The Planarity Theorems of MacLane and Whitney for Graph–like Spaces

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Abstract

The planarity theorems of MacLane and Whitney are extended to compact graph–like spaces. This generalizes recent results of Bruhn and Stein (MacLane's Theorem for the Freudenthal compactification of a locally finite graph) and of Bruhn and Diestel (Whitney's Theorem for an identification space obtained from a graph in which no two vertices are joined by infinitely many edge-disjoint paths).

1 Introduction

The theorems of MacLane and Whitney characterize planarity of a graph G in different ways. The former characterizes planar graphs by the existence of a basis for the cycle space of G in which every edge appears at most twice, while the latter characterization is in terms of the cycle matroid of G having a graphic dual matroid. Naturally, these are closely connected to Kuratowski's Theorem: G is planar if and only if G does not contain a subdivision of either $K_{3,3}$ or K_5.

Recent work has treated extensions of these theorems to infinite graphs. Bruhn and Diestel [1] generalize Whitney's Theorem in the case of an infinite graph in which no two vertices are joined by infinitely many edge-disjoint paths. Bruhn and Stein [2] provide MacLane's Theorem in the case of locally finite graphs. In both cases, the theorem is proved for a topological space obtained from the graph by adding its ends and, in the former case, identifying a vertex with each end that is not separated from the vertex by any finite set of edges. In the central case of 2–connected graphs, these spaces are compact, connected, and Hausdorff.

Thomassen and Vella [9] have recently introduced the notion of a graph–like space. Their concept is more general than what we present here, but for our purposes, a *graph–like space* is a compact metric space G that contains a totally–disconnected subset V so that $G - V$ consists of disjoint open subsets of G, each homeomorphic to the real line and having precisely two limit points in V. The elements of V are the *vertices* of G and the components of $G - V$ are the *edges* of G. Graph–like spaces as we have defined them are locally connected Peano spaces, and Thomassen and Vella show they satisfy a form of Menger's Theorem.

Graph–like spaces appeal as an object of study because they extend many of the combinatorial properties of finite graphs while still being quite general. The class of graph–like spaces contains the class of finite graphs, the compactifications of infinite graphs considered by Casteels, Richter and Vella [3, 10], and the spaces considered by Bruhn, Diestel and Stein in [1, 2]. The graph–like space depicted in Figure 1, in which there are two points that have degree 1 but also are the ends of a ladder, is an example of a space that does not fit into any of the aforementioned categories.

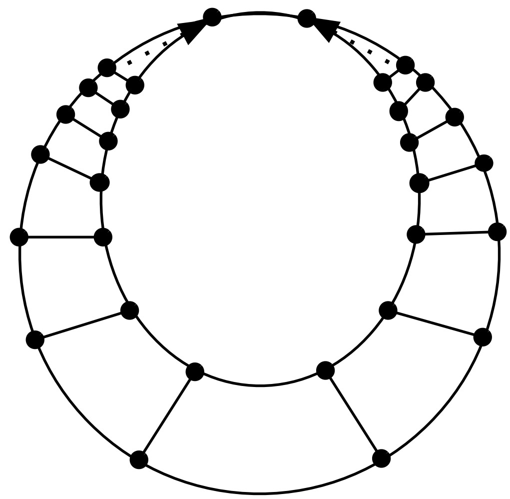

Figure 1: Graph–like space formed by joining the two ends of the Freudenthal compactification of the doubly infinite ladder with an edge.

For finite graphs, the theorems of MacLane and Whitney can be easily derived from Kuratowski's Theorem. Fortunately, graph–like spaces satisfy the following deep version of Kuratowski's Theorem, due to Thomassen [8].

Theorem 1.1 *Let X be a compact, 2–connected, locally connected metric space. Then X is homeomorphic to a subset of the sphere if and only if X contains no subspace homeomorphic to either $K_{3,3}$ or K_5.*

We will use this result to prove MacLane's and Whitney's Theorems for graph–like spaces. We note the assumption here that X is *2–connected*. This means X is connected and, for each $x \in X$, $X - x$ is also connected. Thomassen mentions that the thumbtack space, consisting of a disc together with a compact interval with one end identified with the centre of the disc, is not planar. It satisfies all the hypotheses of Theorem 1.1 except 2–connection, and yet does not contain either $K_{3,3}$ or K_5. There are graph–like spaces that have this same property, so we cannot hope to have planarity be the characterization in our generalizations of MacLane's and Whitney's Theorems. Rather, our versions of MacLane's and Whitney's Theorems will characterize the graph–like spaces with no subspace homeomorphic to either $K_{3,3}$ or K_5. In both instances, the 2–connected spaces are the interesting ones and this is, by Theorem 1.1, the same as planarity.

