Crossing numbers

R. B. Richter G. Salazar

July 28, 2008

The crossing number of a graph G is the smallest number of pairwise crossings of edges among all drawings of G in the plane. In the last decade, there has been significant progress on a true theory of crossing numbers. There are now many theorems on the crossing number of a general graph and the structure of crossing–critical graphs, whereas in the past, most results were about the crossing numbers of either individual graphs or the members of special families of graphs. This chapter highlights these recent advances and some of the open questions that they suggest.

1 Introduction

Historically, the study of crossing numbers has mostly been devoted to the computation of the crossing numbers of particular families of graphs. Given that we still do not know the crossing numbers for basic graphs such as the complete graphs and complete bipartite graphs, this is perhaps not surprising. However, a broader theory has recently begun to emerge. This theory has been used for computing crossing numbers of particular graphs, but has also promulgated open questions of its own. One of the aims of this chapter is to highlight some of these recent theoretical developments.

The study of crossing numbers began during the Second World War with Paul Turán. In [60], he tells the story of working in a brickyard and wondering about how to design an efficient rail system from the 'kilns' to the 'storage yards'. For each kiln and each storage yard, there was a track directly connecting them. The problem he considered was how to lay the rails to reduce the number of crossings, where the cars tended to fall off the tracks, requiring the workers to reload the bricks onto the cars. This is the problem of finding the crossing number of the complete bipartite graph.

It is also natural to try to compute the crossing number of the complete graph. To date, there are only conjectures for the crossing numbers of these graphs (Figure 1 suggests how the $K_{r,s}$ conjecture arises):

$$\mathrm{cr}(K_{r,s}) = \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor$$

and

$$\mathrm{cr}(K_r) = \frac{1}{4} \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{r-3}{2} \right\rfloor .$$

The former is known to be true for $r \leq 6$ and all s, and also for $r = 7$ and 8 when $s \leq 10$. The latter is known for $r \leq 12$.

A third family has played an important role in stimulating the development of some of the theory. The Cartesian product of two cycles makes a toroidal grid. The fact that its crossing number grows with the sizes of the

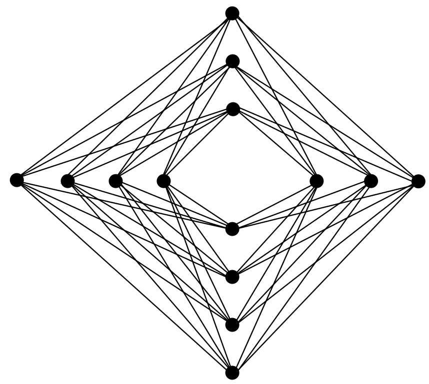

Figure 1:

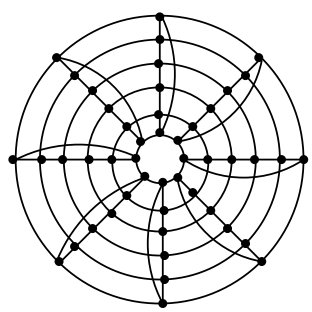

Figure 2:

cycles showed that toroidal graphs have arbitrarily large crossing number. The conjecture (see Figure 2) is that $\text{cr}(C_r \times C_s) = (r-2)s$, for $3 \leq r \leq s$. This has been proved by traditional methods for $r \leq 7$. The pioneering work of Beineke and Ringeisen [47], [4] for the cases $r = 3$ and $r = 4$ stimulated the work that leads to much of the theory outlined in Section 6.

We begin in Section 2 by introducing four different crossing numbers. The rest of this article is about the standard crossing number. Section 3 provides some general lower bounds for the crossing number, while Section 4 gives one of several lovely applications of these bounds to combinatorial geometry. Section 5 introduces the recently developed structure found in crossing–critical graphs. Section 6 describes some theory developed to be better able to determine the crossing numbers of certain families of graphs — this has been especially successful for the Cartesian product of cycles. Some algorithmic questions are discussed in Section 7 and we conclude in Section 8 with a brief look at recent work demonstrating connections between drawings of the same graph on different surfaces.

There is an impressive bibliography of crossing number papers maintained by Vrťo [64].

2 What is the crossing number?

Pach and Tóth [38] ask: what is the 'right' definition of a crossing number? A *drawing* of a graph G in the plane is a set of distinct points in the plane, one for each vertex of G, and a collection of simple open arcs, one for each edge of the graph, such that if e is an edge of G with ends v and w, then the closure (in the plane) of the arc α representing e consists precisely of α and the two points representing v and w. We further require that no edge–arc intersects any vertex point.

We may now try to count crossings of the arcs. We follow Székely's lead [59] and give four different ways of doing this. Let $\mathcal{D}$ be a drawing of a graph G. The (standard) *crossing number* $\mathrm{cr}(\mathcal{D})$ of $\mathcal{D}$ is the number of pairs $(x, \{\alpha, \beta\})$, where x is a point of the plane and α and β are arcs of $\mathcal{D}$ representing distinct edges of G such that $x \in \alpha \cap \beta$.

The *pair crossing number* $\mathrm{pcr}(\mathcal{D})$ of $\mathcal{D}$ is the number of pairs of arcs α and β of $\mathcal{D}$ representing distinct edges of G such that $\alpha \cap \beta \neq \varnothing$. The point here is that two edges may intersect many times, but contribute only one to $\mathrm{pcr}(\mathcal{D})$.

The *odd crossing number* $\mathrm{ocr}(\mathcal{D})$ of $\mathcal{D}$ is the number of pairs α and β of arcs of $\mathcal{D}$ representing distinct edges of G such that $|\alpha \cap \beta|$ is odd.

The *independent odd crossing number* $\mathrm{iocr}(\mathcal{D})$ of $\mathcal{D}$ is the number of pairs

α and β of arcs of $\mathcal{D}$ representing distinct edges of G that are not incident with a common vertex and such that $|\alpha \cap \beta|$ is odd.

