RELATING DIFFERENT CYCLE SPACES OF THE SAME INFINITE GRAPH

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Abstract. Casteels and Richter have shown that if X and Y are distinct compactifications of a locally finite graph G and $f : X \to Y$ is a continuous surjection such that f restricts to a homeomorphism on G, then the cycle space Z_X of X is contained in the cycle space Z_Y of Y. In this work, we show how to produce a basis for the quotient space Z_Y/Z_X.

1. Introduction

Bonnington and Richter [1] introduced the cycle space of a locally finite graph as the edge sets of subgraphs in which every vertex has even degree. Diestel and Kühn [3] introduced a different cycle space for a locally finite graph G as the space "generated" by embeddings of circles into the Freudenthal compactification $\mathbb{F}(G)$ of G. (The space $\mathbb{F}(G)$ is obtained from G by adding one point for each "end" of G. An *end* consists of an equivalence class of rays: two rays R and S are equivalent if, for every finite set W of vertices, R and S have their tails in the same component of $G - W$. A basic neighbourhood of the end ω is the component K of $G - W$ containing the tails of the rays in ω, plus all edges between K and W – but not their incident vertices from W – and all the other ends whose rays' tails lie in K.)

Vella and Richter [6] unified these notions by introducing edge spaces and showing that a nice cycle space theory holds for compact, weakly Hausdorff edge spaces (definitions will follow shortly). In particular, the cycle space of Bonnington and Richter can be viewed as the Diestel-Kühn cycle space, but in the 1-point compactification $\mathbb{A}(G)$ of the locally finite graph G, rather than in $\mathbb{F}(G)$.

An *edge space* is an ordered pair (X, E) consisting of a topological space X and a subset E of X, called the *edges*, so that each $e \in E$ is an open singleton whose boundary set has at most two elements. The motivation for edge spaces is to generalize a natural topology of a graph. Let G be a graph with vertex set V and edge set E. In the

Date: July 1, 2009.

classical topology [6], $\{e\}$ is open for each $e \in E$, and v plus all its incident edges is open for each $v \in V$. With this topology, topological and graph-theoretic connection coincide. We will be interested in compactifications of G, which, for infinite graphs, means adding some additional points and open sets.

We believe there is merit to treating our infinite spaces more combinatorially that would be usual in topology. Thus, we prefer edges to be open singletons, rather than real intervals. In many places in this work, the vertex set will consist of closed singletons making a totally disconnected subspace — this provides a very natural, simultaneous generalization of finite and infinite graphs, as well as the Freudenthal compactification of a locally finite graph. Since most topology texts generally assume all singletons are closed for their more advanced theorems, at this stage it is necessary to develop the topological results more suited to this context. This leads us to the following natural extension of the Hausdorff property. A space X is *weakly Hausdorff* if for all $x, y \in X$ there exist open sets U_x and U_y so that: $x \in U_x$; $y \in U_y$; and $|U_x \cap U_y|$ is finite. In the classical topology on a graph, the only pairs that have no disjoint neighbourhoods are a vertex and an incident edge, and adjacent vertices.

A *cycle* in an edge space (X, E) is a connected subspace Y of X so that: $Y \cap E$ is not empty; for every $e \in Y \cap E$, $Y \setminus \{e\}$ is connected; and, for any distinct $e, f \in Y \cap E$, $Y \setminus \{e, f\}$ is not connected. This definition should be compared with the "embedding of a circle" definition of Diestel and Kühn. We connect the two definitions in Section 4: let C be a cycle in compact weakly Hausdorff edge space (X, E) for which $X \setminus E$ is totally disconnected; as long as C has only countably many edges, then C naturally corresponds to an embedding of a circle in a compact Hausdorff space related to X.

The *cycle space* $Z_{(X,E)}$ of (X, E) is the smallest subset of the power set 2^E of E that contains all the (edge-sets of) cycles and is closed under *thin sums*: a set $\mathcal{C} \subseteq 2^E$ is *thin* if each element of E occurs in only finitely many elements of $\mathcal{C}$ and the "sum" is the set of elements of E that are in an odd number of elements of $\mathcal{C}$.

We assume the reader is familiar with the following results from [6].

