On planarity of compact, locally connected, metric spaces

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Abstract

Thomassen [Combinatorica **24** (2004), 699–718] proved that a 2–connected, compact, locally connected metric space is homeomorphic to a subset of the sphere if and only if it does not contain K_5 or $K_{3,3}$. The "thumbtack space" consisting of a disc plus an arc attaching just at the centre of the disc shows the assumption of 2–connectedness cannot be dropped. In this work, we introduce "generalized thumbtacks" and show that a compact, locally connected metric space is homeomorphic to a subset of the sphere if and only if it does not contain K_5, $K_{3,3}$, or any generalized thumbtack, or the disjoint union of a sphere and a point.

1 Introduction

It is well-known that planarity of finite graphs was characterized in 1930 by Kuratowski [1], who showed that a finite graph is planar if and only if it does not contain a homeomorph of either the complete graph K_5 or the complete bipartite graph $K_{3,3}$. It is natural to wonder how this might generalize. Thomassen [3] proved the following.

Theorem 1.1 *Let X be a compact, locally connected, 2–connected metric space. Then X is homeomorphic to a subspace of the sphere if and only if it does not contain a homeomorph of either K_5 or $K_{3,3}$.*

The *thumbtack space* consists of a closed unit disc in the plane, together with a line segment, one end of which is the centre of the disc. It is a standard fact (and an easy consequence of Lemma 2.1 below) that the thumbtack space is not homeomorphic to a subspace of any 2–manifold and therefore, in particular, is not a subspace of the sphere. Thus, the assumption of 2–connection in Theorem 1.1 is required.

In this work, we completely characterize planarity for compact, locally connected metric spaces. Such a space has only finitely many components. If none of them is a sphere, then the whole space embeds in the sphere if and only if each component embeds in the sphere. (While we are on this small point, we use the word *planar* to mean is homeomorphic to a subspace of the sphere. Except for the sphere itself, this is the same as being homeomorphic to a subspace of the plane.) Thus, it will suffice to work with connected spaces. In the next section, we describe a general version of the thumbtack space and in the following two sections we shall prove our main result:

Theorem 1.2 *A compact, locally connected metric space X is not planar if and only if X contains either K_5 or $K_{3,3}$ or a generalized thumbtack or the disjoint union of a sphere and a single point.*

If X contains one of the non–planar spaces listed in the theorem, then X is not planar. The interesting point is the converse.

2 Webs

In [3] Thomassen suggested that Theorem 1.1 extends to connected, locally connected compact metric spaces if we also exclude certain subspaces of the thumbtack space. However, the description given in [3] does not suffice. In this section we introduce the generalization of the thumbtack space that is the remaining forbidden structure for planarity.

Let X be a topological space and let Y be a subspace of X. A *Y–bridge* in X is the closure B of a component of $X \setminus Y$. The *attachments of* B are the elements of $B \cap Y$.

In the cases of interest, X is a locally connected space and Y is a closed subset of X. In the important case that Y is a homemorph of a circle, the attachments of a Y–bridge B naturally occur in the cyclic order determined by Y. A *residual arc* of B is the closure in Y of a component of $Y \setminus B$. In particular, every point of $Y \setminus B$ is in a unique residual arc of B.

If B and B' are distinct Y–bridges and Y is again a homeomorph of a circle, then either all the attachments of B are in the same residual arc of B' or they are not. In the latter case, B and B' *overlap*. It is not hard to see that this is a symmetric condition, that is, if all the attachments of B are in the same residual arc of B', then all the attachments of B' are in the same residual arc of B.

It is also not hard to see that B and B' overlap if and only if either there are attachments x, y of B and x', y' of B' so that x, x', y, y' are distinct and occur in this cyclic order in Y — in this case B and B' are *skew*, or B and B' both have just three attachments and they are all the same — in this case B and B' are *3–equivalent*. (Here is an argument slightly easier than the one given in [3]. If one of them, say B, has an attachment x that is not an attachment of B', then pick x' and y' to be the ends of the residual arc A of B' containing x. Since not every attachment of B is in A, there is an attachment y of B in $Y \setminus A$ and, therefore, the vertices x, x', y, y' are distinct and occur in this cyclic order in Y. Otherwise the attachments of B and B' are identical. If there are not at least three attachments, then they do not overlap. If there are at least four attachments, then any four may be chosen as x, x', y, y'.)