It is very easy to see that, for any drawing $\mathcal{D}$ of G,

$$\mathrm{iocr}(\mathcal{D}) \le \mathrm{ocr}(\mathcal{D}) \le \mathrm{pcr}(\mathcal{D}) \le \mathrm{cr}(\mathcal{D})\,.$$

For each of these crossing numbers, the corresponding crossing number of G is defined to be the minimum over all drawings of G. For example, $\mathrm{pcr}(G) = \min\{\mathrm{pcr}(\mathcal{D})\}$, where the minimum is over all drawings $\mathcal{D}$ of G. Thus,

$$\mathrm{iocr}(G) \le \mathrm{ocr}(G) \le \mathrm{pcr}(G) \le \mathrm{cr}(G)\,.$$

In particular, Pach and Tóth ask:

Problem *Is it true that, for all graphs G, $\mathrm{pcr}(G) = \mathrm{cr}(G)$?*

The independent odd crossing number was introduced by Tutte [61], who proved that if $\mathrm{iocr}(G) = 0$, then $\mathrm{cr}(G) = 0$. He also proved the nice property that if $\mathcal{D}$ and $\mathcal{D}'$ are two drawings of a graph G, then $\mathrm{iocr}(\mathcal{D}) \equiv \mathrm{iocr}(\mathcal{D}') \pmod 2$. In that paper, Tutte wonders whether $\mathrm{iocr}(G)$ is equal to $\mathrm{cr}(G)$, for all graphs G. Pach and Tóth [38] proved the following interesting result.

Theorem 2.1 $\mathrm{cr}(G) \le 2(\mathrm{ocr}(G))^2$.

The proof consists of the non–trivial fact that, given a drawing $\mathcal{D}$ of G, there is a second drawing $\mathcal{D}'$ of G such that if an edge e of G has a crossing in $\mathcal{D}'$, then, in $\mathcal{D}$, e has an odd number of crossings with some other edge; see Pelsmajer, Schaefer and Štefankovič [40] for a simpler proof and related results.

An important recent work [41] by Pelsmajer, Schaefer and Štefankovič gives the first example of a graph G for which $\mathrm{ocr}(G) < \mathrm{cr}(G)$. This answers Tutte's question in the negative. For each $\varepsilon > 0$, they show that there is a graph G such that $\mathrm{ocr}(G) < (\frac{\sqrt{3}}{2} + \varepsilon)\mathrm{pcr}(G)$. It may be that $\sqrt{3}/2$ is the smallest possible coefficient that can occur here, but this is open.

Pach's question, "Is $\mathrm{pcr}(G) = \mathrm{cr}(G)$?" is still open.

3 General bounds

Erdős and Guy [15] conjectured the following, which has been proved by Ajtai, Chvátal, Newborn, and Szemerédi [2] and independently by Leighton [33].

Theorem 3.1 *There exists a positive constant* c *such that, if* $m \geq 4n$*, then* $\mathrm{cr}(G) \geq cm^3/n^2$.

The proof is an easy induction and generalizes the result slightly to allow $m \geq (3+\varepsilon)n$, for any $\varepsilon > 0$, but then c becomes a function of ε. Note that this result already implies that $\mathrm{cr}(K_n) \geq cn^4$, for some constant $c > 0$.

Pach, Spencer and Tóth [36] generalized this bound as follows.

Theorem 3.2 *Let* g *be a positive integer. There is a positive constant* $c = c(g)$ *such that if* G *has girth at least* $2g+1$ *and* $m \geq 4n$*, then* $\mathrm{cr}(G) \geq cm^{2+g}/n^{1+g}$.

A *graph property* is a set of graphs. A graph property $\mathcal{P}$ is *monotone* if:

- $G \in \mathcal{P}$ and $H \subseteq G$ implies that $H \in \mathcal{P}$; and
- $G_1 \in \mathcal{P}$ and $G_2 \in \mathcal{P}$ implies that the disjoint union $G_1 \cup G_2$ is in $\mathcal{P}$.

For a monotone property $\mathcal{P}$, $ex(n, \mathcal{P})$ denotes the maximum number of edges in a graph G with n vertices such that G has property $\mathcal{P}$. The following general statement is in [36].

Theorem 3.3 *Let* $\mathcal{P}$ *be a monotone graph property and suppose that there are positive constants* α, c_1*, and* c_2 *for which*

$$c_1 n^{1+\alpha} \leq ex(n, \mathcal{P}) \leq c_2 n^{1+\alpha}\,.$$

Then there are positive constants c_3 *and* c_4 *such that if* $G \in \mathcal{P}$*,* G *has* n *vertices and* $m \geq c_3 n \log^2 n$ *edges, then*

$$\mathrm{cr}(G) \geq c_4 \frac{m^{2+1/\alpha}}{n^{1+1/\alpha}}\,.$$

The conclusion of Theorem 3.1 is the same as the conclusion of Theorem 3.3 in the case $\alpha = 1$. However, notice the hypothesis in the latter requires that $m \geq cn \log^2 n$, rather than $m \geq cn$ as in Theorem 3.1.

Problem *Is Theorem 3.3 still true if the hypothesis is weakened to $m \geq cn$?*

One sticking point in the proof of Theorem 3.3 given in [36] is finding a good estimate on the sum of the squares of the degrees of the vertices in a graph in $\mathcal{P}$ having n vertices — the trivial estimate $\sum_{v \in V(G)} deg^2(v) \leq n \sum_{v \in V} deg(v) \leq n^{2+\alpha}$ is used. Füredi and Kündgen [19] have shown that $\sum_{v \in V(G)} deg^2(v) \leq f(n, \alpha)$, where:

$$f(n,\alpha) = \begin{cases} n^2 & \text{if } \alpha < 1/2 \\ n^2 \log n & \text{if } \alpha = 1/2 \\ n^{1+2\alpha} & \text{if } 1/2 < \alpha \leq 1 \end{cases} .$$

Unfortunately, this improved estimate improves the hypothesis of Theorem 3.3 only to $m \geq cn \log n$, for $\alpha < 1/2$, and to $m \geq cn(\log n)^{3/2}$, for $\alpha = 1/2$. The accuracy of the estimates in [19] suggests that other methods may be needed to improve the hypothesis to $m \geq cn$.

