}$, x_{n+1} never precedes x_n on P, each vertex of P appears at most C times in the sequence, and $\delta_P(x_n, x_{n+1}) \leq C$.

R. Halin (1964) defined two rays to be *end-equivalent* if they are joined by infinitely many pairwise-disjoint paths; the resulting equivalence classes are called *ends*. More recently H.A. Jung (1993) defined rays P and Q to be *b-equivalent* if there exist sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ regulated by some constant $C \geq 1$ such that $\delta(x_n, y_n) \leq C$ for all $n \in \mathbb{N}$; he named the resulting equivalence classes *b-fibers*.

Let $\mathcal{F}_0$ denote the set of nondecreasing functions from $\mathbb{N}$ into the set of positive real numbers. The relation $P \sim_f Q$ (called *f-equivalence*) generalizes Jung's condition to $\delta(x_n, y_n) \leq Cf(n)$. As f runs through $\mathcal{F}_0$, uncountably many equivalence relations are produced on the set of rays that are no finer than b-equivalence while, under specified conditions, are no coarser than end-equivalence. Indeed, for every Γ there is an "end-defining function" f such that $P \sim_f Q$ implies that P and Q are end-equivalent.

Say $P \approx Q$ if there exists a sublinear function $f \in \mathcal{F}_0$ such that $P \sim_f Q$. The equivalence classes with respect to $\approx$ are called *bundles*. We pursue the notion of "initially metric'" rays in relation to bundles, and show that in any bundle either all or none of its rays are initially metric. Furthermore, initially metric rays in the same bundle are end-equivalent.

In the case that Γ contains translatable rays we give some sufficient conditions for every f-equivalence class to contain uncountably many g-equivalence classes, (where $\lim_{n\to\infty} g(n)/f(n) = 0$).

We conclude with a variety of applications to infinite planar graphs. Among these, it is shown that two rays whose union is the boundary of an infinite face of an almost-transitive planar map are never bundle-equivalent.

1 Introduction

In his seminal work in 1964, R. Halin [3] introduced the notion of ends of graphs, an analogy to ends of groups: two rays (one-way infinite paths) in a graph belong to the same end if the intersection of some third ray with each of the given rays is infinite. For a locally finite graph Γ, the number of these equivalence classes of rays equals the supremum of the number of infinite components of $\Gamma - \Phi$ as Φ ranges over all finite subgraphs of Γ.

Writing nearly 30 years later, H. A. Jung ([6]), p. 477) said, "By now there are quite a few 'fixed-point' results involving ends. Most of them degenerate to trivialities in the case of graphs with only one end." Thus motivated, Jung defined the equivalence classes of "b-fibers", which decompose thick ends into infinitely many of these finer equivalence classes. In 1995, Jung and P. Niemeyer [7] conceived of d-fibers, which have a simpler definition and coincide with b-fibers when all of the rays contained therein are metric or when they are contained in thin ends. However, d-fibers that contain non-metric rays may simultaneously refine thick ends and be refined by b-fibers.

In the case of both of these kinds of fibers, all of the rays contained within a given fiber, although they may conceivably be arbitrarily far apart, nonetheless are essentially parallel; the distance between two fiber-equivalent rays varies within a fixed interval as the rays progress to infinity. In this work, we loosen this bound; we will define two rays to be f-equivalent if the distance between them (actually between regulated sequences of pairs of vertices on them as they progress toward infinity) is bounded by some constant multiple of a nondecreasing function f.

In Section 2 we formally state the definitions and fix notational conventions used throughout this work.

In Section 3 we show that f-equivalence is indeed an equivalence relation (a nontrivial result), prove some elementary, not-so-surprising consequences of this definition, and give a couple of examples.

In Section 4 we "bundle together" the equivalence classes with respect to sublinear functions to form larger equivalence classes called *bundles*. The notion of "metric rays", first defined independently in [6] and [9] and further developed in [7], is pursued here in relation to bundles. Jung showed (Lemma 5 of [6]) that in any b-fiber either all rays are metric or no ray is metric. We define a weaker condition called "initially metric" and show that in any bundle either all of its rays are initially metric or no ray is initially metric.

Section 5 analyzes the relationship between bundles and ends. It is shown that if initially metric rays belong to the same bundle, then they belong to the same end. The main result of this section is that for every connected, infinite, locally finite graph Γ, there exists an unbounded, nondecreasing, positive-valued function f such that f-equivalent rays belong to the same end.

We consider in Section 6 the actions of translations (i.e., automorphisms with no finite orbit) upon rays and double rays, and show that a translatable ray (one mapped into itself by some translation) is metric if and only if it is initially metric. The main result of this section gives some surprisingly weak sufficient conditions for, when $\lim_{n\to\infty} g(n)/f(n) = 0$, every f-equivalence class to contain uncountably many g-equivalence classes.

In Section 7 the concepts developed earlier are applied to infinite planar maps. It is first shown that the number of infinite faces of a biconnected planar map equals at most the number of ends of the underlying graph. We show that if the double ray bounding an infinite face of a locally finite planar map is translatable, then it is metric. This facilitates the proof of the main result of this section, that any two rays whose union is the boundary of an infinite face of a locally finite, almost-transitive planar map belong to different bundles. In [2] it was shown that such rays belong merely to distinct fibers. Among the three conjectures posed in the concluding Section 8 is that such rays belong to distinct ends.

2 Notation and Definitions

Throughout this article, Γ will denote an infinite, locally finite, connected graph. To say that Γ is locally finite means that all valences are finite, though not necessarily of bounded magnitude.

The symbols $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ will denote, respectively, the set of real numbers, the set of all integers, and the set of nonnegative integers. In particular, $\mathbb{R}^2$ denotes the plane.

A *ray* is a one-way infinite path, and a *double ray* is a two-way infinite path. From this point on, however, a *path* will be understood to have finite length, and $|P|$ will denote the length of the path P. If P is a path or a ray or a double ray and v, $w \in VP$, then $P[v,w]$ denotes the subpath of P that joins v and w, while $P[v,w)$ (respectively, $P(v,w]$) denotes the subpath of $P[v,w]$ with w (v) and the edge of P incident with w (v) deleted.

If Θ_1 and Θ_2 are subgraphs of Γ, then by a $\Theta_1\Theta_2$*-path*, we mean a path with one terminal vertex in Θ_1, the other terminal vertex in Θ_2, and no interior vertex in $\Theta_1 \cup \Theta_2$. As a special case, a vertex of $\Theta_1 \cap \Theta_2$ is a trivial $\Theta_1\Theta_2$-path.

The set of all rays in Γ is denoted by $\mathcal{R}(\Gamma)$. For $P \in \mathcal{R}(\Gamma)$, $VP = \{p_0, p_1, p_2, \cdots\}$, where $[p_i, p_{i+1}] \in EP$. Similarly, $VQ = \{q_0, q_1, \cdots\}$, $VP' = \{p'_0, p'_1, \cdots\}$ for rays Q, P', etc. If $u \in VP$, then $P[u, \infty)$ denotes the subray of P that emanates from u. If u, $v \in VP$, we write (as in [7]) $u <_P v$ if $u \neq v$ and $v \in VP[u, \infty)$.

The (graphical) distance in Γ between vertices u and v is denoted by $\delta(u,v)$. If Ψ is a subgraph of Γ and u, $v \in V\Psi$, then $\delta_\Psi(u,v)$ denotes the distance in Ψ between u and v. A ray P in Γ is *geodetic* if $\delta_P(u,v) = \delta(u,v)$ holds for all $u, v \in VP$.

The notion of an "end" as formulated by Halin [3] is based upon the notion that two rays are end-equivalent if there exists a ray whose intersection with each of them is infinite. An *end* is an equivalence class of rays with respect to this equivalence relation. It is easily shown that two rays are end-equivalent if and only if there exist infinitely many pairwise-disjoint paths joining them. We denote by $\epsilon(\Gamma)$ the number of ends of the graph Γ if this number is finite and set $\epsilon(\Gamma) = \infty$ otherwise. When Γ is locally finite, then $\epsilon(\Gamma)$ has a particularly convenient characterization; it is the supremum over all finite subgraphs Φ of the number of infinite components remaining when Φ is deleted. For an end $\mathcal{E}$ we denote by $\mu(\mathcal{E})$ the maximum cardinality of a set of pairwise-disjoint rays in $\mathcal{E}$. (The existence of $\mu(\mathcal{E})$ was proved in [4].) The end $\mathcal{E}$ is *thin* if $\mu(\mathcal{E})$ is finite and *thick* if $\mu(\mathcal{E})$ is infinite.

We let Δ denote a double ray. For $d \in \mathbb{N}$, Δ^d denotes the Cartesian product of d copies of Δ. Embedding Δ^d in d-dimensional Euclidean space with Cartesian coordinates, we identify $V\Delta^d$ with $\mathbb{Z}^d$ in the natural way, so that every edge projects nontrivially onto exactly one coordinate axis.

Definition 1 *Let $P \in \mathcal{R}(\Gamma)$. A sequence $\{x_n\}_{n=0}^{\infty}$ in VP is* regulated *if there exists a constant $C \geq 1$ such that*
(i) $x_n \leq_P x_{n+1}$ for all $n \in \mathbb{N}$;
(ii) $x_m = x_n \Rightarrow |m - n| \leq C$;
(iii) $\delta_P(x_n, x_{n+1}) \leq C$ for all $n \in \mathbb{N}$.
In this case we say that $\{x_n\}_{n=0}^{\infty}$ is regulated by C *or, more briefly, is* C-regulated.

We have immediately:

Proposition 2.1 *For all m, $n \in \mathbb{N}$, $\left\lfloor \frac{|m-n|}{C} \right\rfloor \leq \delta_P(x_m, x_n) \leq C|m - n|$.*

Clearly, if $\{x_n\}_{n=0}^{\infty}$ is regulated by C_1 and $C_1 < C_2$, then $\{x_n\}_{n=0}^{\infty}$ is also regulated by C_2.

Definition 2 *Rays P, $Q \in \mathcal{R}(\Gamma)$ are* fiber-equivalent *if for some $C \geq 1$ there exist C-regulated sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ such that $\delta(x_n, y_n) \leq C$ for all $n \in \mathbb{N}$. The equivalence classes of rays produced by this equivalence relation are called* fibers.

The notion of "fibers" is due to H.A. Jung. In [6] he proved that fibers (which he called b-fibers) are indeed equivalence classes of rays.

We denote by $\mathcal{F}_0$ the set of all nondecreasing functions from $\mathbb{N}$ into the set $\mathbb{R}^{>0}$ of positive real numbers.

Definition 3 *For $f \in \mathcal{F}_0$ we define the binary relation $\sim_f$ on $\mathcal{R}(\Gamma)$ as follows. If $P, Q \in \mathcal{R}(\Gamma)$ we write $P \sim_f Q$ whenever there exist sequences $\{x_n\}_{n=0}^{\infty}$ in VP and $\{y_n\}_{n=0}^{\infty}$ in VQ both regulated by C such that $\delta(x_n, y_n) \leq Cf(n)$ holds for all $n \in \mathbb{N}$.*

In the foregoing definitions, the vertices in each of the two sequences proceed nondecreasingly away from the initial vertex of their ray, each vertex may be repeated at most C times, and path-distances between consecutive sequence terms are also uniformly bounded by C. If the two rays are f-related for some $f \in \mathcal{F}_0$, then they may in a sense diverge, but for each $n \in \mathbb{N}$ the distance between them at the nth vertex of their corresponding sequences is bounded by $Cf(n)$.