As an historical remark, we note that Tutte developed the theory of bridges (including the notion of overlapping bridges) in graphs and binary matroids in the course of several papers. It is difficult to cite references for particular results in the subject. Bridges appear in [5] under the name J-components and also in [6]. The book [7] is a good reference for material about bridges.

A *web with centre* w in a locally connected metric space X is a closed connected subspace W of X in which there is a sequence $(C_i)_{i\geq 0}$ of disjoint circles with the following two properties:

1. for each $i > 0$, there are two overlapping C_i–bridges in W, one containing $C_0, C_1, \dots,$ C_{i-1}, and the other containing $C_{i+1}, C_{i+2}, \dots$;

2. every neighbourhood of w contains all but finitely many of the C_i.

A *generalized thumbtack* consists of a web W with centre w together with a line segment, one end of which is w. Moreover, the line and the web are *internally disjoint*; that is, they have only w in common. If W is a web with centre w embedded in the sphere, then each of the C_i $(i > 0)$ has the property that the two C_i–bridges in W are embedded in different faces of C_i. Thus, if W is the web of a generalized thumbtack, there is no face of W incident with w. This proves the following.

Lemma 2.1 *No generalized thumbtack embeds in the sphere.*

3 Web-free blocks

Our main result is that if X is a locally connected, compact metric space that has no K_5, $K_{3,3}$, generalized thumbtack or the disjoint union of a sphere and a point, then X is homeomorphic to a subspace of the sphere. By earlier remarks, it suffices to consider the case X is connected. In view of Theorem 1.1, we may further assume that X is not 2–connected, so there is at least one point x of X for which $X \setminus \{x\}$ is not connected. Such an x is a *cut point* of X.

If K and L are subspaces of X with no cut points and $|K \cap L| \geq 2$, then $K \cup L$ also has no cut points. Thus, it is an easy application of Zorn's Lemma to see that X has maximal connected subspaces with no cut points; we shall call such a maximal subspace a *block* of X. The blocks with more than one point are precisely Whyburn's E_0–sets [8]. Obviously, distinct blocks intersect in at most one point, and a block is easily seen to be closed. A block is *non–degenerate* if it has more than one point (which is easily equivalent to saying it contains a simple closed curve).

The following lemma is the first of the two main points of the proof.

Lemma 3.1 *Suppose X is a connected, locally connected, compact metric space and that B is a non–degenerate block of X embeddable in the sphere. Then either B has an embedding in the sphere so that each cut point of X in B is incident with a face of B, or B contains a web centred at some cut point.*

The second main point of the proof, which is in the next section, is to use this lemma to show that if every block of X embeds in the sphere and X has no generalized thumbtack, then X embeds in the sphere. In order to prove Lemma 3.1, we need the following.

Lemma 3.2 *Suppose X is a connected, locally connected, compact metric space and B is a block of X. Then B contains only countably many points that are cut points of X.*

Proof. Let C denote the set of cut points of X in B. For each $v \in C$, let $K(v)$ denote any component of $X - v$ not containing $B - v$.

The result is an immediate consequence of showing the following: for each positive integer k, the number of $v \in C$ for which the distance from v to some point of $K(v)$ is at least $1/k$ is finite.

Otherwise, there is a sequence (v_n) of distinct points in C and points w_n in $K(v_n)$ so that the sequences (v_n) and (w_n) converge to v and w, respectively, and, for all n, $dist_X(v_n, w_n) \geq 1/k$. Then $dist_X(v, w) \geq 1/k$ as well. Let V and W be disjoint, connected open neighbourhoods of v and w, respectively. For n sufficiently large, $v_n, v_{n+1} \in V$ and $w_n, w_{n+1} \in W$.

Let A be an arc in W joining w_n and w_{n+1}. Obviously A is disjoint from $\{v_n, v_{n+1}\}$, a contradiction, since every $w_n w_{n+1}$–arc must contain both v_n and v_{n+1}. ■

The remainder of this section is devoted to proving the following, which clearly combines with Lemma 3.2 to prove Lemma 3.1.