Leighton introduced two other important ideas in [33]. One way to estimate the crossing number of a graph G is to immerse G into another graph H. An *immersion of a graph G into a graph H* is a mapping of the vertices of G to vertices of H and a mapping of the edges of G to paths in H (if $e = vw \in E(G)$, then e is mapped to a path in H whose ends are the images of v and w). There is no requirement that the edge–representing paths be disjoint, but no edge–representing path should have a vertex–representing vertex in its interior. For each edge e of H, we call the *congestion* of e the number of paths of H corresponding to edges of G that pass through e. The crossing number of G can be readily approximated by a function of the crossing number of H and the maximum congestion of the immersion. This idea has been used by Shahrokhi, Székely and Vrťo [55] to show that the n–cube has crossing number at least $c4^n$.

The other of Leighton's ideas in [33] we mention here is that of relating the crossing number of a graph G to its *bisection width* $b(G)$, which is the smallest number of edges in a cutset dividing $V(G)$ into two sets, each with at least a third of the vertices of G. One form of this theorem is the following [35]:

Theorem 3.4 *Let G be a graph with n vertices with degrees d_1, d_2, ..., d_n. Then $b(G) \le 10\sqrt{\mathrm{cr}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}$.*

This type of result is used in the proof of Theorem 3.3 and shows why estimates on $\sum d_i^2$ are wanted.

In a slightly different direction, Pach and Tóth [37] considered a graph having a drawing in the plane for which no edge crosses more than k other edges. This obviously restricts the number of edges and they proved that, for $k \le 4$, $m \le (k+3)(n-2)$ (generalizing the $m \le 3n-6$ bound for planar graphs), while, for general k, there is a universal constant c such that $m \le c\sqrt{k}n$.

4 Applications to geometry

In a beautiful paper, Székely [58] has shown how to use Theorem 3.1 to provide simple proofs of 'hard' problems in combinatorial geometry. The simplest example is the Erdős–Trotter theorem:

Theorem 4.1 *Given n points and ℓ lines in the plane, there is a constant c for which the number of incidences among the points and lines is at most $c\left[(n\ell)^{2/3} + n + \ell\right]$.*

Proof. Let G be the graph with the n points as vertices in which two vertices are adjacent if they are consecutive points on one of the lines. The arrangement of points and lines gives a drawing of G in the plane. Since no two lines cross more than once, we have $\mathrm{cr}(G) \le \ell^2$. On the other hand, in each line the number of vertices is one greater than the number of edges, so the number I of point–line incidences is at most $m + \ell$. Therefore, by Theorem 3.1, either $4n \geq m$ or $\mathrm{cr}(G) \geq cm^3/n^2$, that is, either $I \leq 4n + \ell$ or $\ell^2 \geq c(I - \ell)^3/n^2$. ∎

5 Crossing–critical graphs

A graph G with crossing number k or more is *k–crossing–critical* if every proper subgraph has crossing number less than k, and G is homeomorphically minimal. Note that this last condition, together with Kuratowski's Theorem, implies that the only 1–crossing–critical graphs are $K_{3,3}$ and K_5. We remark that there are graphs G (examples are mentioned below) that are k–crossing–critical for some $k < \mathrm{cr}(G)$.

Although significant progress has been made, it is still not known what the 2–crossing–critical graphs are. Richter [45] determined the eight cubic 2–crossing–critical graphs and Vitray [62] presented the following result:

Theorem 5.1 *If G is 2–crossing–critical, then either* $\mathrm{cr}(G) = 2$ *or* $G = C_3 \times C_3$.

Since $\mathrm{cr}(C_3 \times C_3) = 3$, so not every 2–crossing–critical graph has crossing number 2, Theorem 5.1 can be stated in a less precise way: there is a constant c such that if G is 2–crossing–critical, then $\mathrm{cr}(G) \leq c$. This suggests the following result, proved by Richter and Thomassen [46]:

Theorem 5.2 *If G is k–crossing–critical, then* $\mathrm{cr}(G) \leq \frac{5}{2}k + 16$.

Lomeli and Salazar [34] have shown that there is a constant c such that, if G is sufficiently large and k–crossing–critical, then $\mathrm{cr}(G) \leq 2k + c$.

The important point is that the crossing number of a k–crossing–critical graph is bounded by a function of k. If $n \geq 5$, then the complete graph K_n

(for example) is k–crossing–critical for some (unknown) k. Since k is of the order of n^4, some edge must cross about n^2 edges, so the deletion of any edge reduces the crossing number by about n^2. Thus, K_n is k–crossing–critical for some k that is about $\mathrm{cr}(K_n) - \sqrt{\mathrm{cr}(K_n)}$. This means that there are examples of k–crossing–critical graphs G for which $\mathrm{cr}(G)$ is about $k + \sqrt{k}$.

Question 1 *Is there a positive constant c for which $\mathrm{cr}(G) \leq k + c\sqrt{k}$ whenever G is k–crossing–critical?*

Equivalently, one may ask if there is a constant c' such that every graph G has an edge e for which $\mathrm{cr}(G - e) \geq k - c'\sqrt{k}$. Recently, Fox and Tóth [18] verified this for dense graphs. More precisely, they proved the following.

Theorem 5.3 *For any connected graph G with n vertices and m edges, and for every edge e of G, $\mathrm{cr}(G - e) \geq \mathrm{cr}(G) - 2m + \frac{1}{2}n + 1$.*

Fox and Tóth also investigated the effect on crossing numbers caused by the removal of a substantial number of edges in a sufficiently dense graph:

Theorem 5.4 *For each $\varepsilon > 0$, there is a constant n_0, depending on ε, such that, if G has $n > n_0$ vertices and $m > n^{1+\varepsilon}$ edges, then G has a subgraph G' with at most $(1 - \frac{1}{24\varepsilon})m$ edges for which $\mathrm{cr}(G') \geq (\frac{1}{28} - o(1))\mathrm{cr}(G)$.*

Geelen, Richter and Salazar [22] proved that if G is k–crossing–critical, then G has bounded tree–width, thereby directly relating crossing–number problems to graph minors. Crossing numbers are not monotonic with respect to contraction, since contracting an edge might decrease, increase, or not change the crossing number, so this is a significant relationship.