We define another binary relation $\gg$ on $\mathcal{F}_0$ by

$$f \gg g \quad \text{if and only if} \quad \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0.$$

(Equivalently, one may write $g \ll f$.) The function $f \in \mathcal{F}_0$ is *sublinear* if $I \gg f$, where $I(n) = n$ for all $n \in \mathbb{N}$. We let $\mathcal{F}_1$ denote the subset of sublinear functions in $\mathcal{F}_0$.

3 Elementary Properties of f-Classes

The main result of this section is that for any $f \in \mathcal{F}_0$ and any graph Γ, $\sim_f$ is an equivalence relation on $\mathcal{R}(\Gamma)$. We begin by showing that for f-related rays, the requisite C-regulated sequences may be chosen to include the initial vertices of the rays.

Proposition 3.1 *Suppose that $P \sim_f Q$, where $P, Q \in \mathcal{R}(\Gamma)$ and $f \in \mathcal{F}_0$. Then there exist a number $C > 1$ and C-regulated sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ such that $\delta(x_n, y_n) \leq Cf(n)$ for all $n \in \mathbb{N}$ and $x_0 = p_0$ and $y_0 = q_0$.*

Proof. Let $\{x'_n\}_{n=0}^{\infty}$ in VP and $\{y'_n\}_{n=0}^{\infty}$ in VQ be sequences regulated by C' such that $\delta(x'_n, y'_n) \leq C'f(n)$ for all $n \in \mathbb{N}$. Suppose that $x'_0 = p_h$ and $y'_0 = q_k$. Set $x_0 = p_0,\ y_0 = q_0$ and $x_n = x'_{n-1}$ and $y_n = y'_{n-1}$ for all $n \geq 1$. To establish the regulating constant C, we note that we clearly must have $C \geq \delta(x_0, y_0)/f(0)$. Also, perhaps p_0 already occurs C' times in $\{x'_n\}$ (or q_0 in $\{y'_n\}$) and we must now allow for one more occurrence. If we set

$$C = \max\left\{C' + 1, h, k, \frac{\delta(p_0, q_0)}{f(0)}\right\},$$

then, for all $n \geq 1$, we have $\delta(x_n, y_n) = \delta(x'_{n-1}, y'_{n-1}) \leq C'f(n-1) \leq Cf(n)$, and $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are regulated by C. $\square$

Theorem 3.2 *$\sim_f$ is an equivalence relation for all $f \in \mathcal{F}_0$.*

Proof. It is immediate that $\sim_f$ is reflexive and symmetric.

To prove transitivity, suppose P, Q, $R \in \mathcal{R}(\Gamma)$ and that $P \sim_f Q$ and $Q \sim_f R$. By definition there exist sequences $\{x_n\} \subseteq V(P)$, $\{u_n\} \subseteq V(Q)$, $\{v_n\} \subseteq V(Q)$, and $\{z_n\} \subseteq V(R)$ that we may assume to be regulated by the same constant C and such that, for all $n \in \mathbb{N}$,

$$\delta(x_n, u_n) \leq Cf(n) \qquad \text{and} \qquad \delta(v_n, z_n) \leq Cf(n).$$

By Proposition 3.1, we may assume that $u_0 = v_0$.

Form the sequence $\{y_j\}_{j=0}^{\infty} \subseteq V(Q)$ which includes every vertex u_n and every vertex v_n, indexed in order of nondecreasing distance in Q from $y_0 := u_0 = v_0$. Obviously $0 \leq \delta_Q(y_{j-1}, y_j) \leq C$ holds for all $j \in \mathbb{N}$ and any vertex of Q occurs in the sequence $\{y_j\}$ at most $2C$ times. By (possibly) repeating terms of $\{x_n\}$ at most C times, we form inductively a sequence $\{x'_j\}$ of which $\{x_n\}$ is a subsequence in the following way.

Let $x'_0 = x_0$, and suppose that $x'_0, \ldots, x'_{j-1}$ have been defined for some $j \in \mathbb{N}$. If $y_j = u_n$ for some $n \in \mathbb{N}$, define $x'_j = x_n$. Otherwise, $y_j = v_k$ for some $k \in \mathbb{N}$. Pick the largest index n such that $u_n <_Q y_j$, and define $x'_j = x_n$. Note that by this construction, either $x'_{j-1} = x_n = x'_j$ or $x'_{j-1} = x_{n-1}$. In any case, $\delta(x'_{j-1}, x'_j) \leq C$ for all $j \in \mathbb{N}$. Note that each vertex of P occurs at most C^2 times in the sequence $\{x'_j\}_{j=0}^{\infty}$.

In a similar fashion, with the roles of $\{u_n\}$ and $\{v_n\}$ interchanged, we construct in $V(R)$ a supersequence z'_j of $\{z_n\}$ such that $\delta(z'_{j-1}, z'_j) \leq C$ for all $j \in \mathbb{N}$. Thus for any $j \in \mathbb{N}$ there exist m, $n \leq j$ such that $x'_j = x_m$ and $z'_j = z_n$ and

$$\begin{aligned}\delta(x'_j, z'_j) &\leq \delta(x_m, u_m) + \delta_Q(u_m, y_j) + \delta_Q(y_j, v_n) + \delta(v_n, z_n) \\ &\leq Cf(m) + 2C^2 + Cf(n) \\ &\leq 2C(C+1)f(j),\end{aligned}$$

the last inequality holding because f is nondecreasing. Thus $P \sim_f R$ by virtue of the sequences $\{x'_j\}$ and $\{z'_j\}$ regulated by $2C(C+1)$. □

Definition 4 *Equivalence classes with respect to* $\sim_f$ *will be called* f-classes. *Two rays in the same* f*-class are said to be* f-equivalent.

Example 1 *If* f *is a constant function, then clearly the* f*-classes of* Γ *are precisely the fibers defined in the previous section. In fact, if* f *is any bounded function in* $\mathcal{F}_0$*, say* $f(n) \leq M$ *for all* n*, and* $P \sim_f Q$*, then for appropriate* C*-regulated sequences, we have* $\delta(x_n, y_n) \leq CM$*. This shows that the* f*-classes of bounded functions are fibers.*

Proposition 3.3 *Let* P*,* $Q \in \mathcal{R}(\Gamma)$ *and* f*,* $g \in \mathcal{F}_0$*. Then the following statements hold.*
(i) If Q *is a subray of* P*, then* $P \sim_f Q$*.*
(ii) If $f(n) \leq g(n)$ *for all* $n \in \mathbb{N}$ *and* $P \sim_f Q$*, then* $P \sim_g Q$*.*
(iii) If for constants $a > 0$ *and* $b > -1/[af(0)]$*,* $g(n) = af(n) + b$ *holds for all* $n \in \mathbb{N}$ *and* $P \sim_f Q$*, then* $P \sim_g Q$*.*
(iv) If $\limsup_{n\to\infty} f(n)/g(n) < \infty$ *and* $P \sim_f Q$*, then* $P \sim_g Q$*. In particular,*

$$\{f \ll g \quad \text{and} \quad P \sim_f Q\} \quad \Rightarrow \quad P \sim_g Q.$$

Proof. (i) and (ii) are obvious.

(iii) Suppose that $P \sim_f Q$ by virtue of a pair of sequences regulated by C. Set

$$D = \begin{cases} C/a & \text{if} \quad b \geq 0; \\ Cf(0)/[af(0)+b] & \text{if} \quad b < 0. \end{cases}$$

Since $D(af(n) + b) \geq Cf(n)$ holds for all $n \in \mathbb{N}$, these same vertex sequences are regulated by D and satisfy the definition of $P \sim_g Q$.

(iv) Assume $\limsup_{n\to\infty} f(n)/g(n) < \infty$ and $P \sim_f Q$. There exists some $D > 0$ such that $f(n) < Dg(n)$ for all $n \in \mathbb{N}$. Now apply parts (ii) and (iii). $\square$

Remark By part (i) of this proposition and the transitivity of $\sim_f$, if rays P and Q are subrays of P' and Q', respectively, then $P \sim_f Q \Leftrightarrow P' \sim_f Q'$. Thus when considering arbitrary pairs of rays, we lose no generality by assuming that they emanate from the same vertex.

4 f-Classes and Bundles

For rays $P, Q \in \mathcal{R}(\Gamma)$, write $P \approx Q$ if there exists some function $f \in \mathcal{F}_1$ such that $P \sim_f Q$. Clearly the relation $\approx$ is reflexive and symmetric. To show that $\approx$ is also transitive, suppose that $P \sim_f Q$ and $Q \sim_g R$ for some $f, g \in \mathcal{F}_1$. If $h(n) := \max\{f(n), g(n)\}$ for all $n \in \mathbb{N}$, then $h \in \mathcal{F}_1$ and, by Proposition 3.3(ii), $P \sim_h Q$ and $Q \sim_h R$. Now apply Theorem 3.2.

Definition 5 *An equivalence class with respect to the equivalence relation $\approx$ on $\mathcal{R}(\Gamma)$ will be called a* bundle.

Lemma 4.1 *Let P, $Q \in \mathcal{R}(\Gamma)$. Then $P \approx Q$ if and only if there exist regulated sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ such that*

$$\lim_{n\to\infty} \frac{\delta(x_n, y_n)}{n} = 0.$$

Proof. Assume $P \approx Q$ and let $f \in \mathcal{F}_1$ such that $P \sim_f Q$. There exist vertex sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ regulated by C such that $\delta(x_n, y_n) \leq Cf(n)$. Since f is sublinear,

$$0 \leq \lim_{n\to\infty} \frac{\delta(x_n, y_n)}{n} \leq \lim_{n\to\infty} \frac{Cf(n)}{n} = 0\,.$$

Conversely, assume the existence of regulated sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ such that $\lim_{n\to\infty} \delta(x_n, y_n)/n = 0$. For each $n \in \mathbb{N}$ define $f(n) = \max\{\delta(x_m, y_m) : 0 \leq m \leq n\}$. Since f is nondecreasing, $f \in \mathcal{F}_0$. Furthermore, $\delta(x_n, y_n) \leq f(n)$, and so $P \sim_f Q$.

It remains only to show that $f \in \mathcal{F}_1$. If f is a bounded function, then we are clearly done. So suppose that f is unbounded. There exists a strictly increasing subsequence $\{n_k\}_{k=0}^{\infty}$ of indices such that $n_0 = 0$ and n_{k+1} is the least index m such that $\delta(x_m, y_m) > f(n_k)$. For $n_k \leq m < n_{k+1}$ we have $f(m) = f(n_k)$, and therefore,

$$0 < \frac{f(m)}{m} \leq \frac{f(n_k)}{n_k} = \frac{\delta(x_{n_k}, y_{n_k})}{n_k}.$$

As k approaches ∞, so does m. The right-hand term is a subsequence of a sequence converging to 0, proving that f is sublinear. $\square$

Definition 6 *[9] The* straightness $\sigma(P)$ *of a ray or double ray P is defined to be*

$$\sigma(P) = \liminf_{|n-m|\to\infty} \frac{\delta(p_m, p_n)}{\delta_P(p_m, p_n)}.$$

We call P metric *if $\sigma(P) > 0$. Equivalently (see [7]), P is* metric *if there exists a positive constant K such that $\delta_P(p_m, p_n) \leq K\delta(p_m, p_n)$ for all $m, n \in \mathbb{N}$. (A metric double ray is called a "quasi-axis" in [9].)*

Proposition 4.2 *Let $\{x_n\}_{n=0}^{\infty} \subseteq VP$ be regulated by C. Then*

$$\frac{\sigma(P)}{C} \leq \liminf_{|n-m|\to\infty} \frac{\delta(x_m, x_n)}{|n-m|} \leq C\sigma(P).$$

Thus, if any one of these terms is positive, then all three terms are positive and P is metric.