Lemma 1.1. *Let (X, E) be a compact, weakly Hausdorff edge space.*

1. *There is a minimal connected subset T of X containing $X \setminus E$.*
2. *Each edge e in $E \setminus T$ is in a unique cycle, with edge set $C_T(e)$, contained in $(E \cap T) \cup \{e\}$.*
3. *The set $\mathcal{C} = \{C_T(e) \mid e \in E \setminus T\}$ is thin and, for every element z of $Z_{(X,E)}$, $z = \sum_{e \in z \setminus T} C_T(e)$.*

The set T from Lemma 1.1 (1) is a *spanning tree* of (X, E). The elements of $\mathcal{C}$ in (3) are the *fundamental cycles* (with respect to T). We shall also make use of the following results from Casteels and Richter [2].

Lemma 1.2. *Let (X, E) and (Y, F) be compact, weakly Hausdorff edge spaces and let $f : X \to Y$ be a continuous surjection so that $f : E \to F$ is a bijection and $f(X \setminus E) = Y \setminus F$. Then:*

1. *for any spanning tree T_Y of (Y, F), there is a spanning tree T_X of (X, E) containing $f^{-1}(T_Y)$; and*
2. *identifying each $e \in E$ with $f(e)$, $Z_{(X,E)} \subseteq Z_{(Y,F)}$.*

Our first principal result further clarifies the relationship $Z_{(X,E)} \subseteq Z_{(Y,F)}$ of Lemma 1.2 by showing (perhaps as expected) that it is precisely the fundamental cycles $C_{T_Y}(f(e))$, for $e \in E \cap T_X \setminus f^{-1}(T_Y)$, that generate the quotient $Z_{(Y,F)}/Z_{(X,E)}$.

Theorem 1.3. *Let (X, E) and (Y, E) be compact, connected, weakly Hausdorff edge spaces and let $f : X \to Y$ be a continuous surjection so that, for each $e \in E$, $f(e) = e$ and $f(X \setminus E) = Y \setminus E$. Let T_Y be the edge set of a spanning tree of (Y, E). Let T_X be the edge set of a spanning tree of (X, E) for which $T_Y \subseteq T_X$ and let z be in $Z_{(Y,E)}$. Then there exist unique $z' \in Z_{(X,E)}$ and, for each $e \in T_X \setminus T_Y$, $\alpha_e \in \{0, 1\}$ so that $z = f(z') + \sum_{e \in T_X \setminus T_Y} \alpha_e C_{T_Y}(e)$.*

A main result in [1] considers a locally finite graph G embedded in the sphere with a finite number of accumulation points. If there are $k+1$ accumulation points, then the face boundaries generate a subspace of the cycle space of $\mathbb{A}(G)$ having index k. Our other principal result generalizes this fact. If the graph G can be embedded in the sphere with $k + 1$ accumulation points, performing k identifications converts the closure of the graph into $\mathbb{A}(G)$. The theorem below treats the abstract case of a single identification, while the corollary, proved by a trivial induction from the theorem, provides the case of finitely many identifications. For two points x, y of X, an *xy-path* is a minimal closed connected set containing x and y. The crucial point is that deleting k edges in an xy-path produces a subset having exactly $k+1$ components and, if $k \geq 1$, then x and y are in different components.

Theorem 1.4. *Let (X, E) be a compact, connected, weakly Hausdorff edge space such that $X \setminus E$ is totally disconnected. Let $x, y \in X \setminus E$ be distinct and let Y be the quotient space obtained from X by identifying x and y. Let P be the edge set of an xy-path in X. Then $P \in Z_{(Y,E)} \setminus Z_{(X,E)}$ and, for each $z \in Z_{(Y,E)} \setminus Z_{(X,E)}$, there is a unique element $z' \in Z_{(X,E)}$ so that $z = z' + P$.*