(We comment that Robertson and Seymour [48] have found the 41 forbidden minors for those graphs H that are minors of some graph G with $\mathrm{cr}(G) \leq 1$. However, there are such H for which $\mathrm{cr}(H) > 1$.)

At the same time, Salazar and Thomas made a conjecture that has since been proved by Hliněný [27]:

Theorem 5.5 *There is a function f such that if G is k–crossing–critical, then G has path–width at most $f(k)$.*

We do not wish to get into a discussion of path–width here, but essentially what Theorem 5.5 means is that k–crossing–critical graphs are long and thin. There are several known examples of infinite families of k–crossing–critical graphs, for fixed k, and they are all composed of small pieces 'stuck' together in some circular fashion, with more and more copies of the pieces to make the different examples. These have small path–width, which here is essentially the number of vertices in the pieces.

These examples suggest a refinement of Theorem 5.5. The *bandwidth* $\mathrm{bw}(G)$ of a graph G is the least integer k for which there is an assignment of $1, 2, \ldots, n$ to the vertices such that

$$\max\Big\{|f(v) - f(w)| : vw \in E(G)\Big\} \leq k\,.$$

(Essentially, each neighbour of the vertex i must be one of the numbers $i-k, i-k+1, \ldots, i-1, i+1, i+2, \ldots, i+k$.) We make the following conjecture in a formulation by Carsten Thomassen.

Conjecture 5.1 *For each integer k, there is an integer $B(k)$ for which* $\mathrm{bw}(G) \leq B(k)$ *if G is k–crossing–critical.*

Since $\mathrm{bw}(G) \leq k$ implies that G has path–width at most $2k$, Conjecture 5.1 refines Theorem 5.5. It also implies the following, which itself seems to be challenging.

Conjecture 5.2 *For each integer k, there is an integer $D(k)$ such that the maximum degree of G is at most $D(k)$ if G is k–crossing–critical.*

We do not believe that Theorem 5.5 and Conjecture 5.2 together imply Conjecture 5.1.

Until recently, the following construction gave all known examples of infinite families of crossing–critical graphs.

A *tile* is a triple $T = (G, L, R)$ for which G is a graph and L and R are finite sequences of distinct vertices of G. A *tile drawing* of such a tile is a drawing of G in the unit square $[0,1] \times [0,1]$ so that the vertices of L appear on $\{0\} \times [0,1]$ in order with decreasing y–coordinates and the vertices of R appear on $\{1\} \times [0,1]$ in order with decreasing y–coordinates.

If L and R have the same length, then we can glue two copies (G_1, L_1, R_1) and (G_2, L_2, R_2) of T together by identifying R_1 with L_2 (in order) to create a new tile $T^2 = (G_1 \cup G_2, L_1, R_2)$. Gluing q copies of T together in this linear way gives the tile T^q.

The *tile crossing number* $\mathrm{tcr}(T)$ of a tile T is the minimum value of $\mathrm{cr}(\mathcal{D})$ over all tile drawings $\mathcal{D}$ of T, and T is *planar* if $\mathrm{tcr}(T) = 0$. If $T = (G, L, R)$ is planar, then G is planar, but the converse is not true in general.

Let $T = (G, L, R)$ be a planar tile and consider the planar tile $T^q = (H, L', R')$. By identifying L' with R', but now in the reverse order, we typically create a non–planar graph, denoted by $\otimes(T^q)$. For example, Richter and Thomassen [46] proved that, for the bow–tie B illustrated in Figure 3, $\mathrm{cr}(\otimes(B^q)) = 3$ if $q \geq 3$. This can also be proved by a much simpler method, which we now describe.

It can be shown that if T is a planar tile, then there is a number N_1 such that if $q \geq N_1$ and $\mathcal{D}$ is a drawing of $\otimes(T^q)$ with at most some fixed number c_1 of crossings, then there is another drawing $\mathcal{D}'$ of $\otimes(T^q)$ with c_1 crossings for which some copy of T in $\otimes(T^q)$ is drawn as a tile drawing with no crossings. This means that cutting out that copy of T leaves a tile drawing of a tile $\tilde{T}^{q-1}$ obtained by gluing together $q-1$ tiles, $q-2$ of which are $T = (G, L, R)$ and the other is (G, L, R^{-1}) — that is, T, but with R in the reverse order. Since $\mathrm{cr}(\otimes T^q) \leq \mathrm{tcr}(\tilde{T}^q)$ and $\mathrm{tcr}(\tilde{T}^{q+1}) \leq \mathrm{tcr}(\tilde{T}^q)$, we easily deduce the following result (see [44]):

Theorem 5.6 *If T is a planar tile, then* $\mathrm{tcr}(\tilde{T}^q) = \mathrm{cr}(\otimes(T^q))$ *for large* q.

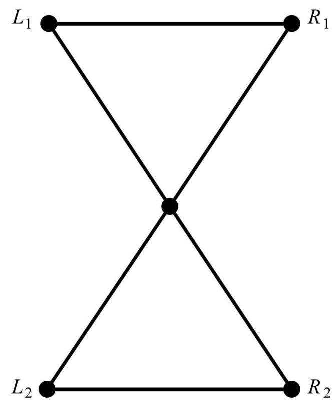

Figure 3:

It is easy to prove that $\mathrm{cr}(\otimes B^q) = 3$. There are four edge–disjoint paths in B from some L_i to R_i: from ‘top’ to ‘bottom’ label these P_1, P_2, P_3, and P_4. Thus, P_1 is disjoint from P_3 and P_4 and P_2 is disjoint from P_4. In any tile drawing of $\tilde{B}^q$, the union of all the P_1–paths must cross both the union of all the P_3–paths and the union of all the P_4 paths, and similarly the P_2s must cross the P_4s. This implies that there are at least three crossings.