Proof. We lose no generality in assuming that $n \geq m$. If $x_m = p_k$ and $x_n = p_\ell$, then from Proposition (2.1) we have $(n-m)/C \leq \ell - k \leq C(n-m)$. Hence

$$\frac{\delta(x_m, x_n)}{n-m} = \frac{\delta(p_k, p_\ell)}{n-m} \geq \frac{\delta(p_k, p_\ell)}{C(\ell - k)}.$$

Clearly as one of $\ell - k$ and $n - m$ approaches ∞, so does the other. Taking lim inf as the appropriate quantity approaches ∞ yields the first inequality.

To obtain the second inequality, we have by definition the existence of a subsequence $\{(k_i, \ell_i)\}_{i=0}^{\infty}$ of subscript pairs such that $x_0 \leq_P p_{k_i} <_P p_{\ell_i}$ and $\sigma(P) = \lim_{i\to\infty} \delta(p_{k_i}, p_{\ell_i})/(\ell_i - k_i)$. For each $i \in \mathbb{N}$ there exist subscripts m_i and n_i such that $\delta(x_{m_i}, p_{k_i}) \leq \delta_P(x_{m_i}, p_{k_i}) \leq C$ and $\delta(x_{n_i}, p_{\ell_i}) \leq \delta_P(x_{n_i}, p_{\ell_i}) \leq C$. If $x_{m_i} = p_a$ and $x_{n_i} = p_b$, then $\ell_i - k_i \leq |\ell_i - b| + (b - a) + |a - k_i| \leq 2C + C(n_i - m_i)$. By the triangle inequality,

$$\begin{aligned}
\sigma(P) &= \lim_{i\to\infty} \frac{\delta(p_{k_i}, p_{\ell_i})}{\ell_i - k_i} \\
&\geq \liminf_{i\to\infty} \frac{\delta(x_{m_i}, x_{n_i}) - 2C}{\ell_i - k_i} \\
&\geq \liminf_{i\to\infty} \frac{\delta(x_{m_i}, x_{n_i})}{2C + C(n_i - m_i)} \\
&= \frac{1}{C} \liminf_{i\to\infty} \frac{\delta(x_{m_i}, x_{n_i})}{n_i - m_i}\ .
\end{aligned}$$

$\square$

Definition 7 *For a ray P, define $\sigma_0(P) = \liminf_{k\to\infty} \delta(p_0, p_k)/k$. The ray P is* initially metric *if $\sigma_0(P) > 0$.*

Since for any $m \in \mathbb{N}$, we have $\delta(p_m, p_k) \geq \delta(p_0, p_k) - \delta(p_0, p_m) \geq \delta(p_m, p_k) - 2\delta(p_0, p_m)$, it follows that

$$\sigma_0(P) = \liminf_{k\to\infty} \frac{\delta(p_m, p_k)}{k}. \tag{1}$$

Clearly,

$$\sigma(P) \leq \sigma_0(P) \leq 1.$$

Example 2 *We show that it is possible to have $\sigma_0(P) - \sigma(P) = 1$, the largest that this difference can be. To avoid messy subscripts, we write $p(k)$ in place of p_k. Let Γ consist of the ray P together with pairwise-disjoint paths of length 2^n joining vertices $p(4^n - 3^n)$ and $p(4^n)$ for all $n \in \mathbb{N}$. Then*

$$0 \leq \sigma(P) \leq \lim_{n\to\infty} \frac{2^n}{3^n} = 0.$$

If $k \neq 4^n$ for some n, then the shortest path from $p(0)$ to $p(k)$ consists of a shortest path from $p(0)$ to the nearest $p(4^n)$ or $p(4^n - 3^n)$ together with a relatively short subpath of P. The latter cannot decrease the ratio $\delta(p(0), p(k))/k$. Thus, assuming that $k = 4^n$, we straightforwardly compute

$$\sigma_0(P) = \lim_{n\to\infty} \frac{1}{4^n}[4^n - \sum_{i=1}^{n}(3^i - 2^i)] = 1.$$

The computation is similar if $k = 4^n - 3^n$.

Theorem 4.3 *In any bundle, either all of the rays are initially metric or no ray is initially metric.*

Proof. Assume $P \approx Q$. By Lemma 4.1 there exist vertex sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ regulated by C such that $\liminf_{n\to\infty} \delta(x_n, y_n)/n = 0$. For each k such that $y_0 \leq_Q q_k$, there exists some index n_k such that $\delta_Q(q_k, y_{n_k}) \leq C$. By the triangle inequality,

$$\delta(x_0, x_{n_k}) \leq \delta(x_0, y_0) + \delta(y_0, q_k) + \delta(q_k, y_{n_k}) + \delta(y_{n_k}, x_{n_k}).$$

Say that $p_h = x_0$. By Proposition (2.1), $k - h \leq Cn_k$. Hence

$$\begin{aligned}
\sigma_0(Q) &= \liminf_{k\to\infty} \frac{\delta(y_0, q_k)}{k} \\
&\geq \frac{1}{C} \liminf_{k\to\infty} \frac{\delta(x_0, x_{n_k})}{n_k + h/C} - \liminf_{k\to\infty} \frac{\delta(x_0, y_0) + C}{k} - \frac{1}{C} \liminf_{k\to\infty} \frac{\delta(x_{n_k}, y_{n_k})}{n_k} \\
&\geq \frac{1}{C}\sigma_0(P)
\end{aligned}$$

Interchanging P and Q in the above argument, one derives that $\sigma_0(P) \geq \sigma_0(Q)/C$, from which the conclusion follows. □

Remark The above proof yields

$$\frac{1}{C}\sigma_0(P) \leq \sigma_0(Q) \leq C\sigma_0(P)$$

for any two rays P and Q in the same bundle. In particular, if $\lim_{k\to\infty} \delta(p_k, q_k)/k = 0$ (in which case we may take $C = 1$), then $\sigma_0(P) = \sigma_0(Q)$.

H.A. Jung ([6], Lemma 5) proved that fibers have an "all or nothing" property with respect to metric rays, making it reasonable to speak of "metric fibers" and "non-metric fibers". Without stronger assumptions, f-classes admit such a dichotomy only with respect to initially metric rays. An f-class (respectively, bundle) that contains only metric rays is called a *metric class* (respectively, *metric bundle*). However, *non-metric f-classes* (respectively, *nonmetric bundles*) contain *only* non-metric rays. The "stronger assumptions" will be considered in the next section, but first we present two examples, one of a non-metric bundle and then an example of an f-class that contains both metric and non-metric rays.

Example 3 *Let $\Gamma = \Delta^2$, coordinatized in Cartesian fashion. Let the ray P start at $A_0 = (0,0)$ and proceed to $A_1 = (0,1)$, then 2^2 units to the left to $A_2 = (-4,1)$. It then proceeds 3^2 units downward to A_3, 4^2 units to the right, 5^2 units upward to $A_5 = (12,17)$, and so on indefinitely, from quadrant to quadrant in the counter-clockwise sense, so that P makes a 90-degree left turn at the the nth "corner" A_{4n+r} lying in the rth quadrant ($r = 1,2,3,4 \equiv 0$). Thus P has the appearance of a squared-off spiral where the distance between the successive revolutions increases. Thus*

$$\delta_P(A_0, A_m) = \sum_{j=0}^{m} j^2 \quad \text{for} \quad m \geq 1;$$

$$\delta(A_0, A_m) \leq \delta(A_{m-2}, A_m) = m^2 + (m-1)^2 \quad \text{for} \quad m \geq 2.$$

If p_k lies on $P(A_{m-1}, A_m]$ then

$$0 \leq \sigma_0(P) \leq \liminf_{m\to\infty} \frac{\delta(A_0, A_m)}{\delta_P(A_0, A_{m-1})} \leq \liminf_{m\to\infty} \frac{m^2 + (m-1)^2}{(m-1)m(2m-1)/6} = 0.$$

It follows from Theorem 4.3 that for any ray $Q \approx P$, we have $0 \leq \sigma(Q) \leq \sigma_0(Q) = \sigma_0(P) = 0$.

Example 4 *We exhibit a metric ray P and a non-metric ray Q such that $P \approx Q$. Let Γ consist of these two rays emanating from $p_0 = q_0$ but otherwise disjoint, together with pairwise-disjoint $p_k q_k$-paths S_k of length $f(k) = \lfloor k^{3/4} \rfloor$ and edges joining q_{k^2-k} and q_{k^2} for each $k \in \mathbb{N}$. With all the vertices of P and of Q forming 1-regulated sequences, it is clear* *that $P \sim_f Q$. Since $\lim_{k\to\infty} \delta(q_{k^2-k}, q_{k^2})/k = 0$, clearly Q is not metric, and we demonstrate a positive lower bound for $\sigma(P)$.*

Let $0 \le k < \ell$ be given. If a shortest $p_k p_\ell$-path $T_{k,\ell}$ is not contained in P, then $T_{k,\ell} = S_k \cup Q' \cup S_\ell$, where Q' is a $q_k q_\ell$-path contained in the union of Q and edges of the form $[q_{r^2-r}, q_{r^2}]$. When such a "shortcut" edge belongs to Q', then clearly at least one of $r^2 - r$ and r^2 lies between k and ℓ. Moreover, at least one such edge must belong to Q'. Let h be the least integer such that $k \le h^2 - h$, and let i be the greatest integer such that $i^2 \le \ell$. For suitable $h' \in \{h-1, h\}$ and $i \in \{i, i+1\}$ and for $\ell - k$ not too small, the length of $T_{k,\ell}$ equals

$$\begin{aligned} \lfloor k^{3/4} \rfloor &+ \lfloor \ell^{3/4} \rfloor + (\ell - k) - \sum_{j=h'}^{i'} (j-1) \\ &\ge \lfloor k^{3/4} \rfloor + \lfloor \ell^{3/4} \rfloor + (\ell - k) - \binom{i+1}{2} + \binom{h-1}{2} \\ &\ge \lfloor k^{3/4} \rfloor + \lfloor \ell^{3/4} \rfloor + \frac{1}{2}(\ell - k). \end{aligned}$$

Thus

$$\sigma(P) \ge \liminf_{\ell - k \to \infty} \frac{\lfloor k^{3/4} \rfloor + \lfloor \ell^{3/4} \rfloor + \frac{1}{2}(\ell - k)}{\ell - k} = \frac{1}{2}.$$

Theorem 4.4 *Let P, $Q \in \mathcal{R}(\Gamma)$. If $D = P \cup Q$ is a metric double ray, then P and Q belong to different bundles. Thus exactly two bundles are represented by the set of rays contained in D.*

Proof. If $P \approx Q$, then for some vertex sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_i\}_{n=0}^{\infty}$ regulated by C we have by Lemma 4.1,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{\delta(x_n, y_n)}{n} = \lim_{n\to\infty} \left(\frac{\delta_D(x_n, y_n)}{n} \cdot \frac{\delta(x_n, y_n)}{\delta_D(x_n, y_n)} \right) \\ &\ge \lim_{n\to\infty} \frac{2n}{Cn} \cdot \liminf_{n\to\infty} \frac{\delta(x_n, y_n)}{\delta_D(x_n, y_n)} \ge \frac{2\sigma(D)}{C} \ge 0. \end{aligned}$$

Hence $\sigma(D) = 0$ and D is not metric. The rest follows by Proposition 3.3(i). □

The following example shows that the converse of this theorem is false.