Lemma 3.3 *Suppose X is a 2-connected, locally connected, compact metric space embeddable in the sphere and S is a countable subset of X. Then either X has an embedding in the sphere so that each point of S is incident with a face of X, or X contains a web centred at some point of S.*

Proof. The following claim is central to the proof.

Claim 1 *Let w be an arbitrary point of X and suppose X has no web centred at w. Then there is a point $w' \in X \setminus \{w\}$ so that, if we add an arc A joining w and w' that is internally disjoint from X, then $X \cup A$ embeds in the sphere. In particular, any embedding of $X \cup A$ in the sphere induces an embedding of X in the sphere having w incident with a face.*

Proof. Suppose we have a connected subspace W_i of X containing simple closed curves $C_1, \dots, C_i$ so that, for each j with $1 < j \le i$, there are two overlapping C_j–bridges, one containing $C_1 \cup \cdots \cup C_{j-1}$, and the other containing $C_{j+1} \cup \cdots \cup C_i \cup \{w\}$. We claim that either we can extend the sequence to C_{i+1}, with C_{i+1} within distance $1/i$ of w or we can find w' and introduce A.

Assume now that X is embedded in the sphere. Consider a closed connected neighbourhood N of w having diameter (in X) less than $1/i$ and less than the distance from w to C_i. From [3, Cor. 4.4], either there is an arc in the sphere that is internally disjoint from X and joins w to some point of the face of N containing C_i or N contains a simple closed curve C separating w from C_i in the sphere. Since the former implies w is incident with a face of X, we may assume it is the latter that occurs.

Now C is disjoint from C_i, and the face F of C containing w is different from the face of C containing C_i. Let M be the C–bridge containing w and let L be the union of the remaining C–bridges contained in F. Then $C \cup L$ is a 2–connected space and so by [2] the face F' of $C \cup L$ containing M is bounded by a simple closed curve C' and there is only one C'–bridge contained in F', namely M.

Let w' be an attachment of the C'–bridge containing C_i. Consider the space $M^+ = M \cup C' \cup \alpha$, where α is an arc internally disjoint from X joining w and w'. If M^+ is planar, then, in any planar embedding, C' bounds a face (since $M^+ \setminus C'$ is connected) and, therefore, the arc α may be added to X, as required. If it is not planar, then Theorem 1.1 implies it has a homeomorph K of either K_5 or $K_{3,3}$. Since $M \cup C'$ is planar, $\alpha \subseteq K$ and, therefore, $M \cup C'$ has a simple closed curve C_{i+1} so that the C_{i+1}–bridges containing w and w' overlap. In particular, $w' \notin C_{i+1}$ and so the C_{i+1}–bridge containing w' is the same as the one containing C_i, and overlaps the C_{i+1}–bridge containing w, as required. □

We can enumerate the elements of S as $a_1, a_2, \dots$. We add to X a sequence (A_i) of arcs. If we have added $A_1, \dots, A_{i-1}$ and i' is least such that $a_{i'} \notin \cup_{j<i} A_j$, Claim 1 implies we can add an arc A_i such that: $a_{i'}$ is an end of A_i; A_i has diameter $< 1/i$; A_i is internally disjoint from X; A_i is totally disjoint from $\bigcup_{j<i} A_j$; and $X \cup \left(\bigcup_{j\le i} A_j\right)$ is planar.

Claim 2 $X \cup \left(\bigcup_{j\ge 1} A_j\right)$ *is planar.*

Proof. Notice that $X \cup \left(\bigcup_{j\ge 1} A_j\right)$ is a compact, 2–connected, locally connected metric space, so Theorem 1.1 implies that either $X \cup \left(\bigcup_{j\ge 1} A_j\right)$ is planar or it contains a subspace

K homeomorphic to either K_5 or $K_{3,3}$. Notice that K has only 5 or 6 branchpoints v_i; these are points of X, so they have disjoint connected (in X) neighbourhoods U_i.

Consider the three or four branches of K having v_i as an end point. If possible, for each such branch β, let $b_{\beta,i}$ be a point of $(\beta \cap U_i) - v_i$. Otherwise, there is an A_j contained in β and containing v_i. In this case, let $b_{\beta,i}$ denote the other end of A_j. Let $\mathfrak{A}_i$ denote the union of such A_j. (It is also possible that β is equal to some A_j; such an A_j will be in $\mathfrak{A}_i$ for both its ends.)