2 Blocks, the cycle space and B–matroids

In both proofs the first step is a reduction to the 2–connected case, which allows us to apply Theorem 1.1 for one direction of the characterization. Thomassen and Vella

[9] prove that each component of a graph-like space is locally connected and has only countably many edges.

A *cut point* in a topological space X is a point u of X so that, if K is the component of X containing u, $K - u$ is not connected. A *block of* X is a maximal non-empty connected subset Y of X so that Y has no cut point. The space X is *2–connected* if it is connected and has no cut point.

To prove the existence of blocks of X, it suffices to make the easy observation that if K and L are connected subsets of X each with no cut point and $|K \cap L| \geq 2$, then $K \cup L$ has no cut point; the existence of blocks now follows from Zorn's Lemma. Furthermore, it is an easy exercise to show that a maximal connected subset of X with no cut point is closed.

Let G be a connected graph–like space. If e is an edge of G so that $G - e$ is not connected, then every point of $\mathrm{cl}(e)$ is a cut point of G. However, we would like $\mathrm{cl}(e)$ to be a block of G. This is achieved by restricting the consideration of cut points to elements of the vertex set V of G. The application of Zorn's Lemma is just as simple.

A graph–like space H contained in G is a *subgraph* of G if, for each edge e of G, either $e \subseteq H$ or $e \cap H = \varnothing$. A *cycle* in G is a connected subgraph C of G containing at least one edge so that, if e is any edge of C, then $C - e$ is connected, while if e and f are distinct edges of C, then $C - (e \cup f)$ is not connected.

Moreover, the connected closed subsets of a graph–like space are arc–wise connected [9]. From this it follows that, in fact, the cycles of G are precisely the homeomorphic images of circles in G [10, Thm. 35]. Therefore, a block of G has at least two edges if and only if it contains a cycle.

The notion of a cycle space for a topological space has been introduced by Vella and Richter [10]. Their results are for spaces more general still than graph–like spaces and so, in particular, a graph–like space has a cycle space. We begin by summarizing the relevant results, as applied to graph–like spaces.

Let $\mathcal{E}$ be a collection of sets of edges of G. Then $\mathcal{E}$ is *thin* if every edge of G is in at most finitely many of the sets in $\mathcal{E}$. The point is that if $\mathcal{E}$ is thin, then the symmetric difference of the sets in $\mathcal{E}$ is well defined: it consists of those edges that are in an odd number of elements of $\mathcal{E}$. This is the *thin sum* of the elements of $\mathcal{E}$.

The *cycle space* Z_G of G is the smallest subset of the power set 2^E of E that contains all edge sets of cycles and is closed under thin sums. From this definition and the earlier remarks, it is easy to see that Z_G is the direct sum of the Z_B, over all the blocks B of G.

A natural way to state Whitney's Theorem is that the cycle matroid of a graph has a graphic dual if and only if the graph is planar. We will prove this form of Whitney's Theorem for graph–like spaces. To do so, we must introduce B–matroids [4].

A *B–matroid* is a pair $(X, \mathcal{I})$ consisting of a set X and a set $\mathcal{I}$ of subsets of X so that:

1. $\mathcal{I} \neq \varnothing$;
2. if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;
3. for each $Y \subseteq X$, there is a maximal element of $\mathcal{I}$ contained in Y; and
4. for each $Y \subseteq X$, if I and J are two maximal elements of $\mathcal{I}$ contained in Y and $x \in I \setminus J$, then there is a $y \in J \setminus I$ so that $(I \cup \{y\}) \setminus \{x\} \in \mathcal{I}$.

Essentially, B–matroids are one way of generalizing matroids to infinite sets. The last axiom corresponds to the basis exchange axiom for finite matroids. Every B–matroid M has a dual B–matroid M^* whose bases are the complements of the bases of M.