Example 5 *Let $h : \mathbb{N} \to (\mathbb{N} \backslash \{0\})$ be increasing and unbounded. Suppose that*

$$\limsup_{k\to\infty} h(k)/h(k+1) < 1.$$

(Many integer-valued functions have these properties including, for example, $k!$ and a^k for $a \ge 2$.) Let P and Q be rays such that $V(P \cap Q) = \{p_0\} = \{q_0\}$. Adjoin all edges of the

form $[p_{h(k)}, q_{h(k)}]$ *for* $k \in \mathbb{N}$. *Thus* P *and* Q *are geodetic rays, and it is clear that* $P \cup Q$ *is not metric.*

Choose arbitrary vertex sequences $\{x_n\}_{n=1}^{\infty} \subseteq VP$ *and* $\{y_n\}_{n=1}^{\infty} \subseteq VQ$ *regulated by (an arbitrary) constant* C. *Since* $h(k+1)-h(k) = h(k+1)[1-h(k)/h(k+1)]$, *there exists* $\epsilon > 0$ *such that for sufficiently large* k, $h(k+1)-h(k) > \epsilon h(k+1)$. *Hence* $\lim_{k\to\infty}[h(k+1)-h(k)] = \infty$. *It follows that one can form a subsequence* $\{x_{n_k}\}_{k=0}^{\infty}$ *of* $\{x_n\}_{n=0}^{\infty}$ *such that for some* $M > 0$ *and all* $k \geq M$, *if* $x_{n_k} = p_m$ *then*

$$\frac{1}{2}[h(k)+h(k+1)] - C \leq m < \frac{1}{2}[h(k)+h(k+1)],$$

i.e., x_{n_k} *lies on* P *almost halfway from* $p_{h(k)}$ *to* $p_{h(k+1)}$ *for each* $k \geq M$. *Observe that* $n_k \leq Ch(k+1)$. *We now use Lemma 4.1 to show that* P *and* Q *belong to different bundles:*

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\delta(x_n, y_n)}{n} &\geq \limsup_{k\to\infty} \frac{\delta(x_{n_k}, Q)}{n_k} \\
&\geq \limsup_{k\to\infty} \frac{1}{Ch(k+1)} \left[\frac{h(k+1)-h(k)}{2} - C + 1\right] \\
&= \frac{1}{2C}\left[1 - \liminf_{k\to\infty} \frac{h(k)}{h(k+1)}\right] \\
&\geq \frac{1}{2C}\left[1 - \limsup_{k\to\infty} \frac{h(k)}{h(k+1)}\right] > 0
\end{aligned}$$

as required.

5 Bundles and Ends

In the previous section we saw an example of a nonmetric bundle. However, there exist no "nonmetric ends", as every end contains a metric ray (see [10], Theorem 3.1). Hence no end contains only nonmetric bundles. As mentioned in the introduction, Jung and Niemeyer ([6] and [7]) showed that ends are refined by fibers and, in the case of thick ends, the refinement is proper. Although the relationship between ends and bundles is more complicated, it will be delineated by Theorem 5.3, which, together with Theorem 5.2, are the main results of this section.

Lemma 5.1 *If* $f \in \mathcal{F}_0$ *and* $\liminf_{n\to\infty} f(n)/n > 0$, *then* $\mathcal{R}(\Gamma)$ *consists of a single* f*-class.*

Proof. Since f is positive-valued, there exists some number $\epsilon > 0$ such that $f(n)/n > \epsilon$ for all $n \in \mathbb{N}$. Choose arbitrary P, $Q \in \mathcal{R}(\Gamma)$. As remarked earlier, we may assume that $p_0 = q_0$. For any $n \in \mathbb{N}$, we have $\delta(p_n, q_n) \leq 2n < \frac{2}{\epsilon} f(n)$. Hence $P \sim_f Q$ via the vertex sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ which are regulated by $2/\epsilon$. □

Theorem 5.2 *Let P, $Q \in \mathcal{R}(\Gamma)$ be initially metric. If $P \approx Q$, then P and Q belong to the same end of Γ.*

Proof. Suppose that P and Q belong to different ends. Hence there exists a finite subset $T \subset V\Gamma$ such that subrays of P and of Q belong to different components of $\Gamma - T$. By considering subrays or superrays of P and of Q, we may assume that $VP \cap T = \{p_0\}$ and $VQ \cap T = \{q_0\}$. The definition of initially metric rays implies that there exist positive constants K_P and K_Q such that $n \leq K_P \delta(p_0, p_n)$ and $n \leq K_Q \delta(q_0, q_n)$.

Let $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ be arbitrary vertex sequences regulated by C. Hence for all $n \in \mathbb{N}$,

$$\begin{aligned}
\delta(x_n, y_n) &\geq \delta(x_n, T) + \delta(T, y_n) \\
&\geq \delta(x_n, p_0) + \delta(q_0, y_n) - 2\,\mathrm{diam}(T) \\
&\geq \frac{1}{K_P}\delta_P(p_0, x_n) + \frac{1}{K_Q}\delta_Q(q_0, y_n) - 2\,\mathrm{diam}(T) \\
&\geq \frac{n}{C}\left(\frac{1}{K_P} + \frac{1}{K_Q}\right) - 2\,\mathrm{diam}(T).
\end{aligned}$$

From this it follows that

$$\liminf_{n\to\infty} \frac{\delta(x_n, y_n)}{n} \geq \frac{1}{C}\left(\frac{1}{K_P} + \frac{1}{K_Q}\right) > 0$$

and by Lemma 4.1, P and Q belong to different bundles. □

Remark It follows from Theorem 5.2 that for initially metric rays, *bundle-equivalence is a refinement of end-equivalence*. In the opposite direction, bundles are refined by f-classes for any $f \in \mathcal{F}_1$. Suppose $f, g \in \mathcal{F}_1$, where $\limsup_{n\to\infty} f(n)/g(n) = 0$, and let $0 < t < 1$. Define $h_t(n) = [f(n)]^t[g(n)]^{1-t}$ for all $n \in \mathbb{N}$. Then $h_t \in \mathcal{F}_1$ and $\limsup_{n\to\infty} f(n)/h_t(n) = \limsup_{n\to\infty} h_t(n)/g(n) = 0$. Thus by Proposition 3.3(iv) there exists an uncountable chain of equivalence relations totally ordered by refinement from bundles down to fibers, the finest equivalence relation of all among those considered in this work.

Example 6 *Consider the 2-ended graph Γ in Figure 3.1, where the rays P and Q are determined by the labeling of the vertices. Clearly neither ray is initially metric. If $2^{k-1} \leq n < 2^k$, then $\delta(p_n, q_n) = 2k$. If $f(n) = 1 + \log_2 n$, then $f \in \mathcal{F}_1$ and $\delta(p_n, q_n) < 2f(n)$ for all $n \in \mathbb{N}$. Letting $C = 2$ and using each vertex of P and Q exactly once, we have $P \sim_f Q$. However, P and Q belong to different ends. This shows that Theorem 5.2 fails for rays that are not initially metric.*

Figure 5.1: Example 6.

By way of motivation for our next result, let us observe some other properties of Example 6. Let $S = \{p_0\}$ and let Λ_P and Λ_Q be the components of $\Gamma - S$ containing P and Q, respectively. Then the number of vertices in each component at distance n from S is $2^{n-1} = f^{-1}(n)$. Secondly, let g, $h \in \mathcal{F}_0$ such that $\lim_{n\to\infty} g(n)/f(n) = \lim_{n\to\infty} f(n)/h(n) = 0$. Then there always exist rays P' in Λ_P and Q' in Λ_Q such that $P' \sim_h Q'$, but for every ray P' in Λ_P and every ray Q' in Λ_Q, we have $P' \nsim_g Q'$. In other words, the function f (or any other function f^* such that $0 < \lim_{n\to\infty} f(n)/f^*(n) < \infty$) is a "defining function" for the ends of Γ.

Theorem 5.3 *For every connected, infinite, locally finite graph* Γ*, there exists an unbounded function* $f \in \mathcal{F}_1$ *such that* $P \sim_f Q$ *implies* P *and* Q *are in the same end of* Γ*.*

Proof. We first observe that if the desired function $f \in \mathcal{F}_0$ exists and if Γ has more than one end (for otherwise this theorem is trivial), then by Lemma 5.1, $\liminf_{n\to\infty} f(n)/n = 0$. While we would need $\lim_{n\to\infty} f(n)/n = 0$ to deduce that $f \in \mathcal{F}_1$, in the present context it is obvious that if a satisfactory function exists in $\mathcal{F}_0$, then one can be found in $\mathcal{F}_1$ as well.

We start by proving two claims to be used in the course of the proof of this theorem.

Claim 1: Let $f : [1, \infty) \to (0, \infty)$ *be an unbounded nondecreasing function such that* $f(1) = 1$*. Then there is an unbounded, nondecreasing function* $g : [1, \infty) \to (0, \infty)$ *such that, for all* $t \geq 1$ *and for all* $C > 1$*,* $g(t) \leq f(t)$ *and* $g(Ct) \leq Cg(t)$*.*

We let $g(t) = \inf\{af(\frac{t}{a}) : 1 \leq a \leq t\}$ and show that g has the required properties. Clearly $g(t) \leq 1f(\frac{t}{1}) = f(t)$ and $g(t) \leq tf(\frac{t}{t}) = t$.

To compare $g(s)$ with $g(t)$ when $1 \leq s \leq t$, observe that, if $s \leq a \leq t$, then $af(\frac{t}{a}) \geq sf(1) = s \geq g(s)$. Furthermore, if $1 \leq a \leq s$, then $af(\frac{s}{a}) \leq af(\frac{t}{a})$, so that

$$g(t) = \inf\{af(\frac{t}{a}) : 1 \leq a \leq t\} \geq \min\{g(s), \inf\{af(\frac{t}{a}) : 1 \leq a \leq s\}\} \geq g(s).$$

Thus, g is non-decreasing.

Fix $\epsilon > 0$. For each $k \in \mathbb{N}$, there is some $a_k \in [1, k]$ such that $g(k) \geq a_k f(\frac{k}{a_k}) - \epsilon$. If $\{\frac{k}{a_k} : k \in \mathbb{N}\}$ is bounded then $\lim_{k\to\infty} a_k = \infty$. Since $g(k) \geq a_k - \epsilon$, g is unbounded. If, on the other hand, $\{\frac{k}{a_k} : k \in \mathbb{N}\}$ is unbounded, then $f(\frac{k}{a_k})$ is unbounded and $g(k) \geq f(\frac{k}{a_k}) - \epsilon$ is unbounded.

Finally, we prove that $g(Ct) \leq Cg(t)$ for arbitrary $C > 1$ and $t \geq 1$. By definition, $g(Ct) = \inf\{af(\frac{Ct}{a}) :\leq a \leq Ct\}$. Since $\{Ca : 1 \leq a \leq t\} \subseteq \{a : 1 \leq a \leq Ct\}$, we see that $g(Ct) \leq \inf\{Caf(\frac{Ct}{Ca}) : 1 \leq a \leq t\} = C\inf\{af(\frac{t}{a}) : 1 \leq a \leq t\} = Cg(t)$, as required.

We remark in passing that the restriction that $f(1) = 1$ is mere convenience. For arbitrary f, one computes g for $f(t)/f(1)$ and then uses $f(1)g(t)$.

Claim 2: Let $\{f_i\}$ be a countable set of nondecreasing unbounded functions $f_i : \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$. Then there is a nondecreasing, unbounded function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$ such that, for every index $i \in \mathbb{N}$, (i) there exists $N_i \in \mathbb{N}$ such that if $t \geq N_i$, then $f_i(t) \geq f(t)$, and (ii) $f \ll f_i$.