For each branch β of K, with ends v_i and $v_{i'}$, let β' be the subarc of β joining $b_{\beta,i}$ and $b_{\beta,i'}$. Every point x of X in β' has a connected neighbourhood V_x in X, which may be chosen so that, if x, x' are in distinct β', then V_x and $V_{x'}$ are disjoint. Since β' is compact, $X \cap \beta'$ is covered by finitely many of the V_x. Together with finitely many of the arcs A_j, the union of these finitely many V_x provides a connected subspace of $X \cup \left(\bigcup_{j \geq 1} A_j\right)$ containing the ends of β'. This subspace contains an arc β'' joining the ends of β', and β'' contains only finitely many of the arcs A_j. Note the arcs β'' are pairwise disjoint.

In U_i there is a tree T_i contained in X and containing the $b_{\beta,i}$ that are in U_i. The space L consisting of all the T_i, all the $\mathfrak{A}_i$, and all the β'' is a subspace of $X \cup \left(\bigcup_{j \leq J} A_j\right)$, for some positive integer J, and, therefore, embeds in the sphere. However, L contains either K_5 or $K_{3,3}$, a contradiction. Hence $X \cup \left(\bigcup_{j \geq 1} A_j\right)$ is planar. □

The proof of Lemma 3.3 is completed by observing that any embedding of $X \cup \left(\bigcup_{j \geq 1} A_j\right)$ in the sphere provides an embedding of X in the sphere so that every point of S is incident with a face of X. ■

4 Proof of Theorem 1.2

We are now prepared to prove our main theorem. We start with the following.

Claim 1 *For each positive integer n, there are only finitely many non–degenerate blocks with diameter at least $1/n$.*

Proof. Otherwise, there is an infinite sequence $(B_i)_{i \geq 0}$ of distinct blocks of X all having diameter at least $1/n$, for some positive integer n. Each B_i contains two points v_i and w_i at distance $1/n$, and the B_i, v_i and w_i may be chosen so that the sequences (v_i) and (w_i) both converge, say to v and w.

There exist disjoint connected neighbourhoods V and W of v and w, respectively. For i sufficiently large, $v_i, v_{i+1} \in V$ and $w_i, w_{i+1} \in W$. There are disjoint arcs, one in V joining v_i and v_{i+1} and one in W joining w_i and w_{i+1}. But this is impossible, since any $v_i v_{i+1}$–arc and any $w_i w_{i+1}$–arc must have a cut–point of X in common. □

Let $B_1, B_2, \dots$ be an enumeration of the non–degenerate blocks of X and let Y be a countable dense subset of X. Let $Y' = Y \backslash \mathrm{cl}\left(\bigcup_i B_i\right)$. We note that $X = \mathrm{cl}\left((Y' \cup \left(\bigcup_i B_i\right)\right)$. Enumerate Y' as $y_1, y_2, \dots$ and now consider the sequence $B_1, y_1, B_2, y_2, \dots$. We begin with H_0 being an embedding of B_1 so that every cut point of X in B_1 is incident with a face of H_0.

After a given iteration, we have an embedding in the sphere of a connected, compact subspace H_i of X so that every cut point of X in H_i is incident with a face of H_i and, if some B_j has more than one point in H_i, then $B_j \subseteq H_i$. We now explain how to obtain H_{i+1}.

Consider the first term of the sequence $B_1, y_1, B_2, y_2, \dots$ that is not in H_i. We will describe how to proceed assuming this term is a B_j; the same description applies to the case this term is a y_j.

Let A be an arc in X joining B_j to H_i that is internally disjoint from $H_i \cup B_j$ with ends $a \in H_i$ and $b \in B_j$. Let $\mathcal{B}$ denote the set of non–degenerate blocks that intersect A in more than one point. For each $B \in \mathcal{B}$, $B \cap A$ is an arc with ends a_B and b_B. Let α_B be a new arc internally disjoint from X with ends a_B and b_B. Let $\mathcal{B}'$ denote the set of $B \in \mathcal{B}$ for which $B \cup \alpha_B$ is not planar.