If G is a graph–like space or if G is a graph, then there is a natural B–matroid to associate with G, namely all the sets of edges that do not contain the edge set of a circuit. For graph–like spaces, there may be infinite circuits, but a graph has only finite circuits.

For a finite matroid M it is well–known (although not trivial) that the relation $\sim$ on the elements of M, defined by $e \sim f$ if and only if e and f are in a circuit together, is an equivalence relation. The matroid M is connected when $\sim$ has just one equivalence class. This is also true for B–matroids (Bruhn and Wollan, personal communication); fortunately we do not need it, as it holds true in an obvious way for graph–like spaces.

In particular, the B–matroid M associated with a graph–like space G is the direct sum of the B–matroids associated to each of the blocks of G. This implies that the dual M^* is also a direct sum of the B–matroids of the duals of the blocks of G.

3 MacLane's Theorem

A *2-basis* for the cycle space Z_G is a set B of elements of Z_G so that each element of E is in at most 2 elements of B and every element of Z_G is a symmetric difference of some (possibly infinite) subset of B. Notice that every 2-basis is thin.

The following is MacLane's Theorem for graph–like spaces.

Theorem 3.1 *Let G be a graph–like space. Then G has a 2-basis if and only if G does not contain a homeomorph of either K_5 or $K_{3,3}$.*

Proof. It suffices to prove the theorem in the case G is 2–connected. Suppose first that G has no homeomorph of either $K_{3,3}$ or K_5. By Theorem 1.1, G is planar. By [6], we know that the face boundaries of any embedding of G in the sphere are circles. We take these circles to be the elements of B and note that every edge of G is in at most 2 elements of B.

Let $\mathcal{B}$ be the smallest subset of 2^E containing the elements of B and closed under thin sums. As $\mathcal{Z}_G$ is such a set, $\mathcal{B} \subseteq \mathcal{Z}_G$. On the other hand, if C is a cycle in G, then C is a circle in the sphere and so C is the symmetric difference of the elements of B on one side of C. Therefore, $C \in \mathcal{B}$, so every cycle of G is in $\mathcal{B}$. We conclude that $\mathcal{Z}_G \subseteq \mathcal{B}$, whence $\mathcal{Z}_G = \mathcal{B}$.

It remains to show that every element of $\mathcal{Z}_G$ is a thin sum of the elements of B. To do this, we shall require the following lemma (which is also required for Whitney's Theorem), whose proof is given below. Note that this lemma implies the geometric dual is connected.

Lemma 3.2 *Let G be a 2-connected graph-like space, with vertex set V, embedded in the sphere $\mathbb{S}^2$ and let C be any circle in G. If F and F' are distinct faces of G, then there is an arc α in $\mathbb{S}^2 \setminus V$ having one end in F, one end in F' and such that $\alpha \cap G$ is finite. Moreover, if F and F' are in the same component of $\mathbb{S}^2 \setminus C$, then α may be chosen so that $\alpha \cap C = \emptyset$.*

Let $z \in \mathcal{Z}_G$ and partition z into disjoint cycles; let $\{C_\pi\}$ be the corresponding circles, that is, the edge sets of the C_π partition z. Pick any point u of $\mathbb{S}^2 \setminus G$. For any other point v of $\mathbb{S}^2 \setminus G$, if α is any uv-arc in $\mathbb{S}^2 \setminus V$ meeting G only finitely often, then the parity of $\alpha \cap G$ is independent of α: it crosses each C_π a number of times whose parity is odd if u and v are on different sides of C_π and even otherwise.

It follows that we can partition the faces of G into those that have odd parity paths from u and those that have even parity paths from u. It is now easy to see that z is the sum of the boundaries of all the faces in one of these two sets, as required.

Conversely, suppose G has a 2-basis B. We use (without significant change) the proof of [2] to show that G has no subdivision of either $K_{3,3}$ or K_5. We show that every finite 2–connected graph that has a homeomorph H in G also has a 2-basis. Since neither $K_{3,3}$ nor K_5 has a 2-basis, G does not contain a homeomorph of either of these, as claimed.