Inductively define a sequence of non-negative integers N_i by: $N_0 = 0$ and, given N_i, N_{i+1} is the least $n \in \mathbb{N}$ such that

$$\min\{f_1(n), f_2(n), \ldots, f_i(n), f_{i+1}(n)\} \geq 1 + \min\{f_1(N_i), f_2(N_i), \ldots, f_i(N_i)\}\,.$$

Since all the functions f_j are unbounded, such an N_{i+1} exists. Because the functions are nondecreasing, $N_{i+1} > N_i$.

Define the function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$ by: for $N_i \leq t < N_{i+1}$, $f(t) = i$. (For $N_0 \leq t \leq N_1$, we should actually define $f(t)$ to be 1. This will make no difference to the argument that follows.) Evidently f is nondecreasing and unbounded. By the choice of N_i we see that $\min\{f_1(N_i), f_2(N_i), \ldots, f_i(N_i)\} \geq i = f(N_i)$, so the fact that f_i is nondecreasing implies that, for all $n \geq N_i$, $f_i(n) \geq f(n)$, proving part (i) of Claim 2.

To obtain part (ii), we note that one may replace the function f that we just obtained by, for example, the function $1 + \lfloor \log f \rfloor$. Then $(1 + \lfloor \log f(n) \rfloor)/f_i(n)$ is the same as

$$\frac{(1 + \lfloor \log f(n) \rfloor)}{f(n)} \frac{f(n)}{f_i(n)}\,.$$

The limit of the first factor is 0, while the second factor is bounded above by 1.

To begin the proof of the theorem proper, we note first that for each finite subset S of $V\Gamma$, there are only finitely many infinite components K of $G - S$, each of which has a growth function $h_K(n)$ denoting the number of vertices of K at distance at most n from S. This is a strictly increasing function and so has an inverse function f_K, i.e., $f_K(h_K(n)) = n$. We extend the domain of f_K to $\mathbb{N}$ by defining $f_K(i) = n$ when $h_K(n) \leq i < h_K(n+1)$. We now define

$$f_S(n) = \min\{f_K(n) : K \in \mathcal{K}\},$$

where $\mathcal{K}$ is the set of infinite components of $\Gamma - S$.

We may apply Claim 1 to obtain an unbounded function $g_S(n) \in \mathcal{F}_0$ such that: for all n, $f_S(n) \geq g_S(n)$; and, for all $C \in \mathbb{N}^+$, $g_S(Cn) \leq C g_S(n)$.

Apply Claim 2 to the countable collection $\{g_S : S \subset V\Gamma;\ |S| < \infty\}$ to get an unbounded function $f \in \mathcal{F}_0$ such that $\lim_{x\to\infty} f(x)/g_S(x) = 0$ for every function g_S in the collection.

Choose arbitrary rays $P, Q \in \mathcal{R}(\Gamma)$ belonging to distinct ends of Γ. There exists a finite subset S of $V\Gamma$ that separates subrays of P and Q. We lose no generality in assuming that $VP \cap S = \{p_0\}$ and $VQ \cap S = \{q_0\}$. If $P \sim_f Q$, then there exist vertex sequences $\{x_n\}_{n=0}^{\infty} \subseteq VP$ and $\{y_n\}_{n=0}^{\infty} \subseteq VQ$ regulated by C such that $\delta(x_n, y_n) \leq Cf(n)$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, there is a first j_n such that $\delta(p_{j_n}, S) = n$. Thus, $\delta(p_{j_n}, S) = n \geq f_S(j_n)$. There is an index i_n such that $\delta(p_{j_n}, x_{i_n}) \leq C$; in fact, there is such an i_n with $j_n \geq i_n/C$. It follows that $\delta(x_{i_n}, y_{i_n}) \geq \delta(x_{i_n}, S) \geq \delta(p_{j_n}, S) - C \geq f_S(j_n) - C \geq f_S(i_n/C) - C$, i.e., for infinitely many k, $Cf(k) \geq f_S(k/C) - C$.

Since $f_S(k/C) \geq g_S(k/C) \geq (1/C)g_S(k)$, we see that, for infinitely many k,

$$\frac{f(k)}{g_S(k)} \geq \frac{1}{C^2} - \frac{1}{g_S(k)}\,.$$

But the limit as $k \to \infty$ of the left-hand side is 0, while the right-hand side tends to $1/C^2 > 0$, a contradiction. □

6 f-Classes under Translation

Up to this point, while we have discussed metric rays, we haven't imposed any conditions involving graph symmetry. The following example indicates why it may be necessary to do so.

Example 7 *We return to Δ^2 with Cartesian coordinates but consider only the first quadrant together with the nonnegative x- and y-axes. Now subdivide each edge $[(i,j),(i,j+1)]$ by the insertion of i new vertices. Clearly this new graph Γ is still 1-ended, but its automorphism group contains no element of infinite order. All rays containing only finitely many horizontal edges belong to the same fiber. Since at least one of these rays is metric, all of them are metric; a ray containing all but finitely many edges of the form $[(i,m),(i,m+1)]$ for any fixed i has straightness $1/(i+1)$. By contrast, the horizontal rays are geodetic and diverge from each other linearly. In fact, any two rays having infinitely many horizontal edges but finite intersection belong to different f-classes for every $f \in \mathcal{F}_1$. Hence every bundle contains at most a single f-class for every $f \in \mathcal{F}_1$. The paucity of structure in this "discrete" bundle decomposition of Γ serves only to motivate consideration of graphs admitting "translations".*

An automorphism of an arbitrary graph Γ is a *translation* of Γ if it has an infinite orbit in $V\Gamma$. For locally finite graphs, clearly all orbits are finite or all orbits are infinite.

Definition 8 *The double ray D is said to be τ-*translatable *if τ is a translation of Γ and $\tau(D) = D$. Clearly D would then be τ^k-translatable for all $k \in \mathbb{N}$. A ray P is τ-translatable if it is a subray of a τ-translatable double ray. More briefly, D is* translatable *if D is τ-translatable for some translation τ.*

A clever example by Jung and Niemeyer (see [7], p. 195) shows that a ray or double ray being merely translatable is not sufficient for it to be metric. In their example there are *translatable* double rays which are the union of two rays belonging to the same bundle, unlike the case with *metric* double rays (cf. Theorem 4.4). This matter will be pursued in Section 7.

Let $P \in \mathcal{R}(\Gamma)$ and let $\tau \in \mathrm{Aut}(\Gamma)$ satisfy $\tau(P) \subset P$. Thus τ is a translation. By Proposition 3.3(i), $P \sim_f \tau(P)$ for any $f \in \mathcal{F}_0$. It follows that

Proposition 6.1 *If an f-class $\mathcal{C}$ contains a τ-translatable ray, then $\tau(\mathcal{C}) = \mathcal{C}$.*

Proposition 6.2 *If P is a translatable ray, then $\sigma(P) = \sigma_0(P)$.*

Proof. As noted above, $\sigma(P) \leq \sigma_0(P)$ always holds. To prove the reverse inequality, suppose that, for some translation $\tau \in \mathrm{Aut}(\Gamma)$, $\tau(p_k) = p_{k+h}$ holds for all $k \in \mathbb{N}$ and some $h \in \mathbb{N}$.

There exists a sequence $\{(k_i, \ell_i)\}_{i=0}^{\infty}$ of integer pairs such that $\lim_{i\to\infty}(\ell_i - k_i) = \infty$ and $\sigma(P) = \lim_{i\to\infty} \delta(p_{k_i}, p_{\ell_i})/(\ell_i - k_i)$. For each $i \in \mathbb{N}$, there exists some $t_i \in \mathbb{N}$ such that $\tau^{t_i}(p_0) \leq_P p_{k_i} <_P \tau^{t_i+1}(p_0)$. Equivalently, $t_i h \leq k_i < (t_i + 1)h$. Form a new sequence $\{(k'_i, \ell'_i)\}_{i=0}^{\infty}$, where $k'_i = k_i - t_i h$ and $\ell'_i = \ell_i - t_i h$ for all $i \in \mathbb{N}$. Note that $\delta(p_0, p_{\ell'_i}) \leq \delta(p_{k'_i}, p_{\ell'_i}) + h$. We now have

$$\begin{aligned} \sigma(P) &= \lim_{i\to\infty} \frac{\delta(p_{k_i}, p_{\ell_i})}{\ell_i - k_i} \\ &= \lim_{i\to\infty} \frac{\delta(p_{k'_i}, p_{\ell'_i})}{\ell'_i - k'_i} \\ &\geq \liminf_{i\to\infty} \frac{\delta(p_0, p_{\ell'_i}) - h}{\ell'_i} \geq \sigma_0(P). \end{aligned}$$

□

In [5] (Theorem 9(d)), Halin showed that if D is a τ-translatable double ray in a thick end, then that end contains a countably infinite family of pairwise-disjoint τ-translatable double rays that includes D.

The first part of the following proposition extends Corollary 4.2 of [1], wherein it was shown that the deletion of a translatable double ray from a locally finite, connected, *planar* graph leaves at most two infinite components.

Proposition 6.3 *Let Γ be locally finite and 1-ended, and let D be a translatable double ray in Γ. Then the following hold.*
(i) $\Gamma - D$ has only finitely many infinite components.
(ii) Some component of $\Gamma - D$ is a 1-ended subgraph of Γ.

Proof. Suppose that D is τ-translatable. Index $VD = \{d_i : i \in \mathbb{Z}\}$ in the natural way, and suppose that $\tau(d_i) = d_{i+n}$ for all $i \in \mathbb{Z}$. Assume that $n > 0$; otherwise replace τ by τ^{-1}. Let Y be an infinite component of $\Gamma - D$. Since Γ is 1-ended, there are infinitely many vertices of D that are adjacent to vertices in Y. In particular, there exists some $m \in \mathbb{N}$ such that $\tau^m(Y) = Y$. It follows that, for some $i \in \{0, 1, \ldots, n-1\}$ and for infinitely many values of $k \in \mathbb{Z}$, the vertex $\tau^{km}(d_i)$ is adjacent to Y. Let us therefore fix k as the least positive integer such that $\tau^{km}(d_i)$ is adjacent to Y for all $m \in \mathbb{Z}$. Without loss of generality we may suppose that $i = 0$. (Because of local finiteness, there are only finitely many components of $\Gamma - D$ that are adjacent to d_0.)

Let $1 \le j \le k-1$, and consider the component $Z = \tau^j(Y)$ of $\Gamma - D$. Then

$$\tau^k(Z) = \tau^k(\tau^j(Y)) = \tau^j(\tau^k(Y)) = \tau^j(Y) = Z.$$

This implies that a set of components of $\Gamma - D$ adjacent to, say, $h > 0$ vertices in

$$\{d_0, \tau(d_0), \cdots, \tau^{k-1}(d_0)\}$$

is the same as a set adjacent to h vertices in $\{\tau^m(d_0), \tau^{m+1}(d_0), \cdots, \tau^{m+k-1}(d_0)\}$ for all $m \in \mathbb{Z}$. Hence there are only finitely many components of $\Gamma - D$ adjacent to a vertex in the set $\{\tau^m(d_0) : m \in \mathbb{Z}\}$. As this argument would be identical were d_0 replaced by any of $d_1, \ldots, d_{n-1}$, the set of components of $\Gamma - D$ is finite, proving (i).