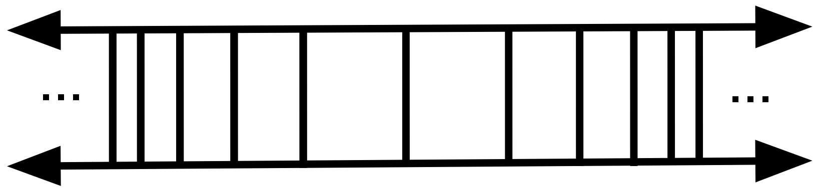

FIGURE 1. 2-way infinite ladder

Corollary 1.5. *Let (X, E) be a compact, connected, weakly Hausdorff edge space such that $X \setminus E$ is totally disconnected. For $i = 1, 2, \dots, k$, let $\{x_i, x_i'\}$ be pairs of elements of $X \setminus E$ so that, for any nonempty subset I of $\{1, 2, \dots, k\}$, $|\cup_{i \in I} \{x_i, x_i'\}| > |I|$. Let Y be the quotient space obtained from X by identifying x_i with x_i', $i = 1, 2, \dots, k$. For each $i = 1, 2, \dots, k$, let P_i be an $x_i x_i'$-path in X. If $z \in Z_{(Y,E)}$, then there exist unique $z' \in Z_{(X,E)}$ and, for $i = 1, 2, \dots, k$, $\alpha_i \in \{0, 1\}$ so that $z = z' + \sum_{i=1}^{k} \alpha_i P_i$.*

We remark that Theorem 1.4 and Corollary 1.5 apply to the case of distinct compactifications of a graph G. This point is discussed in Section 2. Section 3 has the proof of Theorem 1.3, while Theorem 1.4 is proved in Section 5. Section 4 is devoted to a deeper understanding of cycles that is used in Section 5.

2. Examples and remarks

In this section, we show that the hypotheses of Theorem 1.4 and Corollary 1.5 hold in cases of interest.

As a simple example, consider the 2-way infinite ladder L (see Figure 1). There are two ends to L, which we denote by a to the left and b to the right. Then $\mathbb{F}(L)$ is L plus these two additional points. Let $L_{a=b}$ denote the space obtained from $\mathbb{F}(L)$ by identifying a and b. Let v and w be any two vertices of L. We can set $L_{a=v}$ to be the space obtained by identifying a with v, $L_{b=w}$ the space obtained from $\mathbb{F}(L)$ by identifying b with w, and $L_{a=v,b=w}$ the space obtained from $L_{a=v}$ by identifying b with w. All these spaces have cycle spaces, which we naturally denote by $Z_{\mathbb{F}}$, $Z_{a=b}$, $Z_{a=v}$, $Z_{b=w}$, and $Z_{a=v,b=w}$. All of these contain $Z_{\mathbb{F}}$, but, for example, none of $Z_{a=b}$, $Z_{a=v}$, and $Z_{b=w}$ is contained in another of these three, while the last two of these three are both contained in $Z_{a=v,b=w}$. All the containments mentioned have index 1 except $Z_{\mathbb{F}}$ has index 2 in $Z_{a=v,b=w}$.

In all cases, when we delete the set of edges from the superspace of L, the result is a Hausdorff subspace. This recalls the fact that if $f : X \to Y$ is a continuous map from a compact space X to a Hausdorff space Y, then f is a quotient map. In our context, Y is not Hausdorff. Here is a small generalization of the standard fact which is more appropriate for our context.

Lemma 2.1. *Let (X, E) be a compact edge space and let (Y, F) be a weakly Hausdorff edge space such that $Y \setminus F$ is Hausdorff. Let $f : X \to Y$ be a continuous function for which: $f : E \to F$ is a bijection; for $e \in E$, $\mathtt{Cl}(f(e)) = f(\mathtt{Cl}(e))$; and $f(X \setminus E) = Y \setminus F$. Then f is a quotient map.*

Proof. It suffices to show that if $C \subseteq X$ is closed, then $f(C)$ is closed in Y. Since C is closed and X is compact, C is compact. Thus, $f(C)$ is compact. We claim it is closed in Y. Clearly $C \setminus E$ is closed in X and so is also compact. Thus, $f(C \setminus E)$ is compact in $Y \setminus F$ and, since $Y \setminus F$ is Hausdorff, is closed in $Y \setminus F$ and, therefore, is closed in Y.

Let V denote the elements of $Y \setminus F$ incident with elements of $f(C \cap E)$. Clearly $V \subseteq f(C)$, so $\mathtt{Cl}(V) \subseteq f(C)$ by earlier remarks. We claim that $\mathtt{Cl}(f(C \cap E)) = f(E) \cup \mathtt{Cl}(V)$.