Claim 2 *$\mathcal{B}'$ is finite.*

Proof. Otherwise, there is an infinite sequence (B_k) of elements of $\mathcal{B}$ so that the ends a_k, b_k of $B_k \cap A$ are strictly monotonic in A, that is, they are distinct and they occur in the order $a_1, b_1, a_2, b_2, \dots$ in A. Because A is an interval, these points converge to some w in A. We claim that X has a web centred at w.

For each i, Theorem 1.1 implies that $B_i \cup \alpha_{B_i}$ contains a subspace K that is a homeomorph of either K_5 or $K_{3,3}$. Then $\alpha_{B_i} \subseteq K$, and $K - \alpha_{B_i}$ contains a simple closed curve C_i so that a_i and b_i are in distinct, overlapping C_i–bridges in K. Thus, the sequence $(C_i)_{i \geq 1}$ is the sequence of simple closed curves of a web W with centre w (from Claim 1, the diameters of the C_i tend to 0). If w is not equal to y_j, then $W \cup A \cup H_i \cup B_j$ contains a generalized thumbtack, a contradiction. On the other hand, if $w = y_j$, then y_j is in the closure of the union of the non–degenerate blocks (namely those in $\mathcal{B}'$ used to constuct the web), which contradicts the definition of Y'. □

As we traverse A from H_i to B_j, we encounter the elements of $\mathcal{B}'$ in the order $B'_1, B'_2, \dots, B'_k$. Reorder our sequence so that these blocks move in front of B_j in the ordering $B_1, y_1, B_2, y_2, \dots$. Then, instead of trying to introduce B_j, we will first introduce B'_1, then B'_2, …, B'_k and finally B_j. These will be the steps we take to introduce B_j.

The point is that now, in order to introduce the next term in our enumeration, we may assume that every block B in $\mathcal{B}$ has an embedding of $B \cup \alpha_B$ in the unit disc so that α_B is in the boundary of the disc. Set $B^* = A \cup \bigcup_{B \in \mathcal{B}} B$ and let α be a new arc internally disjoint from B^* joining a and b. Then $B^* \cup \alpha$ is 2–connected and planar (it is easily seen that it contains neither K_5 nor $K_{3,3}$) and so has an embedding in the sphere in which every cut point of X in B^* is incident with a face. We use this embedding to extend H_i to H_{i+1}.

Let F be a face of H_i incident with a and let γ_i be a simple closed curve bounding a closed disc Δ_i contained in $F \cup \{a\}$. We embed $B^* \cup B_j$ in Δ_i by embedding $B^* \cup \alpha$ in the closed unit disc so that a corresponds to $(-1, 0)$, b corresponds to the origin, and $B^* \cup \alpha$ is contained inside the circle with radius 1/2 and centre at $(-\frac{1}{2}, 0)$; likewise we embed B_j in the circle with radius 1/4 and centre at $(\frac{1}{4}, 0)$. We then homeomorphically transfer this embedding into Δ_i.

The only other points to make about γ_i and Δ_i are that: for $k < i$, either $\Delta_i \subseteq \Delta_k$ or $\Delta_i \cap \Delta_k = \emptyset$; the diameters of the Δ_i go to 0; if, for several i, we use the same point y to attach the new block to H_i, we choose (for $k < i$) $\Delta_i \subseteq \Delta_k$; and, γ_i and γ_k should be disjoint except for such a y.

Now let H^* denote $\bigcup_{i \geq 1} H_i$, so that H^* is a connected subset of the sphere. We shall prove that the closure of H^* is homeomorphic to X.

Claim 3 *If z is a cut point of X, then $z \in H^*$.*

Proof. As $X \setminus \{z\}$ is not connected, it partitions into two open sets, both of which are open in X. Thus, each of these open sets has a point in $Y' \cup \bigcup_i B_i$ and, therefore, some H_i has a point in each of these open sets. These two points are connected by an arc in H_i and this arc contains z. □

Henceforth, we assume that $X \neq H^*$. By the claim, the only points of X that are not contained in H^* are non–cut points of X not in any non–degenerate block. Therefore, every non–degenerate block has a cut point and the diameters of these blocks converge to 0, so every $x \in X \setminus H^*$ is necessarily the limit of a sequence $(x_i)_{i \geq 0}$ of cut points x_i. We may choose the sequence to be monotonic, in the sense that they occur in this order in any arc joining x_0 to x.