To prove (ii), suppose on the contrary that all of the infinite components of $\Gamma - D$ are multi-ended. By part (i) of this lemma, we may list them as $Y_1, \cdots, Y_c$. Let $VD = \{d_i : i \in \mathbb{Z}\}$ be indexed as before. For each $j = 1, \ldots, c$, there exists a finite subset $T_j \subset VY_j$ such that $Y_j - T_j$ has at least two (but finitely many) infinite components. No component of $Y_j - T_j$ contains all of the vertices in Y_j adjacent to D, for then $\Gamma - T_j$ would contain at least two infinite components (one containing D), contrary to 1-endedness of Γ. Quite specifically, for each $j = 1, \ldots, c$, there exist integers $m_j < n_j$ and infinite components $Y_j^{(1)}$ and $Y_j^{(2)}$ of $Y_j - T_j$ such that if d_i is adjacent to Y and if $i < m_j$, then d_i is adjacent to $Y_j^{(1)}$ but not $Y_j^{(2)}$, while if $i > n_j$, then d_i is adjacent to $Y_j^{(2)}$ but not $Y_j^{(1)}$. Let $m = \min\{m_1, \ldots, m_c\}$ and $n = \max\{n_1, \ldots, n_c\}$. Then $T_1 \cup \cdots \cup T_c \cup D[d_m, d_n]$ separates Γ into at least two infinite subgraphs, again contrary to the 1-endedness of Γ. □

A function $f \in \mathcal{F}_0$ is called *subhomogeneous* if for all $a \ge 1$ and all $n \in \mathbb{N}$ it holds that $f(an) \le af(n)$. A wide variety of familiar sublinear algebraic and transcendental functions are subhomogeneous. Here is an example of a sublinear function g that is not subhomogeneous.

Example 8 *For $n \in \mathbb{N}$ and $n! \le t < (n+1)!$ let $g(t) = \sqrt{n!}$. Then for $n \ge 4$ and $t = (n+1)!/2$ we have $g(2t) = \sqrt{(n+1)!} = g(t)\sqrt{n+1} > 2g(t)$.*

Theorem 6.4 *Let $\mathcal{E}$ be an end of Γ such that $\mu(\mathcal{E}) \ge 3$. Let $f, g \in \mathcal{F}_1$ be subhomogeneous functions such that $f \gg g$. Then any f-class that contains a translatable metric ray in $\mathcal{E}$ contains uncountably many g-classes.*

Proof. Let P be a τ-translatable metric ray in $\mathcal{E}$. Since $\mu(\mathcal{E}) \geq 3$, there exist at least three pairwise disjoint τ-translatable rays in $\mathcal{E}$, and we let P_0, P_1, P_2 be three such rays. If $\mathcal{E}$ is thin, this follows from [7], Lemma 4.4; if $\mathcal{E}$ is thick, then this follows from [5], Theorem 9(d). Moreover, P_0, P_1, P_2 are all in the same fiber as P and therefore are metric (cf. [7], Corollary 4.5).

There is a labeling of P_0, P_1, P_2 and, for $i = 1, 2$, shortest P_0P_i-paths S_i such that S_i has terminal vertices $a_i \in VP_0$ and $b_i \in VP_i$ and is disjoint from P_{3-i}. Furthermore, we may choose the labeling so that $a_1 \leq_{P_0} a_2 <_{P_0} \tau(a_1)$.

Suppose that $f \in \mathcal{F}_1$ is subhomogeneous. We construct the ray R_f inductively as follows. Its initial vertex is $c_0 = b_1$ on P_1. Suppose that for some $i \in \mathbb{N}$, the vertex $c_i = \tau^{j_i}(b_1)$ on P_1 has been determined. We describe the subpath $R_{f,i}$ of R_f from c_i to c_{i+1}, thereby determining c_{i+1}. (The reader may wish to refer to Figure 6.1.)

Figure 6.1: the subpath $R_{f,i}$.

Beginning at $c_i = \tau^{j_i}(b_1)$, $R_{f,i}$ follows $\tau^{j_i}(S_1)$ to its other terminus $\tau^{j_i}(a_1)$,
then advances along P_0 to $\tau^{j_i}(a_2)$,
then includes $\tau^{j_i}(S_2)$, to $\tau^{j_i}(b_2)$,
then advances along P_2 to $\tau^{j_i+f(j_i)}(b_2)$,
then includes $\tau^{j_i+f(j_i)}(S_2)$, to $\tau^{j_i+f(j_i)}(a_2)$,
then backtracks along P_0 to $\tau^{j_i+1}(a_1)$,
then includes $\tau^{j_i+1}(S_1)$, to $\tau^{j_i+1}(b_1)$,
and finally advances along P_1 to $c_{i+1} = \tau^{j_i+f(j_i)+1}(b_1)$, so that $j_{i+1} = j_i + f(j_i) + 1$.

We define $R_f = \bigcup_{i=0}^{\infty} R_{f,i}$.

In the proofs of the next two claims we use the following notation: $\ell_0 = \delta_{P_0}(a_1, a_2)$; for $i = 1, 2$, ℓ_i denotes the length of S_i; and, for $i = 0, 1, 2$, $d_i = \delta_{P_i}(x, \tau(x))$, for any $x \in VP_i$.

Claim 1. $P \sim_f R_f$.

Proof: Since P and P_1 are in the same fiber, it suffices to show that $P_1 \sim_f R_f$. We choose the following regulated sequences. First define $\{x_n\}_{n=0}^{\infty} \subseteq VP_1$ by $x_n = \tau^n(b_1)$. To select $\{y_n\}_{n=0}^{\infty} \subseteq VR_f$, we consider only vertices on R_f of the form $\tau^i(b_1)$, $\tau^i(b_2)$, and $\tau^i(a_2)$. Taking these particular vertices in their order on R_f, y_n is the $(3n)^{\text{th}}$ one of them.

Set $C = 3(d_0 + d_1 + d_2 + \ell_0 + \ell_1 + \ell_2)$. Clearly $\{x_n\}_{n=0}^{\infty}$ is C-regulated. Also, y_n and y_{n+1} are at distance at most C apart on R_f, since going from one of the types of vertices considered for y_n to the next requires traversing three of the following distances along R_f: d_0, d_1, d_2, $\ell_0 + \ell_1$, ℓ_2. Furthermore, all of the vertices in each of these two C-regulated sequences are distinct.

Observe that $R_{f,i}$ uses all the vertices of the forms $\tau^j(b_1)$, $\tau^j(b_2)$, and $\tau^j(a_2)$, for $j_i \leq j < j_{i+1}$. Thus, the number of vertices of these forms in $R_{f,0} \cup \cdots \cup R_{f,i}$ is $3j_{i+1}$. It follows that, if y_n is in $R_{f,i}$, then $j_i \leq n < j_{i+1}$. Thus, $x_n = \tau^n(b_1)$ satisfies $\tau^{j_i}(b_1) \leq_{P_1} x_n <_{P_1} \tau^{j_{i+1}}(b_1)$. Therefore, $\delta(x_n, y_n) \leq f(j_i)(d_0 + d_1 + d_2) + \ell_0 + \ell_1 + \ell_2 \leq Cf(j_i) \leq Cf(n)$, as required.

Let $g \in \mathcal{F}_1$ be subhomogeneous, and suppose $f \gg g$.

Claim 2. $P \not\sim_g R_f$.

Proof: It suffices to show $P_1 \not\sim_g R_f$. Suppose to the contrary that $P_1 \sim_g R_f$. By Proposition 3.1, there are sequences $\{x_n\} \subseteq VP_1$ and $\{y_n\} \subseteq VR_f$ regulated by C such that $y_0 = x_0 = b_1$ and, for all $n \in \mathbb{N}$, $\delta(x_n, y_n) \leq Cg(n)$. Consider i such that $f(j_i) > C$.

There exists some n such that $y_n \in VP_2 \cap R_{f,i}$ and $\delta_{P_2}(\tau^{j_i}(b_2), y_n) \leq C$. There exists some ℓ such that $y_{n+\ell}$ is on the "long" P_0-portion of $R_{f,i}$ and $\delta_{P_0}(\tau^{j_{i+1}}(a_2), y_{n+\ell}) \leq C$. Evidently, $\delta_{R_f}(y_n, y_{n+\ell}) \geq f(j_i)d_2 - C + \ell_2 + (f(j_i) - 1)d_0 - C$, so that

$$\ell \geq \frac{1}{C}\big[f(j_i)d_2 - C + \ell_2 + (f(j_i) - 1)d_0 - C\big].$$

By the triangle inequality we have $\delta(x_n, x_{n+\ell}) \leq \delta(x_n, y_n) + \delta(y_n, y_{n+\ell}) + \delta(y_{n+\ell}, x_{n+\ell})$. Therefore, $\delta(x_n, x_{n+\ell}) \leq Cg(n) + (2C + \ell_2 + d_0) + Cg(n + \ell)$.

On the other hand, since P_1 is metric, there is a positive constant K such that, for any vertices z, z' of P, $\delta(z, z') \geq K\delta_{P_1}(z, z')$. The two preceding paragraphs combine to yield

$$\begin{aligned} Cg(n) + Cg(n+\ell) + 2C + \ell_2 + d_0 &\geq K\delta_{P_1}(x_n, x_{n+\ell}) \\ \geq \frac{K\ell}{C} &\geq \frac{K}{C^2}\big[f(j_i)d_2 - C + \ell_2 + (f(j_i) - 1)d_0 - C\big] \\ &= \frac{K}{C^2}(d_0 + d_2)f(j_i) + D\,, \end{aligned}$$

where D is an easily computed constant.

Notice that $n + \ell \leq C[(d_0 + d_1 + d_2)j_i + 2f(j_i) + \ell_0 + \ell_1 + 2\ell_2] \leq Lj_i$, where $L = (d_0 + d_1 + d_2)j_i + 2f(j_i) + \ell_0 + \ell_1 + 2\ell_2$. (Recall that, since f is assumed to be subhomogeneous, $f(j_i) \leq j_i f(1)$.) Furthermore, $f(Lj_i) \leq Lf(j_i)$, and so $f(j_i) \geq f(Lj_i)/L \geq f(n+\ell)/L$.

The coup de grace now follows:

$$2Cg(n+\ell) + \ell_2 + d_0 + 2C \geq \frac{K}{C^2}(d_0 + d_2)f(j_i) + D$$

$$\geq \frac{K}{LC^2}(d_0+d_2)f(n+\ell)+D\,.$$

For any $\epsilon>0$, there is an N such that $n+\ell\geq N$ implies $g(n+\ell)<\epsilon f(n+\ell)$. Choosing ϵ so that $2C\epsilon<\frac{K}{LC^2}(d_0+d_2)$ shows that f is bounded, which is a contradiction, proving Claim 2.

It is interesting to note that, even though P_1 is metric, R_f is not, since $\delta(\tau^{j_i}(a_1),\tau^{j_i+1}(a_1))\leq d_0$, while $\delta_{R_f}(\tau^{j_i}(a_1),\tau^{j_i+1}(a_1))=2(\ell_0+\ell_2)+f(j_i)(d_0+d_2)-d_0$. Since f is unbounded, the ratio tends to 0. However, R_f is initially metric: $\sigma_0(R_f)=1/3$.

To see that there are uncountably many g-classes contained in the f-class that contains P, observe that, since g and f are subhomogeneous, then, for every $t\in(0,1)$, the function $h_t=f^tg^{1-t}$ is also subhomogeneous. Let the ray R_t be constructed with respect to h_t exactly as R_f was constructed above with respect to f. By Claim 1, for each $t\in(0,1)$, the ray R_t belongs to the same h_t-class as P, and hence, by Proposition 3.3(iv), to the same f-class, i.e., all of the rays R_t are f-equivalent.