If $y \in \mathtt{Cl}(V)$ and U is an open set in Y containing y, then U contains an element v of V. Since U is open and contains v, it contains all the elements of F incident with v; at least one of these is in $f(C \cap E)$, so $U \cap f(C \cap E) \neq \emptyset$. Thus, $y \in \mathtt{Cl}(f(C \cap E))$.

For the opposite containment, let $(f(e_\lambda))$ be a net in $f(C \cap E)$ converging to y. If $y \in F$, then, since $\{y\}$ is open in Y, there is a λ_0 so that, for $\lambda \geq \lambda_0$, $f(e_\lambda) = y$. In this case, $y \in f(C \cap E)$.

So now suppose $y \notin F$. For each λ, let v_λ be an end of $f(e_\lambda)$. Because Y is compact, there is a limit point z of the net (v_λ). Clearly $z \in \mathtt{Cl}(V)$. If $y = z$, then we are done, so we suppose $y \neq z$ and show that $y \in V$.

Let U_y and U_z be neighbourhoods in X so that $U_y \cap U_z$ is finite and contained in F. (The former can be achieved because Y is weakly Hausdorff and the latter can be achieved because $Y \setminus F$ is Hausdorff – taking appropriate intersections yields U_y and U_z.) Let λ_0 be such that, if $\lambda > \lambda_0$, then $f(e_\lambda) \in U_y$.

Let U be an open set in Y containing z. For each $\lambda > \lambda_0$, there is a $\lambda' > \lambda$ so that $v_{\lambda'} \in U \cap U_z$ and, therefore, $f(e_{\lambda'}) \in U \cap U_z$. But we also have $f(e_{\lambda'}) \in U_y$, so $f(e_{\lambda'})$ is one of the finitely many elements of $U_y \cap U_z$. Thus there is an e so that $f(e) \in U_y \cap U_z$ and, for every λ, there is a $\lambda' > \lambda$ with $e = e_{\lambda'}$, in which case we conclude that y is incident with e and so is obviously in V. $\square$

We remark that if (Y, F) is weakly Hausdorff, then $Y \setminus F$ is also weakly Hausdorff. A weakly Hausdorff space X is Hausdorff if and only if X is T_1, which in turn is equivalent to every point in X is closed. Thus, the assumption in Lemma 2.1 that $Y \setminus F$ is Hausdorff can be "weakened" to the assumption that each point in $Y \setminus F$ is closed. It would be interesting to know if there is a generalization of Lemma 2.1 that does not require the assumption that $Y \setminus F$ is Hausdorff.

We conclude with two simple examples. For the first example, let X_1 be the space consisting of continuumly many points with the discrete topology. Let X be the space obtained from X_1 by adding two new points x and y, each joined by an edge to each point of X_1. Let Y be the space consisting of the real line $\mathbb{R}$, plus one new point z, joined by parallel edges $e_{1,r}, e_{2,r}$ to each point r in $\mathbb{R}$. If $b : X_1 \to \mathbb{R}$ is any bijection, then b is continuous, as is the function $f : X \to Y$ defined by

$$f(t) = \begin{cases} b(t) & t \in X_1 \\ z & t = x, y \\ e_{1,r} & t \text{ is an edge incident with } x \text{ and } x', \text{ and } b(x') = r \\ e_{2,r} & t \text{ is an edge incident with } y \text{ and } x', \text{ and } b(x') = r\,. \end{cases}$$

Then f is clearly not a quotient map and X is not compact. The cycle space of X consists of all sets z of edges such that, for each pair of edges e, e' incident with a common vertex of X_1, either both or neither of e and e' are in z. On the other hand, the cycle space of Y consists of all subsets of the set of edges. Thus $Z_{(X,E)}$ has infinite index in $Z_{(Y,E)}$.

Our other example is a graph G with three vertices u, v and w so that the pairs of vertices $\{u, v\}$ and $\{v, w\}$ are each joined by a countable infinity of parallel edges. We use the classical topology, so that G is compact. In this case, Z_G consists of all subsets of edges. If we identify u with w to get (Y, E), then $Z_{(Y,E)}$ is still all subsets of E. Thus some condition, such as weakly Hausdorff, is required in Theorem 1.4 in order to grow the cycle space after the identification.