Since H is 2–connected, its cycle space $\mathcal{Z}_H$ has non-empty elements and each of these is a symmetric difference of the elements of a subset of B. For each element z of $\mathcal{Z}_H$, let B_z denote such a subset of B. Now, among the finitely many elements of $\mathcal{Z}_H$, let $z_1, z_2, \dots, z_k$ be those so that the B_{z_i} are all the inclusion-wise minimal elements of $\{B_z \mid z \in \mathcal{Z}_H\}$.

Let $z \in \mathcal{Z}_H$ be such that, for some i, $B_z \cap B_{z_i} \neq \varnothing$. We claim that $B_{z_i} \subseteq B_z$.

To see this, let z' be $\sum_{b \in B_z \cap B_{z_i}} b$. If e is an edge of G not in H, then e is in an even number of the elements of each of B_z and B_{z_i}. Since e is in at most two elements of B, it follows that e is in either no element of $B_z \cap B_{z_i}$ or two elements of $B_z \cap B_{z_i}$. In either case, e is not in z'. Therefore, $z' \subseteq E(H)$ and we deduce $z' \in \mathcal{Z}_H$. But $B_z \cap B_{z_i} \subseteq B_{z_i}$ and B_{z_i} is minimal. Thus, $B_{z_i} \subseteq B_z$, as required.

Notice that this claim proves that $B_{z_1}, B_{z_2}, \dots, B_{z_k}$ are pairwise disjoint. Thus, every edge of H appears in at most two of the z_i. We claim that $z_1, \dots, z_k$ spans $\mathcal{Z}_H$; this will complete the proof.

Let $z \in \mathcal{Z}_H$ and let I be the set of indices i for which $B_z \cap B_{z_i} \neq \varnothing$. By the preceding remarks, the B_{z_i}, for $i \in I$, are pairwise disjoint and each is contained in B_z.

Let $B' = B_z \setminus \left(\bigcup_{i \in I} B_{z_i}\right)$ and consider $z' = \sum_{b \in B'} b$. Then $z' = z + \sum_{i \in I} z_i \in \mathcal{Z}_H$. If $z' \neq \varnothing$, then there is a $j \in \{1, 2, \dots, k\}$ so that $B_{z_j} \subseteq B_{z'}$. But then $j \in I$, a contradiction. So $B' = \varnothing$ and $z = \sum_{i \in I} z_i$. □

Proof of Lemma 3.2. We prove the moreover version, since the other is an easy consequence. Let R be the region of $\mathbb{S}^2 \setminus C$ containing both F and F'. Note that R and $R \setminus V$ are both open sets; the former is connected by definition. We claim that $R \setminus V$ is also connected.

If we identify all the points of C to a single point c, then the space $R \cup C$ quotients to the sphere. Now $\{c\} \cup (R \cap V)$ is a closed, totally disconnected subset V' of the sphere; therefore, $\mathbb{S}^2 \setminus V'$ is connected. (This is not at all trivial; one way to prove it is to use the classification of non-compact surfaces [5].)

Now each point of $\mathbb{S}^2 \setminus V'$ has a disc neighbourhood so that any two points in the neighbourhood are joined by an arc in $\mathbb{S}^2 \setminus V'$ that meets $G \cap R$ only finitely often (in fact at most twice and twice only if both points are in G). It follows that the set of points of $\mathbb{S}^2 \setminus V'$ joined to a specific point u by an arc that meets G only finitely often is both open and closed and, therefore, is all of $\mathbb{S}^2 \setminus V'$. □

4 Whitney's Theorem

In this section we prove Whitney's Theorem for graph–like spaces. It is important to realize that every cocircuit in a graph–like space is finite [10]. Thus, any dual of a graph–

like space will have only finite circuits. On the other hand, a graph–like space can have infinite circuits and so its dual can have infinite cocircuits.

Thinking next about planar duality, if we embed a connected graph–like space into the sphere, then the faces are open discs, the planar dual has these as vertices, and each edge of the graph–like space has a dual edge joining the two dual vertices on either side of the primal edge. Thus, the planar dual is simply a (possibly infinite) graph.

We are now ready for Whitney's Theorem for graph–like spaces.

Theorem 4.1 *Let G be a graph–like space with corresponding B–matroid M. Then M^* is the B–matroid of a graph if and only if G contains no homeomorph of either $K_{3,3}$ or K_5.*

Proof. We first assume G is 2–connected. If G contains no homeomorph of either $K_{3,3}$ or K_5, then Theorem 1.1 implies G has an embedding in the sphere $\mathbb{S}^2$. In this case, it suffices to prove that the geometric dual has the right B-matroid.