If $0<s<t<1$, then $g\ll h_s\ll h_t\ll f$. By Claim 2, $R_t\nsim_{h_s}P$. But $R_s\sim_{h_s}P$ by Claim 1, and so by transitivity, $R_s\nsim_{h_s}R_t$. Again by Proposition 3.3(iv), $R_t\nsim_g R_s$. It follows that the g-classes represented by the rays R_t for $t\in(0,1)$ are all distinct, and there exist uncountably many of them. □

Example 9 *Let $f(n)=n^t$, where $0<t<1$. If Γ is 1-ended and contains a translatable ray P, then by the above theorem:*
(i) If $g(n)=n^s$, where $0\leq s<t$, then the f-class containing P contains uncountably many g-classes.
(ii) If $g(n)=\log n$, then the f-class containing P contains uncountably many g-classes.

7 Bundles in Infinite Planar Maps

In this section we consider planar maps. Our main results are Theorem 7.1, that the number of infinite faces is bounded above by the number of ends, and Corollary 7.4, that any two rays whose union is a translatable facial double ray belong to different bundles.

We remind the reader that a *facial walk* $(\ldots,v_{i-1},e_i,v_i,e_{i+1},\ldots)$ is a walk determined by the following two conditions: (i) in the clockwise rotation about each vertex v_i, the edge e_{i+1} immediately follows the edge e_i, and (ii) the walk is maximal with respect to the property that no ordered pair (v_i,e_{i+1}) appears more than once. Thus, a facial walk all of whose vertices are distinct is either a double ray or a circuit. A double ray is *facial* if it is also a facial walk. A ray R is *facial* if it is a subray of a facial double ray. In a 2-connected map every facial walk is either a circuit or a double ray.

Theorem 7.1 *If a 2-connected locally finite planar map has exactly m ends, then it has at most m facial double rays.*

Proof. Since a locally finite graph with no ends is finite, the result is trivial if $m = 0$.

We next consider the case $m = 1$, the initial step for our proof by induction on m. Suppose that the map M in question is 1-ended but that D_1 and D_2 are distinct facial double rays of M. Clearly D_1 has a vertex of valence at least 3; otherwise $M = D_1$ would be neither 2-connected nor 1-ended. Let R_0 be a shortest D_1D_2 path; for definiteness R_0 joins $u_1 \in VD_1$ to $u_2 \in VD_2$. If $D_1 \cap D_2 \neq \emptyset$, then R_0 reduces to a single vertex $u := u_1 = u_2 \in V(D_1 \cap D_2)$, and we require u to be at least 3-valent.

For $i = 1, 2$, let e_i, f_i, consecutively in clockwise order, be the two edges of D_i incident with u_i. Let P_1 be the ray in D_1 with origin u_1 and first edge e_1, let Q_1 be the other ray in D_1 with origin u_1. Let P_2 be the ray in D_2 with origin u_2 and first edge f_2 and let Q_2 be the other ray in D_2 with origin u_2. If $D_1 \cap D_2 \neq \emptyset$, then at most one of the equalities $e_1 = f_2$ and $f_1 = e_2$ may hold. In every case, $e_1 \neq e_2$ and $f_1 \neq f_2$.

Claim 1: *$P_1 - u_1$ and $Q_1 - u_1$ are in different components of $M - (R_0 \cup VD_2)$.*

Proof. Suppose R is a path in $M - (R_0 \cup VD_2)$ joining a vertex $p \in VP_1$ to a vertex $q \in VQ_1$ that is otherwise disjoint from $P_1 \cup Q_1$. Then $C = R \cup D_1[p, q]$ is a circuit in M through u_1. The facial sides of D_1 and D_2 are on opposite sides of C at u_1 (this is demonstrated by R_0), showing that C separates P_1 from P_2. This contradicts the fact that M is 1-ended. □

Claim 2: *Both $(P_1 - u_1) \cap (Q_2 - u_2)$ and $(Q_1 - u_1) \cap (P_2 - u_2)$ are empty.*

Proof. By symmetry, it suffices to consider just $P_1 \cap Q_2$. If the claim fails, then $D_1 \cap D_2 \neq \emptyset$, and we let q be the first vertex of $Q_2 - u$ that is also on P_1. Consider the circuit $C = D_2[u, q] \cup D_1[u, q]$. As we traverse C from u to q, the facial side of D_1 is on the right if we proceed along D_1, while the facial side of D_2 is also on the right if we proceed along D_2. Thus C separates the facial sides of D_1 and D_2, and again P_1 and P_2 are separated by a finite subgraph, a contradiction. □

Continuing with the assumption that $m = 1$, *suppose that $D_1 \cap D_2$ is infinite.* By Claim 2 $P_1 \cap P_2$ or $Q_1 \cap Q_2$ is infinite. We may choose the labeling so that $P_1 \cap P_2$ is infinite.

Because M has only one end, there exist infinitely many pairwise-disjoint P_1Q_2-paths. If a P_1Q_2-path were to have a vertex in $Q_1 \cup P_2$, then by Claim 1 it would contain a Q_1P_2-subpath. Because this subpath is disjoint from $P_1 \cup Q_2$, Claim 1 again implies that the subpath has no nonterminal vertex in $D_1 \cup D_2$. Thus, either there is an infinite set of P_1Q_2-paths disjoint from $D_1 \cup D_2$ except for their terminal vertices or there is an infinite set of Q_1P_2-paths disjoint from $D_1 \cup D_2$ except for their terminal vertices. Without loss of generality, we assume the former.

It follows that there is a P_1Q_2-path S joining $p \in VP_1 \setminus \{u\}$ and $q \in VQ_2 \setminus \{u\}$ but otherwise disjoint from $D_1 \cup D_2$. Let w be the vertex in $D_1 \cap D_2$ nearest to p on $D_1[u, p)$, and let $z \neq p$ be the vertex in $D_1 \cap D_2$ nearest to p in the subray of P_1 with origin p. The paths $D_1[p, w]$ and $D_1[p, z]$ are non-trivial with no interior vertex on D_2.

If $p \notin VD_2$, then $C = D_1[w, z] \cup D_2[w, z]$ is a circuit. The edge of S incident with p cannot lie on the side of C containing the facial side of D_1. But q is not on C by Claim 2 and S cannot cross C by Claim 1. Thus S contains edges on opposite sides of C, a contradiction. If $p \in VD_2$, then the same argument applies, except now the edge of S incident with p is separated from q by one of the circuits $D_1[p, w] \cup D_2[p, w]$ and $D_1[p, z] \cup D_2[p.z]$, again a contradiction.

Suppose therefore that $D_1 \cap D_2$ *is finite*, say contained in $D_1[a_1, b_1]$ and contained in $D_2[a_2, b_2]$, chosen so that, for $i = 1, 2$, $u_i \in VD_i[a_i, b_i]$. Let Φ_1 denote the finite subgraph $D_1[a_1, b_1] \cup D_2[a_2, b_2] \cup R_0$.

Since P_1 and P_2 are in the same end of M, there exist infinitely many pairwise-disjoint P_1P_2-paths that are disjoint from Φ_1. Either infinitely many of these paths are disjoint from $Q_1 \cup Q_2$ or (since any that meets $Q_1 \cup Q_2$ contains, by Claims 1 and 2, a Q_1Q_2-subpath) infinitely many that contain Q_1Q_2-subpaths. Thus, either there are infinitely many pairwise-disjoint P_1P_2-paths with no interior vertex in $\Phi_1 \cup D_1 \cup D_2$, or there are infinitely many pairwise-disjoint Q_1Q_2-paths with no interior vertex in $\Phi_1 \cup D_1 \cup D_2$. By symmetry, we may assume the former.

One may choose three such paths R_1, R_2 and R_3 whose terminal vertices are successively further away from Φ_1 along both P_1 and P_2. For $i = 1, 2, 3$ and $j = 1, 2$, let R_i meet P_j at the vertex $r_{i,j}$. There is a P_1Q_2-path R_4 that meets P_1 at a vertex $r_{4,1}$ farther from Φ_1 along P_1 than $r_{3,1}$ and meets Q_2 at the vertex $r_{4,2}$ farther along Q_2 from u than any vertex of Φ_1. We may specify that R_4 have no interior vertex in $\Phi_2 = \Phi_1 \cup R_1 \cup R_2 \cup R_3 \cup D_1 \cup D_2$. Similarly, there is a Q_1P_2-path R_5 joining $r_{5,1} \in VQ_1$ to $r_{5,2} \in VP_2$ that has no interior vertex in $\Phi_3 = \Phi_2 \cup R_4$. Finally, without destroying planarity one may adjoin a path R_6 to the facial side of D_1 joining $r_{1,1}$ to $r_{5,1}$. The subspace of the plane or sphere consisting of $D_1[r_{4,1}, r_{5,1}] \cup D_2[r_{4,2}, r_{5,2}] \cup \Phi_3 \cup R_6$ contains a homeomorph of $K_{3,3}$, contrary to planarity. This concludes the initial step of the induction.

As induction hypothesis, assume that for some $m \geq 1$ every 2-connected locally finite connected planar map having exactly m ends has at most m facial double rays. Suppose that M has exactly $m + 1$ ends, and let $\mathcal{E}$ be an end of M. Since M has only finitely many ends, every end is a "free end" (*freies Ende* in Halin [3], p. 129), which means that there exists a finite subgraph $T \subset M$ that separates $\mathcal{E}$ from all other ends, i.e., there exists an infinite component M_1 of $M - T$ such that any ray in M is all but finitely in M_1 if and only if it belongs to $\mathcal{E}$. Let M_2 be the union of the remaining components of $M - T$. It is easy to see that we may choose T so that each of T, $M_1 \cup T$ and $M_2 \cup T$ is connected. Among all such finite subgraphs T, choose T to be minimal.

We claim that $M_1 \cup T$ is in fact 2-connected. Otherwise, it has a cut vertex u. Notice that $(M_1 \cup T) - u$ has a unique infinite component I. If $u \notin VT$, then T, being connected, is contained in a component of $(M_1 \cup T) - u$. But then u would be a cut vertex of M. So $u \in VT$ and $T \not\subseteq I + u$. Let $T' = T \cap (I + u)$. Then T' is connected, as is $I + u$ and $M_2 \cup T' \cup (M_1 - I)$. This violates the minimality of T. So $M_1 \cup T$ is 2-connected.

If $M_2 \cup T$ is not 2-connected, as in the second paragraph preceding, any cut vertex of $M_2 \cup T$ is in T. So there are only finitely many cut vertices of $M_2 \cup T$ and we may finitely often add to $M_2 \cup T$ a path from $M_1 \cup T$ having terminal vertices in T but otherwise disjoint from $M_2 \cup T$ to eliminate these cut vertices. That is, there is a 2-connected supergraph $M_2' \cup T$ of $M_2 \cup T$ so that $(M_2' \cup T) - (M_2 \cup T)$ is finite. So we add these paths to T, thereby obtaining a finite, connected subgraph T so that both $M_1 \cup T$ and $M_2 \cup T$ are 2-connected.

Every facial double ray of M has its two facial subrays as facial rays in either $M_1 \cup T$ or in $M_2 \cup T$. Since T is finite, no such ray is contained in T, and so there is a bijection from the set of facial rays in M to the union of the sets of facial rays in $M_1 \cup T$ and in $M_2 \cup T$. Since every facial double ray is the union of two facial rays, we see by induction that there are in sum at most $2 + 2m$ facial rays in $M_1 \cup T$ and $M_2 \cup T$, and so M has at most $2(m + 1)$ facial rays and, therefore, at most $m + 1$ facial double rays, as required. □

We are not confident of the necessity of the hypothesis of 2-connectedness in Theorem 7.1; we believe that mere connectedness would suffice. However, there are technicalities that arise in its absence that we did not easily see how to overcome, and so we chose to present the 2-connected version above. In [1] (Theorem 2.3), it was shown that if a locally finite, 2-connected, 1-ended graph Γ is also almost-transitive (i.e., $\mathrm{Aut}(\Gamma)$ acts with finitely many orbits on $V\Gamma$), then there is no facial double ray in any planar embedding of Γ.