It follows that, for each i, x_{i+1}, $x_{i+2}, \ldots$ are all contained in the same x_i–bridge L_i. Since x is never added to any H_j, we know that, for each j, only finitely many x_i are contained in H_j (because H_j is closed). When x_i is added, it is inside some Δ_{j_1}, and so all of $x_{i+1}, x_{i+2}, \ldots$ will be put inside Δ_{j_1}. Thus, there is a nested sequence Δ_{j_k} of disjoint closed discs, each containing a tail of the sequence (x_i). Since the diameters of the Δ_{j_k} are converging to 0, there is a unique point in $\bigcap_k \Delta_{j_k}$; this is the point to which we map x.

Notice that every point in H^* is in some H_j and, therefore, is in the interior of only finitely many of the Δ_i's. In particular, x is not mapped onto any point of H^*.

Furthermore, the argument given shows that this point in the sphere is independent of the particular sequence (x_i).

In order to conclude the proof, we must show that we now have an embedding of X in the sphere. Suppose z and z' are distinct points of $X \setminus H^*$ and that A is an arc in X joining them. Every point of $A \setminus \{z, z'\}$ is either a cut point of X separating z and z' in X or is in a non–degenerate block of X. Since z and z' are neither cut points nor in non–degenerate blocks, there are cut points of X in A. In particular, z and z' are mapped to distinct points of the sphere. It follows that our map is one-to-one.

To prove continuity, it will be a help to distinguish the original space X and its elements from their images in the sphere. For their images in the sphere, we shall place $'$s on the symbols, as in X', H_i' and so on. Let (z_n) be a sequence in X converging to z. We shall prove (z_n') converges to z'. We distinguish two cases, depending on z.

Case 1. z is not in H^*.

In this case, z' is the unique point in $\bigcap \Delta_{j_i}$, for some nested sequence (Δ_{j_i}). Let y_i' be the point of $\gamma_{j_i} \cap H^{*\prime}$. Obviously all but finitely many of the z_n are in the same component of $X \setminus \{y_i\}$ as z, and therefore all but finitely many of the z_n' will appear in Δ_{j_i}. Since this is true for all i, (z_n') converges to z', as required.

Case 2. z is in H^*.

Let $\varepsilon > 0$ be given. Let $k_0 < k_1 < \cdots$ be the indices k so that γ_k contains z'. If this sequence is finite, then let k^* be the largest term in the sequence (or $k^* = 0$ if there is no such k_i). If this sequence is infinite, then let k^* be any one of these for which Δ_{k_i} has diameter less than ε. At the moment Δ_{k^*} is created, it is to extend H_{k^*-1} to H_{k^*}.

Partition the positive integers into the two sets $N_1 = \{n \mid z_n \in H_{k^*}\}$ and $N_2 = \{n \mid z_n \notin H_{k^*}\}$. If the former is infinite, then evidently the sequence $(z_n')_{n \in N_1}$ converges to z'. Now assume the latter is infinite.

For each $n \in N_2$, there is a unique point y_n of H_{k^*} so that the component of $X \setminus H_{k^*}$ containing z_n attaches at y_n. If N is a connected neighbourhood of z containing z_n, then, since y_n is in every arc in X joining z_n and z and N contains such an arc, it follows that y_n is in N. Therefore, the sequence (y_n) converges to z. Thus, (y_n') converges in H_{k^*}' to z'.

For sufficiently large n, y_n' is within ε of z'. Partition N_2 into $N_3 = \{n \in N_2 \mid y_n = z\}$ and $N_4 = \{n \in N_2 \mid y_n \neq z\}$. If the latter is infinite, then choose n sufficiently large so that the Δ_j that attaches at y_n' has diameter less than ε. Thus, for n sufficiently large and $n \in N_4$, z_n' is within distance 2ε of z'.

Finally, suppose N_3 is infinite and consider $n \in N_3$. Since $z_n \notin H_{k^*}$, and $y_n = z$, it follows that, for n sufficiently large, $z_n' \in \Delta_{k^*}$ and, therefore, z_n' is within distance ε of z'.