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3. To this end, let $z \in Z_{(Y,E)}$, let T_Y be the edge set of a spanning tree of (Y, E) and let T_X be the edge set of a spanning tree of (X, E) so that $T_Y \subseteq T_X$.

The fundamental cycles $C_{T_Y}(e)$, with respect to T_Y, generate the cycle space $Z_{(Y,E)}$. Thus, for $z \in Z_{(Y,E)}$, we have $z = \sum_{e \in z \setminus T_Y} C_{T_Y}(e)$. We would like to express this in terms of the fundamental cycles of T_X. Consider $e \in E \setminus T_Y$. Then either $e \in T_X \setminus T_Y$ or $e \in E \setminus T_X$. We partition $E \setminus T_X$ into E_d and E_s: E_d consists of those $e \in E \setminus T_X$

having its ends in distinct components of T_Y, while E_s consists of those $e \in E \setminus T_X$ having both ends in the same component of T_Y.

We claim that, for $e \in E_s$, $C_{T_X}(e) = C_{T_Y}(e)$. To see this, note that $C_{T_X}(e) \in Z_{(X,E)}$, so that Lemma 1.2 implies $C_{T_X}(e) \in Z_{(Y,E)}$. Therefore, Lemma 1.1 implies $C_{T_X}(e) = \sum_{e' \in C_{T_X}(e) \setminus T_Y} C_{T_Y}(e')$. But, for example by Lemma 10 in [6], $C_{T_X}(e) \subseteq T_Y \cup \{e\}$, so $C_{T_X}(e) \subseteq T_Y \cup \{e\}$, and, therefore, $C_{T_X}(e) \setminus T_Y$ contains only $\{e\}$. Thus, $C_{T_X}(e) = C_{T_Y}(e)$, as required.

For $e \in E_d$, we have that $C_{T_X}(e)$ contains edges in $T_X \setminus T_Y$, so that $C_{T_X}(e) = \sum_{e' \in C_{T_X}(e) \setminus T_Y} C_{T_Y}(e')$, and the sum has more than one summand. So we have

$$z = \sum_{e \in z \setminus T_Y} C_{T_Y}(e) = \sum_{e \in z \cap E_s} C_{T_X}(e) + \sum_{e \in z \cap E_d} C_{T_Y}(e) + \sum_{e \in z \cap (T_X \setminus T_Y)} C_{T_Y}(e)\,.$$

We analyze $\sum_{e \in z \cap E_d} C_{T_Y}(e)$.

We saw above that, for $e \in E_d$,

$$C_{T_X}(e) = \sum_{e' \in C_{T_X}(e) \setminus T_Y} C_{T_Y}(e') = C_{T_Y}(e) + \sum_{e' \in T_X \cap C_{T_X}(e) \setminus T_Y} C_{T_Y}(e')\,.$$

Rearranging,

$$C_{T_Y}(e) = C_{T_X}(e) + \sum_{e' \in T_X \cap C_{T_X}(e) \setminus T_Y} C_{T_Y}(e')\,.$$

Summing over $e \in z \cap E_d$ yields

$$\sum_{e \in z \cap E_d} C_{T_Y}(e) = \sum_{e \in z \cap E_d} C_{T_X}(e) + \sum_{e \in z \cap E_d} \left(\sum_{e' \in T_X \cap C_{T_X}(e) \setminus T_Y} C_{T_Y}(e') \right)\,.$$

We claim that $\sum_{e \in z \cap E_d} \left(\sum_{e' \in T_X \cap C_{T_X}(e) \setminus T_Y} C_{T_Y}(e') \right)$ is defined. To see this, note that Lemma 1.1 implies $\{C_{T_X}(e) \mid e \in E \setminus T_X\}$ is thin, so each e' occurs in only finitely many of the $T_X \cap C_{T_X}(e)$, for $e \in z \cap E_d$. Thus, each $C_{T_Y}(e')$ occurs only finitely often and, since $\{C_{T_Y}(e') \mid e' \in E \setminus T_Y\}$ is thin, every edge occurs in only finitely many terms in

$$\sum_{e \in z \cap E_d} \left(\sum_{e' \in T_X \cap C_{T_X}(e) \setminus T_Y} C_{T_Y}(e') \right)\,.$$