In the previous section we noted that two rays whose union is a translatable (though not metric) double ray may belong to the same bundle. We also noted that translatable rays need not be metric. We will show that neither of these statements holds for facial double rays in a planar map, obtaining therefrom a generalization of Theorem 2.1 of [2] from fibers to bundles.

Lemma 7.2 *Let M be a locally finite planar map and let D be a τ-translatable facial double ray in M. Let $a, b \in VD$ and $m \in \mathbb{N}$. If $\tau^m(a)$ lies in $D(a, b)$, then $\delta(a, b) \geq m + 1$.*

Proof. Let $M, D, a, b.m$ and τ satisfy all the conditions in the hypothesis. For any shortest ab-path S, the submap $S \cup D$ of M contains only finitely many finite faces. The *interior* of S denotes the union of the subgraphs of M inside and on the boundary of these faces, but does not include the vertices and edges of S itself. By local finiteness, there exist only finitely many shortest ab-paths in M. From among such shortest ab-paths, we select one with minimal interior (with respect to inclusion) and denote it henceforth by S. The complement in M of the interior of S is the *exterior* of S, but similarly does not include the vertices and edges of S itself. By the minimality of the interior of S, no shortest ab-path in M contains an edge in the interior of S that is not an edge of S.

To reduce notation, let $S_i := \tau^i(S)$ for $i = 0, 1, \ldots, m$. Thus $S = S_0$. For each $1 \le i \le m$, we have $S_i \cap S \neq \emptyset$. This is because $\tau^i(a)$ and $\tau^i(b)$ are separated in the plane by $S \cup D[a, b]$. By application of the translation τ^{j-i} for $0 \le i < j \le m$, it follows that $S_i \cap S_j \neq \emptyset$. Therefore, it makes sense to define $z_{i,j}$ to be the first vertex of $S_i \cap S_j$ encountered when starting from $\tau^j(a)$ and proceeding along S_j.

Remarks. We note that the minimality of the interior of S implies:

1. the subpath $S_j[z_{i,j}, \tau^j(b)]$ of S_j is on S or in the exterior of S_i;
2. $S_i \cap S_j$ is contained in either the subpath $S_i[\tau^i(a), z_{i,j}]$ or in the subpath $S_i[z_{i,j}, \tau^i(b)]$ of S_i; and
3. in the latter case, $S_i \cap S_j$ is a path.

Let $z := z_{0,1}$. Thus $\tau(z)$ lies on S_1, but $\tau(z) \neq z$, because τ is a translation.

Case 1: $\tau(z)$ lies on the subpath $S_1(z, \tau(b))$. We show by induction that, for $i = 1, 2, \ldots, m$, the vertex $\tau^i(z)$ is on S or in the exterior of S. This is true for $i = 1$ by the case assumption and the above remarks. Suppose it is true for some $i < m-1$. Then $\tau^i(z)$ is the first vertex of $S_i \cap S_{i+1}$ encountered when starting from $\tau^{i+1}(a)$ and proceeding along S_{i+1}. Thus, S_{i+1} must first encounter S at or before $\tau^i(z)$, that is, S_{i+1} encounters S no later than it encounters S_i. But the case assumption implies that $\tau^{i+1}(z)$ lies on the subpath $S_{i+1}[\tau^i(z), \tau^{i+1}(b)]$ of S_{i+1}, and so by the above remarks is on S or in the exterior of S, completing the induction.

Consider the three vertices $z_{0,i}$, $\tau^i(z)$, and $\tau^{i+1}(z)$ on S_i. The preceding paragraph shows that, starting from $\tau^i(a)$, $z_{0,i}$ is encountered no later than $\tau^i(z)$. The case assumption imples that $\tau^{i+1}(z)$ is strictly later on S_i than $\tau^i(z)$. For $i = 1, 2, \ldots, m$, let P_i denote the subpath $S_i[\tau^i(a), z_{0,i}]$ of S_i. Since $P_i \cap P_{i+1} = \emptyset$, the m paths $P_1, P_2, \ldots, P_m$ are pairwise disjoint. Thus, the m vertices $z_{0,1}, z_{0,2}, \ldots, z_{0,m}$ of S are distinct from each other and from a and b, and S has length at least $m + 1$, as required.

Case 2: $\tau(z)$ lies on the subpath $S_1(\tau(a), z)$. We show by induction that, for $i = 1, 2, \ldots, m$, the vertex $\tau^i(z)$ is in the interior of S. This is true for $m = 1$ by the case assumption, and we suppose that it is true for some $i < m - 1$. Starting at $\tau^{i+1}(a)$, the path S_{i+1} first encounters S_i at $\tau^i(z)$ and this vertex lies in the interior of S. The case assumption implies that $\tau^{i+1}(z)$ lies between $\tau^{i+1}(a)$ and $\tau^i(z)$ on S_{i+1}. The above remarks imply that $\tau^{i+1}(z)$ must be in the interior of S, completing the induction.

We have shown that on S_i the four vertices $\tau^i(a)$, $\tau^i(z)$, $\tau^{i-1}(z)$, and $z_{0,i}$ are encountered in precisely this order.

Subcase 2.1: $S_1 \cap S[a,z]$ contains a vertex different from z. In this case, $S_1 \cap S \subset S(a,z)$. This implies that, if $1 \leq i < j \leq m$, then all the intersections of $S_i \cap S_j$ are enirely in the interior of S. Therefore, as in Case 1, the m vertices $z_{0,i}$ are distinct, implying that S has length at least $m+1$.

Subcase 2.2: $S_1 \cap S[a,z] = \{z\}$. In this case, $S_1 \cap S$ is a path (possibly consisting only of z). If $1 \leq i < j \leq m$, then the subpath $S_i(\tau^i(a), z_{0,i})$ must contain a vertex of S_j. Let i^* be the least index i such that $S_i \cap S_1 \cap S = \{z\}$. (It may be that $i^* = 1$ or that i^* does not exist, in which case we define $i^* = m+1$.) For $m \geq j \geq i^*$ and all $i \in \{1, \ldots, j-1, j+1, \ldots, m\}$, $S_i \cap S_j$ is contained entirely in the interior of S. For $1 \leq i < k < i^*$, the intersection $S_i \cap S_k$ is a path. Proceeding from $\tau^i(a)$ along S_i, let w_i be the last vertex of $S \cap S_i$ encountered. Then the m vertices w_i are distinct, again showing that S has length at least $m+1$. □

Corollary 7.3 *Let M be a locally finite, planar map, and let the double ray D be a facial double ray in M. If D is translatable, then D is metric.*

Proof. Suppose that $\tau(d_i) = d_{i+t}$ for all $i \in \mathbb{Z}$. For $d_i, d_j \in VD$, write $|i-j| = m_{i,j}t + r_{i,j}$, where $m_{i,j}, r_{i,j} \in \mathbb{Z}$ and $0 \leq r_{i,j} < t$. By Lemma 7.2,

$$\sigma(D) = \liminf_{|i-j|\to\infty} \frac{\delta(d_i,d_j)}{\delta_D(d_i,d_j)} \geq \liminf_{|i-j|\to\infty} \frac{m_{i,j}+1}{m_{i,j}t + r_{i,j}} = \frac{1}{t} > 0.$$

□

Now we have an immediate consequence of the preceding result followed by Theorem 4.4:

Corollary 7.4 *Let M be a locally finite, planar map, and let P and Q be rays whose union is a facial double ray of M. If $P \cup Q$ is translatable, then P and Q belong to different bundles in $\mathcal{R}(M)$.*

A graph or map is said to be *almost-transitive* if its automorphism group acts on its vertex set with only finitely many orbits.

Lemma 7.5 *The facial double rays in a planar, locally-finite, almost-transitive map are translatable.*

Proof. Let the map M be as described, and let D be a facial double ray in M. Since M is almost-transitive, some orbit O of the orientation-preserving subgroup H of $\mathrm{Aut}(M)$ contains an infinite sequence $v_1, v_2, \ldots$ of distinct elements of $O \cap VD$. For each $i \in \mathbb{N}$, let e_i denote the edge incident with both v_i and D that follows D in the clockwise ordering about v_i. Choose $\alpha_i \in H$ such that $\alpha_i(v_i) = v_1$. Since the valence of v_1 is finite, there exist distinct $m, n \in \mathbb{N}$ such that $\alpha_m(e_m) = \alpha_n(e_n)$. Then $\tau := \alpha_n^{-1}\alpha_m$ maps v_m onto v_n and e_m onto e_n. Since τ also preserves orientation, $\tau(D) = D$ and hence is a translation of D. □

Putting together the two previous results yields the following strengthening of Theorem 2.1 of [2].

Corollary 7.6 *Any two rays whose union is a facial double ray in a planar, locally-finite, almost-transitive map belong to different bundles.*

8 Concluding Questions, Conjectures, and Remarks

We believe that in Theorem 6.4 the hypothesis "metric" may be dropped, provided that the function f "distinguishes ends". Specifically:

Conjecture 1. Let the subhomogeneous function $f \in \mathcal{F}_1$ have the property that f-equivalence implies end-equivalence on $\mathcal{R}(\Gamma)$. Then every thick end of Γ that contains a translatable ray contains uncountably many f-classes.

In the same vein:

Conjecture 2. Let $f \in \mathcal{F}_0$. Every end of Γ either is contained within an f-class or is the union of uncountably many f-classes.

Question: Given a locally finite multiended graph Γ, does there exist a function $h \in \mathcal{F}_0$ such that for any $f \in \mathcal{F}_0$, if $h \gg f$, then f satisfies the end-separating property of Theorem 5.3, but if $f \gg h$, then there exist f-equivalent rays in $\mathcal{R}(\Gamma)$ belonging to different ends of Γ? If such a function h does exist, then does h also satisfy the end-separating property of Theorem 5.3?

We have sought to emphasize that the two properties "metric" and "translatable" of rays are independent, although in conjunction with other assumptions, one may imply the other (viz. Corollary 7.3). Other assumptions may also offer an alternative to the assumption of "metric" or "translatable". For example, in comparing the hypothesis of the following result with that of Theorem 6.4, we let the existence of a periodic subgraph stand in for "metric" to obtain a weaker but similar conclusion.

Proposition 8.1 *Let α, $\beta \in Aut(\Gamma)$ be translations such that $\langle \alpha, \beta \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Let $f \in \mathcal{F}_0$ be unbounded and let $P \in \mathcal{R}(\Gamma)$ be a ray translated by some translation in $\langle \alpha, \beta \rangle$. Then the f-class containing P contains a set of pairwise-disjoint rays representing uncountably many fibers of Γ.*

We omit the proof of this proposition, as it is long and tedious. It uses techniques analogous to those used to prove Lemma 5 of [8]. Of significance is that the ray P in this proposition need not be metric, as may be illustrated in a higher dimensional variant of the above-cited example on p. 195 of [7]. The vertex set is $\mathbb{N} \times \mathbb{Z} \times \mathbb{Z}$. Each vertex is adjacent to a vertex where exactly one coordinate has been increased by 1. Additionally, each vertex (a, b, c) is adjacent to the vertices $(a, b \pm a, c)$ and $(a, b, c \pm a)$. We let $\alpha(a, b, c) = (a, b + 1, c)$ and $\beta(a, b, c) = (a, b, c + 1)$. No ray with only finitely many different first coordinates is metric, although infinitely many such rays may be translated by an automorphism of the form $\alpha^m \beta^n$.

We believe that Corollary 7.6 can be strengthened as follows.

Conjecture 3. Any two rays whose union is a facial double ray in a planar, locally-finite, almost-transitive map belong to different ends.

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