All examples of infinite families of k–crossing–critical graphs known before 2006 can be explained by variations of these arguments; in particular, this is true of the infinite family described in [32]. Oporowski (unpublished) has shown that there is a number N such that, if G is 2–crossing–critical and has more than N vertices, then G is constructed in just the manner of $\otimes T^q$, except that now combinations of 13 different tiles are used.

In recent work, Bokal [7] has introduced the *zip product* of graphs. Among other things, he has used this to give infinite families of k–crossing–critical graphs, enough such families to show that, for each rational number $r \in (3, 6)$, there are infinitely many numbers k for which each member of an infinite family of simple k–crossing–critical graphs has average degree r. Salazar [51] observed that such a rational r must be in interval $(3, 6]$, so the only open case is $r = 6$. Richter and Thomassen [46] showed that, if the infinite family of k–crossing–critical graphs are all r–regular, then r is 4 or 5. No example of a 5–regular family or a family with average degree 6 is known. Bokal's families are not made up only of tiles: rather, they decompose into smaller crossing–critical graphs across edge–cuts, which is the operation inverse to the zip product.

6 Other families of graphs

We continue the tile theme. It is clear that, if T is a tile and p and q are positive integers, then

$$\operatorname{tcr}(T^{p+q}) \leq \operatorname{tcr}(T^p) + \operatorname{tcr}(T^q)\,.$$

A simple standard result about such sequences, sometimes referred to as Fekete's Lemma, gives a nice limit result:

Theorem 6.1 *For each tile T, $\lim_{q\to\infty} \operatorname{tcr}(T^q)/q$ exists.*

The *average crossing number* $\operatorname{acr}(T)$ of a tile T is $\lim_{q\to\infty} \operatorname{tcr}(T^q)/q$. We note that $\operatorname{acr}(T) \leq \operatorname{tcr}(T)$. In the next result, from [43], the graph $\circ(T^q)$ is obtained from T^q by identifying L and R in the same order. (If T is not planar, then we do not need to reverse R in order to create crossings.)

Theorem 6.2 *Let T be any tile for which T^q exists, for every q. Then there is a constant $c = c(T)$ such that, for all q,*

$$\operatorname{tcr}(T^q) \geq \operatorname{cr}(\circ(T^q)) \geq \operatorname{tcr}(T^q) - c\,.$$

In particular, $\lim_{n\to\infty} \operatorname{cr}(\circ(T^q))/q = \operatorname{acr}(T)$.

If we knew $\operatorname{acr}(T)$, then it would follow that $\operatorname{cr}(\circ(T^q))$ is asymptotically equal to $n \cdot \operatorname{acr}(T)$.

Problem *Is there any general method for computing (or estimating)* $\mathrm{acr}(T)$*?*

Let T_p be a tile for which $\circ(T_p^q)$ is the Cartesian product $C_p \times C_q$ of two cycles. It follows from [1] that $\mathrm{acr}(T_p) = \mathrm{tcr}(T_p) = p-2$. Thus, for all $q \geq p$, $\mathrm{cr}(C_p \times C_q)$ lies between $(p-2)q - c(p)$ and the conjectured value $(p-2)q$.

An important improvement for this graph has been obtained by Glebskiĭ and Salazar [24]. It is based on [1], but includes some additional arguments:

Theorem 6.3 *If* $q \geq p(p+1)$*, then* $\mathrm{cr}(C_p \times C_q) = (p-2)q$*.*

Theorem 6.2 opens up the possibility of getting the right order of magnitude for the crossing numbers of families of graphs. One natural candidate is the generalized Petersen graph $P(p,k)$, obtained from C_p by adding a pendant edge to each vertex and then joining each pair of pendant vertices that are k apart. Theorem 6.2 implies that there is a function $f(k)$ such that $\mathrm{cr}(P(p,k))$ is essentially $pf(k)$. Salazar [50] has proved that $f(k) \leq 2 + o(k)$ and $f(k) \geq \frac{1}{4} + o(k)$. It follows that, roughly, $\frac{1}{4}p \leq \mathrm{cr}(P(p,k)) \leq 2p$.

In a different direction, it would be interesting to get some handle on $\mathrm{cr}(P(q^2,q))$, as q gets large. It is an easy consequence of Theorem 3.4 and the main result of [11] that the crossing number of the random cubic graph with n vertices is at least cn^2. It would be nice to have actual examples of such graphs.

Finally, Bokal [8] has used his zip product to compute the crossing number of the Cartesian product of the star S_p with p leaves and the path P_q with q vertices. His proof is the most succinct argument we know of to show that a particular graph has a given crossing number. An extension of this method to compute the crossing numbers of other Cartesian products of $K_{1,n}$ or wheels with trees or paths is given in [9].

7 Algorithmic questions

In this section, we consider algorithmic aspects of crossing numbers. The basic result in the area is the following, due to Garey and Johnson [21]:

Theorem 7.1 *The problem, 'Given a graph G and an integer k, is* $\mathrm{cr}(G) \leq k$*?' is NP–complete.*

Hliněný [28] has recently proved that computing the crossing number of cubic graphs is NP–complete, thereby answering a long–standing question. This proves that the minor–monotone version of crossing number is also NP–hard (see [10]). The NP–completeness of crossing number for cubic graphs also follows from a recent result by Pelsmajer, Schaefer and Štefankovič, who proved that determining the crossing number of a graph with a given rotation system is NP–complete [42].

A graph is *almost planar* if there is an edge whose removal leaves a planar graph. Hliněný and Salazar [30] have proved that there is a polynomial–time algorithm for approximating (within a constant factor) the crossing number of an almost planar graph with bounded degree. Gutwenger, Mutzel and Weiskircher [26] have shown that if G is planar and if e is an edge not in G, then there is a linear–time algorithm for finding a planar embedding of G such that the number of crossings that result from the insertion of e is minimized. Cabello and Mohar [13] have proved the surprising result that computing the crossing number of an almost planar graph is NP–complete.