It follows that

$$\begin{aligned} z &= \sum_{e\in z\cap E_s} C_{T_X}(e) + \sum_{e\in z\cap T_X\setminus T_Y} C_{T_Y}(e) + \sum_{e\in z\cap E_d} C_{T_Y}(e) \\ &= \sum_{e\in z\cap E_s} C_{T_X}(e) + \sum_{e\in z\cap T_X\setminus T_Y} C_{T_Y}(e) + \sum_{e\in z\cap E_d} C_{T_X}(e) \\ &\qquad + \sum_{e\in z\cap E_d}\left(\sum_{e'\in T_X\cap C_{T_X}(e)\setminus T_Y} C_{T_Y}(e')\right). \end{aligned}$$

We comment that all four sums in the last equation are over thin families, so the entire expression is over a thin family. We note that the first and third of these sums yield elements of $Z_{(X,E)}$, while the other two sums are linear combinations of the cycles $C_{T_Y}(e)$, for $e \in T_X \setminus T_Y$, so this last sum for z can be rewritten as

$$z = z' + \sum_{e\in T_X\setminus T_Y} \alpha_e C_{T_Y}(e)\,,$$

with $z' \in Z_{(X,E)}$ as claimed.

To prove uniqueness, suppose $z'' \in Z(X,E)$ and $\alpha'_e \in \{0,1\}$ are such that we also have

$$z = z'' + \sum_{e\in T_X\setminus T_Y} \alpha'_e C_{T_Y}(e)\,.$$

Then

$$z' + z'' = \sum_{e\in T_X\setminus T_Y} (\alpha_e + \alpha'_e) C_{T_Y}(e)\,.$$

The left-hand side of this equation is an element of $Z(X,E)$, while the right-hand side is a subset of T_X. The only element of $Z(X,E)$ contained in a spanning tree of X is the empty set, from which we conclude that $z' = z''$. Since each element of $Z(Y,E)$ is uniquely expressible as a sum of fundamental cycles, this further implies that, for $e \in T_X \setminus T_Y$, $\alpha_e = \alpha'_e$.

4. Cycles

Before we can prove Theorem 1.4, we need to prove some general facts about cycles. The main point of this section is to prove that, in the special case of a compact, weakly Hausdorff edge space (X,E) with $X \setminus E$ totally disconnected (which holds for the Freudenthal compactification of a connected, locally finite graph, or for $\tilde{G}$ in the case G is 2-connected and no two vertices are joined by infinitely many edge-disjoint paths), the edge-set of a cycle determines the cycle and, furthermore, no cycle is contained in another cycle. These issues have been considered in various guises, such as the discussion in [4, P. 838] or [6, Thm. 35].

We begin with an easy result.

Lemma 4.1. *Let (X, E) be a compact edge space so that $X \setminus E$ is totally disconnected. Let $F \subseteq E$ and let $x \in X \setminus E$ be such that $x \notin \mathtt{Cl}(F)$. Then x is a component of $(X \setminus E) \cup F$.*

Proof. For each $y \in \mathtt{Cl}(F) \setminus F$, let U_y and W_y be open sets in X so that $(U_y \setminus E, W_y \setminus E)$ is a separation of $X \setminus E$ with $x \in U_y$ and $y \in W_y$. The sets W_y form an open cover of the compact set $\mathtt{Cl}(F)$ and, therefore, there is a finite subcover $W_{y_1}, W_{y_2}, \ldots, W_{y_k}$. Let $W = \bigcup_{i=1}^{k} W_{y_i}$ and let $Z = \bigcap_{i=1}^{k} U_{y_i}$.

We observe that $\mathtt{Cl}(F) \subseteq W$ and that $Z' = Z \setminus \mathtt{Cl}(F)$ is open. Furthermore $x \in Z'$ and (W, Z') is a separation of $(X \setminus E) \cup F$. Therefore, the component of $(X \setminus E) \cup F$ containing x is a contained in $Z' \setminus E$ and, therefore, is x. $\square$

Our next step is to show that cycles are closed. We wonder if there is a simpler proof of this fact. The reader might wish to compare this theorem with the example [6, P. 137] of a cycle, in a compact, weakly Hausdorff edge space, that is not closed and with the fact that the edges in a circle are required by Diestel and Kühn to be dense in the circle [3, p. 74 (3)] and [4, p. 838 (1)].