Let H be the geometric dual of G and let C be a cycle of G. Lemma 3.2 implies that the subgraphs of H on either side of C are connected and, therefore, the edges of H dual to the edges of C form a bond of H.

To complete the proof that the geometric dual has the right B-matroid, we must also show that every bond of H is dual to a cycle of G. Let U be a subset of $V(H)$ so that the set $\delta(U)$ of edges of H with precisely one end in U is a bond. For each $u \in U$, let b_u denote the edge set of the cycle of G bounding the face of G containing u. Then $\sum_{u\in U} b_u$ is in the cycle space of G and is equal to the set $(\delta(U))^*$ of edges in G dual to the edges of $\delta(U)$. Since it is in the cycle space of G, it partitions into the edge sets of cycles of G. If C is one of these edge sets, then the preceding paragraph proves that the set C^* of edges of H dual to the edges of C is a bond. Since $C^* \subseteq \delta(U)$, we deduce that $C^* = \delta(U)$ and, therefore $(\delta(U))^*$ is the edge set of a cycle of G, as required.

Conversely, suppose H is a graph whose corresponding B–matroid is the dual of G's B–matroid. We proceed by proving that the cycle matroid of any finite graph homeomorphic to a subspace of G has a graphic dual. So suppose K is a subspace of G that is homeomorphic to a finite graph F — which we may assume has no vertices of degree 2. Then K is a subgraph of G and so is obtained by deleting all the edges of G not in K. We obtain F by contracting the arcs of K representing edges of F to single edges.

In H, the dual operations are: contract all the edges of H not dual to an edge in K and delete the edges of H dual to edges of K that are contracted to obtain F. This produces a minor of H, which, since H is a graph, is graphic. Thus, the cycle matroid of a finite graph homeomorphically contained in G has a graphic dual and, therefore, the subgraph is planar. In particular, it cannot be either $K_{3,3}$ or K_5, as claimed.

We conclude by treating the case G is not 2–connected. In this case, M is the direct sum of the B–matroids of the blocks of G. Each of these is the B–matroid of a graph–like space. If G contains no $K_{3,3}$ or K_5, then neither does any block of G. From the 2–connected case, each block of G has a B–matroid whose dual is the B–matroid of a graph. The dual is the direct sum of the B–matroids of the graphs and so is the B–matroid of a graph.

On the other hand, suppose the dual M^* of M is the B–matroid of a graph H. Since M is the direct sum of the B–matroids of the blocks of G, the dual is the direct sum of the B–matroids dual to those of the blocks of G. The B–matroid for the block K of

G is obtained by deleting all the edges of G not in K; its dual is obtained from M^* by contracting all the elements of G not in K. Since contractions done in H result in a graph, the B–matroid of K has the B–matroid of a graph as its dual. From the 2–connected case, K has no $K_{3,3}$ or K_5 and, therefore, neither does G. □

5 Concluding Remarks

Two of the current authors, together with Thomassen, have proven the following extension of Kuratowski's Theorem [7].

Theorem 5.1 *A compact, locally connected metric space X is not planar if and only if X contains either K_5 or $K_{3,3}$, or a generalized thumbtack, or the disjoint union of the sphere and a point.*

This theorem gives the precise forbidden structure for graph–like spaces (and more general spaces) to be planar. (A *generalized thumbtack* is a graph–like space that approximates a thumbtack. There are five canonical generalized thumbtacks, so Theorem 5.1 gives a finite number of obstructions to planar embedding.) In particular, this theorem explains why our versions of the theorems of MacLane and Whitney do not reference planarity.

It is also interesting to contemplate another variation of Whitney's Theorem: Suppose G is an infinite graph with corresponding B–matroid M. Then M^* is the B–matroid of a graph–like space if and only if G contains no homeomorph of either $K_{3,3}$ or K_5 (which *is* equivalent to G being planar).

The difficulty with this is that G is not compact and, therefore, it is not so obvious how to make use of a planar embedding to obtain a graph–like space geometric dual. However, using quite different techniques, the first author has proved this result; this will appear in his doctoral thesis.

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