5 Conclusion

Generalized thumbtacks may well provide further insight into embeddability into higher genus surfaces. Thomassen [4] proves that if a 2–connected, locally 2–connected compact metric space M does not contain the infinite complete graph as a subspace, then there is a unique compact surface M' so that M is homeomorphic to a subspace of M' and the set of finite graphs that embed in M is the same as the set of finite graphs that embed in M'. The locally 2–connected condition is very strong and was introduced to exclude spaces obtained by identifying two (or more) points in a locally connected space. For example, if x and y are distinct points of the sphere, then the space Σ obtained by identifying x and y is not a subspace of any compact surface.

We remark that the space Σ contains a thumbtack as a subspace. It is natural to wonder about the following.

Conjecture 5.1 *Let X be a connected, compact, locally connected metric space not containing a generalized thumbtack. If X contains none of the (finitely many) minimal graphs that do not embed in a given compact surface Σ, then X embeds in a surface with genus at most that of Σ.*

More weakly, one might conjecture that if the genera of the finite graphs contained in X is bounded, then X embeds in some compact surface.

A necessary condition for a space X to embed in a 2-manifold is that every point have a neighbourhood that embeds in the plane. For this to fail, every neighbourhood of some point (of a compact, locally connected metric space) must contain a generalized thumbtack or a K_5 or a $K_{3,3}$.

Even if we exclude all generalized thumbtacks, there are several ways a point can have no planar neighbourhood. Here are some natural examples:

- a sequence of disjoint $K_{3,3}$'s converging to x;
- a sequence of $K_{3,3}$'s converging to x and having only x in common; and
- a sequence of $K_{3,3}$'s converging to x that are disjoint except that there is one edge of each that properly contains an edge of the next (and these edges are nested).

Is there a finite list $\mathfrak{L}$ of structures, centred on a point x, so that a point x of a compact, locally connected metric space has no planar neighbourhood if and only if x is the centre of a structure in $\mathfrak{L}$?

Finally, we note that one does not need to exclude infinitely many different subspaces from X to ensure planarity. By considering a subsequence of the simple closed curves making up the web, we can assume the overlapping bridges are either all skew or all

3-equivalent. If the overlaps are all skew, the subweb is a unique cubic graph (see Figure 1). If they are all 3-equivalent, then for each simple closed curve either all the vertices have degree 4, producing one transition, or some vertices have degree 3, giving rise to three additional transitions (see Figure 2). Now we can find a subweb so that all of the transitions are of the same type. Thus, there are only five further obstructions to planarity.

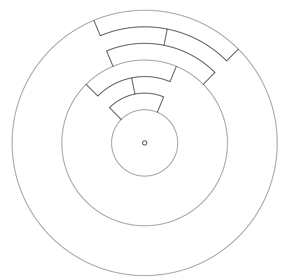

Figure 1: The thumbtack arising from skew transitions.

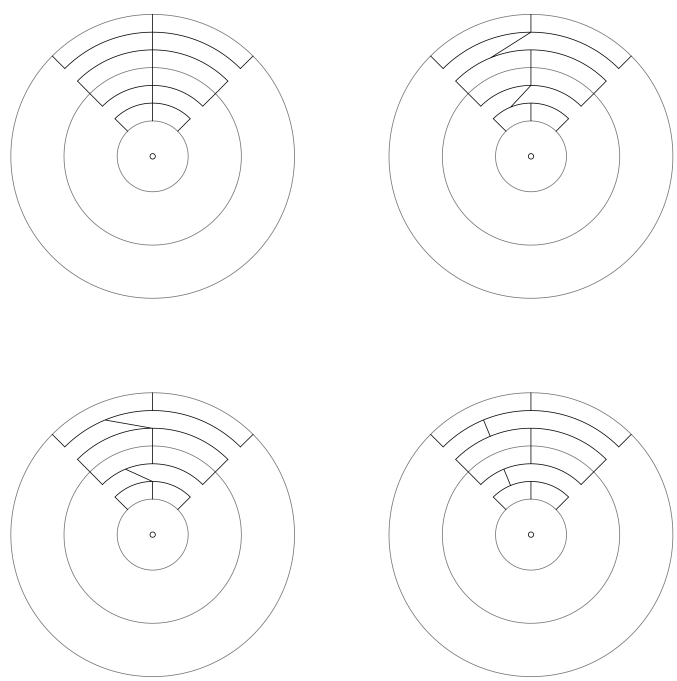

Figure 2: The four thumbtacks arising from 3-equivalent transitions.

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