For fixed k, there is an easy polynomial–time algorithm to determine whether $\mathrm{cr}(G) \leq k$: try all possible ways of inserting k vertices of degree 4 as the crossings and then test each possibility for planarity. Much better, Kawarabayashi and Reed [31] have shown the following.

Theorem 7.2 *For each integer k, there is a linear–time algorithm for determining whether* $\mathrm{cr}(G) \leq k$*.*

They improve Grohe's [25] original quadratic–time algorithm. Both begin by reducing G to a graph with bounded tree–width but with the same crossing number, and then test whether the crossing number of G is at most k. (The first part works because crossing–critical graphs have bounded tree–width [22].) Closely related is one of the most important algorithmic problems involving crossing numbers, posed by Seese (see the open problems section in [49]).

Question 2 *Does there exist a polynomial–time algorithm for computing the crossing numbers of graphs with bounded tree–width?*

It is interesting that the problem 'Is $\mathrm{cr}(G) \leq k$?' is trivially in NP, while related problems are not obviously so. For example, consider the *rectilinear crossing number* $\mathrm{rcr}(G)$ of a graph G; this is the minimum number of crossings in any drawing of G in which each edge is a straight–line segment.

Bienstock [5] has shown that there is an infinite family G_n of graphs with the property that any straight–line drawing of G_n with $\mathrm{rcr}(G_n)$ crossings must have a vertex whose coordinates require more than polynomially many bits. Thus, if we wish to prove that the problem 'Is $\mathrm{rcr}(G) \leq k$?' is in NP, it does not suffice to exhibit a suitable drawing. However, a suitable drawing does suffice to show the problem 'Is $\mathrm{cr}(G) \leq k$?' is in NP.

While we are on the subject of rectilinear crossing numbers, Bienstock and Dean [6] have proved the following remarkable results.

Theorem 7.3 *(a) If* $\mathrm{cr}(G) \leq 3$, *then* $\mathrm{rcr}(G) = \mathrm{cr}(G)$.
(b) For each positive integer $n \geq 4$, *there is a graph* G_n *such that* $\mathrm{cr}(G_n) = 4$ *and* $\mathrm{rcr}(G_n) \geq n$.

The pair crossing number problem is also not trivially in NP, since it is not known in advance that if $\mathrm{pcr}(G) \leq k$, then there is a polynomially describable drawing of G in the plane that demonstrates this. It is now known that there is such a drawing for which each edge has at most 2^m

crossings [54]. Note that $\mathrm{pcr}(G)$ is closely related to the *weak realizability* problem, which provides as input both G and a set R of pairs of edges, so that crossings may only occur between pairs that are in R. The following is a consequence of [53]:

Theorem 7.4 *The problem, 'Given a graph G and an integer k, is $\mathrm{pcr}(G) \leq k$?' is in NP.*

Pach and Tóth [38] have observed that the proof of Theorem 7.1 carries over to prove that 'odd crossing number' is also NP–hard.

8 Drawings in other surfaces

In this section, we consider crossing numbers for drawings of graphs in surfaces other than the plane.

Böröczky, Pach and Tóth [12] recently investigated the (plane) crossing number of a graph G embeddable in a surface with Euler characteristic χ. Settling a conjecture of Peter Brass, they proved that G can always be drawn in the plane with at most $c_\chi \Delta n$ crossings, where c_χ is a constant depending on χ and Δ is the maximum degree of G. Examples show that, up to the value of c_χ, this is best possible. They actually proved a stronger statement: there is a constant c'_χ such that, if $d_1, d_2, \ldots, d_n$ are the degrees of the vertices of G, then G can be drawn with at most $c'_\chi \sum_{i=1}^{n} d_i^2$ crossings.

This last result is also implied by a recent result of Wood and Telle [63], who proved that every graph with bounded degree that excludes a fixed graph as a minor has linear crossing number. They also proved that every graph with bounded degree and bounded tree–width has linear rectilinear crossing number (cf. Question 2 above).

The following was proved by Salazar [52], building upon results of Garcia–Moreno and Salazar [20]. The *representativity* of a graph G embedded in a surface Σ is the minimum number of intersections with G of a non–

contractible closed curve γ in Σ, the minimum being taken over all possible curves γ.

Theorem 8.1 *Let Σ and Σ' be surfaces. Then either every Σ–embeddable graph is Σ'–embeddable, or there is a constant c such that, if G is a graph embedded in Σ with representativity r, then $cr_{\Sigma'}(G) \geq cr^2 + O(r)$.*

In the case Σ is the real projective plane and Σ' is the sphere, it should be possible to get a better result. The *dual–width* of an embedding of a graph in a surface is the length of the shortest non–contractible cycle in the surface dual graph.

Problem *Find a good estimate for the (plane) crossing number of a projective planar graph in terms of the dual–width of the embedding.*

In [23], it is proved that the desired estimates exist for projective graphs of bounded degree.

An algorithmic version of this question can be given.

Problem *Is computing the planar crossing number of projective planar graphs NP–complete?*

Part of the motivation for this problem is the fact that the genus of projective planar graphs can be computed in polynomial time (see [17]).

Problem *Find a good estimate for the (plane) crossing number of a toroidal graph in terms of the dual–width of the embedding.*

Within a constant factor, this is done for toroidal graphs with bounded degree [29].

One expects that the right estimate for the planar crossing number of a toroidal graph is $d_1 d_2$, where d_1 and d_2 are the lengths of transverse non–contractible dual cycles, chosen to minimize the product $d_1 d_2$.

Širáň [56] introduced the (orientable and non–orientable) crossing sequence of a graph G. The orientable surface of genus g is denoted by S_g and the non–orientable surface of genus g is denoted by N_g. The *orientable crossing sequence* is $\mathrm{cr}_{S_0}(G), \mathrm{cr}_{S_1}(G), \mathrm{cr}_{S_2}(G), \ldots$ and the *non–orientable crossing sequence* is $\mathrm{cr}_{S_0}(G), \mathrm{cr}_{N_1}(G), \mathrm{cr}_{N_2}(G), \ldots$. Evidently, if the genus of G is $\gamma(G)$ and if $\gamma \geq \gamma(G)$, then $\mathrm{cr}_{\Sigma_g}(G) = 0$, and, if $g < \gamma(G)$, then $\mathrm{cr}_{\Sigma_g}(G) > \mathrm{cr}_{\Sigma_{g+1}}(G)$.

The main result of [56] is that if the (finite) sequence $a_0 > a_1 > a_2 > \ldots > a_k = 0$ of integers is convex (that is, $a_{i+1} - a_{i+2} \leq a_i - a_{i+1}$ for each i), then there is a graph G whose orientable crossing sequence is $(a_0, a_1, a_2, \ldots, a_k, 0, 0, 0, \ldots)$. There is also a graph H whose non–orientable crossing sequence is $(a_0, a_1, a_2, \ldots, a_k, 0, 0, 0, \ldots)$.

Archdeacon, Bonnington and Širáň [3] showed by example that not every crossing function is convex. Furthermore, they proved that if $a_0 > a_1 > 0$, then there is a graph G that embeds in the Klein bottle satisfying $\mathrm{cr}_{N_i}(G) = a_i$, for $i = 0, 1$. In the orientable case, DeVos, Mohar and Šámal [14] have proved that if $a_0 > a_1 > 0$, then there is a graph G that embeds in the double torus and $\mathrm{cr}_{S_i}(G) = a_i$, for $i = 0, 1$. Archdeacon et al. [3] suggested the following conjecture:

Conjecture 8.1 *Any strictly decreasing sequence of positive integers is the orientable crossing sequence of some graph and the non–orientable crossing sequence of some graph.*

9 Conclusion

We have attempted to highlight recent work about crossing numbers that demonstrate the growth of the theory. We have seen:

- relations with extremal graph theory;
- Székely's simple proof of the Erdös–Trotter theorem, which connects crossing numbers with several areas of mathematics;

- our understanding of k–crossing–critical graphs grow substantially;
- intimations of how to get good estimates for the crossing numbers of the members of families of graphs;
- many advances in algorithmic and complexity questions.

Despite these exciting results, there are still many easily stated problems to motivate further work.

References

[1] J. Adamsson and R. B. Richter, Arrangements and the crossing number of $C_m \times C_n$, *J. Combin. Theory (B)* **90** (2004), 21–39.

[2] M. Ajtai, V. Chvátal, M. M Newborn and E. Szemerédi, Crossing–free subgraphs, *Theory and Practice of Combinatorics*, North–Holland Math. Stud. **60**, North–Holland, (1982), 9–12.

[3] D. Archdeacon, C. P. Bonnington and J. Širáň, Trading crossings for handles and crosscaps, *J. Graph Theory* **38** (2001), 230–243.

[4] L. W. Beineke and R. D. Ringeisen, On the crossing numbers of products of cycles and graphs of order four, *J. Graph Theory* **4** (1980), 145–155.

[5] D. Bienstock, Some provably hard crossing number problems, *Discrete Comput. Geom.* **6** (1991), 443–459.

[6] D. Bienstock and N. Dean, Bounds for rectilinear crossing numbers, *J. Graph Theory* **17** (1993), 333–348.

[7] D. Bokal, paper on (3,6), to appear, *J. Graph Theory.*

[8] D. Bokal, On the crossing numbers of Cartesian products with paths. *J. Combin. Theory (B)* **97** (2007), 381–384.

[9] D. Bokal, On the crossing numbers of Cartesian products with trees. J. Graph Theory 56 (2007), 287–300.

[10] D. Bokal, G. Fijavž and B. Mohar, The minor crossing number, *SIAM J. Discrete Math.* **20** (2006), 344–356.

[11] B. Bollobás, The isoperimetric number of random regular graphs, *European J. Combin.* **9** (1988), 241–244.

[12] K. Böröczky, J. Pach and G. Tóth, Planar crossing numbers of graphs embeddable in another surface, *Internat. J. Found. Comput. Sci.* **17** (2006), 1005–1015.

[13] S. Cabello and B. Mohar, Crossing and weighted crossing number of near–planar graphs, manuscript (2008).

[14] M. DeVos, B. Mohar and R. Šámal, Unexpected behaviour of crossing sequences, manuscript, 2007.

[15] P. Erdős and R. Guy, Crossing number problems, *Amer. Math. Monthly* **80** (1973), 52–58.

[16] G. Even, S. Guha and B. Schieber, Improved approximations of crossings in graph drawings and VLSI layout areas, *SIAM J. Comput.* **32** (2002/03), 231–252.

[17] J. Fiedler, J. P. Huneke, R. B. Richter and N. Robertson, Computing the orientable genus of projective graphs, *J. Graph Theory* **20** (1995), 297–308.

[18] J. Fox and C. Tóth, On the decay of crossing numbers, *J. Combin. Theory (B)* **98** (2008), 33–42.

[19] Z. Füredi and A. Kündgen, Moments of graphs in monotone families, *J. Graph Theory* **57** (2006), 37–48.

[20] E. Garcia–Moreno and G. Salazar, Bounding the crossing number of a graph in terms of the crossing number of a minor with small maximum degree, *J. Graph Theory* **36** (2001), 168–173.

[21] M. R. Garey and D. S. Johnson, Crossing number is NP–complete, *SIAM J. Algebraic Discrete Methods* **4** (1983), 312–316.

[22] J. Geelen, R.B. Richter and G. Salazar, Embedding grids in surfaces, *European J. Combin.* **25** (2004), 785–792.

[23] I. Gitler, P. Hliněný, J. Leanos and G. Salazar, The crossing number of a projective graph is quadratic in the face–width, Electron. J. Combin. **15** (2008) #R46.

[24] L. Glebskiĭ and G. Salazar, The crossing number of $C_m \times C_n$ is $(m-2)n$ for all but finitely many n for each m, *J. Graph Theory* **47** (2004), 53–72.

[25] M. Grohe, Computing crossing numbers in quadratic time, *J. Comput. System Sci.* **68** (2004), 285–302. .

[26] C. Gutwenger, P. Mutzel and R. Weiskircher, Inserting an edge into a planar graph, *Algorithmica* **41** (2005), 289–308.

[27] P. Hliněný, Crossing–number critical graphs have bounded path–width, *J. Combin. Theory (B)* **88** (2003), 347–367.

[28] P. Hliněný, Crossing number is hard for cubic graphs, *J. Combin. Theory (B)* **96** (2006), 455–471.

[29] P. Hliněný and G. Salazar, Approximating the crossing number of toroidal graphs, in ISAAC 2007, 148–159, Lecture Notes in Comput. Sci.4835, Springer, Berlin 2007.

[30] P. Hliněný and G. Salazar, On the crossing number of almost planar graphs, in Graph Drawing 2006, 162–173, Lecture Notes in Comput. Sci., 4372, Springer, Berlin, 2007.

[31] K. Kawarabayashi and B. Reed, Computing crossing number in linear time, in STOC '07, 382 - 390, Proc. 39th ACM Symp. Theory of Computing, ACM, New York, 2007.

[32] M. Kochol, Construction of crossing–critical graphs, *Discrete Math.* **66** (1987), 311–313.

[33] F. T. Leighton, *Complexity Issues in VLSI: Optimal Layouts for the Shuffle–Exchange Graph and Other Networks*, MIT Press, 1983.

[34] M. Lomeli and G. Salazar, Nearly light cycles in embedded graphs and crossing–critical graphs, *J. Graph Theory* **53** (2006), 151–156.

[35] J. Pach, F. Shahrokhi and M. Szegedy, Applications of the crossing number, *Algorithmica* **16** (1996), 111–117.

[36] J. Pach, J. Spencer and G. Tóth, New bounds on crossing numbers, ACM Symposium on Computational Geometry (Miami, 1999), *Discrete Comput. Geom.* **24** (2000), 623–644.

[37] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17** (1997), 427–439.

[38] J. Pach and G. Tóth, Which crossing number is it anyway?, *J. Combin. Theory (B)* **80** (2000), 225–246.

[39] J. Pach and G. Tóth, Thirteen problems on crossing numbers, *Geombinatorics* **9** (2000), 194–207.

[40] M. Pelsmajer, M. Schaefer and D. Štefankovič, Removing even crossings, *J. Combin. Theory (B)* **97** (2007), 489–500.

[41] M. Pelsmajer, M. Schaefer and D. Štefankovič, Odd crossing number and crossing number are not the same, Discrete Comput. Geom. 39 (2008), 442–454.

[42] M. Pelsmajer, M. Schaefer and D. Štefankovič, *Crossing Numbers of Graphs with Rotation Systems*, DePaul University Technical Report, TR 05–017, 2005.

[43] B. Pinontoan and R. B. Richter, Crossing numbers of sequences of graphs I: general tiles, *Australas. J. Combin.* **30** (2004), 197–206.

[44] B. Pinontoan and R. B. Richter, Crossing numbers of sequences of graphs II: planar tiles, *J. Graph Theory* **42** (2003), 332–341.

[45] B. Richter, Cubic graphs with crossing number two, *J. Graph Theory* **12** (1988), 363–374.

[46] R. B. Richter and C. Thomassen, Minimal graphs with crossing number at least k, *J. Combin. Theory (B)* **58** (1993), 127–224.

[47] R. D. Ringeisen and L. W. Beineke, The crossing number of $C_3 \times C_n$, *J. Combin. Theory (B)* **24** (1978), 134–136.

[48] N. Robertson and P. D. Seymour, Excluding a graph with one crossing, *Graph Structure Theory* (Seattle, 1991), *Contemp. Math.*, **147**, (eds. N. Robertson and P. D. Seymour) Amer. Math. Soc. (1993), 669–675.

[49] N. Robertson and P.D. Seymour (eds), *Graph Structure Theory: Proc. AMS–IMS–SIAM Joint Summer Research Conference on Graph Minors*, (Seattle, 1991) *Contemp. Math.* **147** Amer. Math. Soc., 1993.

[50] G. Salazar, On the crossing numbers of loop networks and generalized Petersen graphs, *Discrete Math.* **302** (2005), 243–253.

[51] G. Salazar, Infinite families of crossing–critical graphs with given average degree, *Discrete Math.* **271** (2003), 343–350.

[52] G. Salazar, Drawing an embedded graph in an alternative surface, manuscript, 2005.

[53] M. Schaefer, E. Sedgwick and D. Štefankovič, Recognizing string graphs is in NP, *J. Comput. System Sci.* **67** (2003), 365–380.

[54] M. Schaefer and D. Štefankovič, Decidability of string graphs, *J. Comput. System Sci.* **68** (2004), 319–334.

[55] F. Shahrokhi, L. Székely and I. Vrťo, Crossing numbers of graphs, lower bound techniques and algorithms: a survey, *Graph Drawing* (Princeton, 1994), Lecture Notes in Comput. Sci., **894**, Springer (1995), 131–142,.

[56] J. Širáň, The crossing function of a graph, *Abh. Math. Sem. Univ. Hamburg* **53** (1983), 131–133.

[57] J. Širáň, Infinite families of crossing–critical graphs with a given crossing number, *Discrete Math.* **48** (1984), 129–132.

[58] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Probab. Comput.* **6** (1997), 353–358.

[59] L. Székely, A successful concept for measuring non–planarity of graphs: the crossing number, *Discrete Math.* **276** (2004), 331–352.

[60] P. Turán, A note of welcome, *J. Graph Theory* **1** (1977), 1–5.

[61] W. T. Tutte, Toward a theory of crossing numbers, *J. Combin. Theory* **8** (1970), 45–53.

[62] R. P. Vitray, Graphs containing graphs of crossing number 2, presentation at AMS Summer Conference, Ohio State University, August 1990.

[63] D. Wood and J. A. Telle, Planar decompositions and the crossing number of graphs with an excluded minor. *New York J. Math.* **13** (2007), 117–146.

[64] I. Vrťo. Crossing Numbers Bibliography, available on–line at `www.ifi.savba.sk/∼imrich/`.