Lemma 4.2. *Let (X, E) be a compact, weakly Hausdorff edge space so that $X \setminus E$ is totally disconnected. If C is a cycle in (X, E), then $C = \mathtt{Cl}(C \cap E)$.*

Proof. We prove C is closed; the result then follows from Lemma 4.1. Fix $e^* \in E \cap C$ and let $Y = C \setminus e^*$. Clearly $\mathtt{Cl}(C) = \mathtt{Cl}(Y) \cup \mathtt{Cl}(e^*)$. Since, for each $e \in E \cap C$, $\mathtt{Cl}(e) \subseteq C$, it follows that C is closed if and only if Y is closed, so we prove the latter.

By way of contradiction, suppose there is an $x \in \mathtt{Cl}(Y) \setminus Y$. Let a and b be the ends of e^* and notice that, for each $e \in E \cap Y$, a and b are in distinct components K_a^e and K_b^e, respectively, of $Y \setminus e$. Set $A = \{e \in E \cap Y \mid x \in \mathtt{Cl}(K_a^e)\}$ and let $B = (E \cap Y) \setminus A$. Evidently, if $e \in B$, then $x \in \mathtt{Cl}(K_b^e)$.

We claim that $A \neq \varnothing$ and $B \neq \varnothing$. The proofs are similar, so we prove only $B \neq \varnothing$. Otherwise, for every $e \in E \cap Y$, $x \in \mathtt{Cl}(K_a^e)$. Thus, $\bigcap_{e \in E \cap Y} \mathtt{Cl}(K_a^e)$ is closed, connected, contains a and x, and contains no edge of X. This contradicts the assumptions that $X \setminus E$ is totally disconnected and $x \notin Y$.

Next, observe that, if $e \in A$ and $e' \in B$, then x is in $\mathtt{Cl}(K_a^e) \setminus \mathtt{Cl}(K_a^{e'})$, and so $e \in K_b^{e'}$. We now set $J = \bigcap_{e\in A} \mathtt{Cl}(K_a^e)$. Note that $x \in J$ and the previous sentence implies $B = J \cap E$. From an earlier remark, $x \in \cap_{e\in B}\mathtt{Cl}(K_b^e)$. It is also straightforward to see that $A = E \cap \bigcap_{e\in B} \mathtt{Cl}(K_b^e)$.

For $e \in A$ and $e' \in B$, let $K_{e,e'}$ be the component of $Y \setminus \{e, e'\}$ containing neither a nor b. Set $L = \bigcap_{e,e'} \mathtt{Cl}(K_{e,e'})$.

A main point of the proof is to show L is just x. For $e \in A$ and $e' \in B$, $x \notin \mathtt{Cl}(K_a^{e'})$, $x \in \mathtt{Cl}(K_a^e)$ and $K_a^e = K_a^{e'} \cup \{e'\} \cup K_{e,e'}$. Since $x \notin \mathtt{Cl}(K_a^{e'}) \cup \mathtt{Cl}(e')$, we deduce that $x \in \mathtt{Cl}(K_{e,e'})$. Therefore, $x \in L$.

It is easy to see that no edge of X is in L. We claim L is connected, from which it follows that L is just a single point. Firstly, for $e \in A$, set $L_e = \bigcap_{e'\in B} \mathtt{Cl}(K_{e,e'})$. This is the nested intersection of closed, connected sets all containing x and so is closed, connected, and contains x. Likewise, $L = \bigcap_{e\in A} L_e$ is closed, connected, and contains x. That is, L is just x.

We are now in a position to show that Y is not connected, which is the final contradiction. The separation we are looking for is

$$\left(\bigcup_{e\in A} K_b^e, \bigcup_{e\in B} K_a^e \right) ,$$

so we must show its two sets are open and partition Y. The disjointness is trivial: for $e \in A$ and $e' \in B$, $K_b^e \cap K_a^{e'} = \varnothing$. Since L is just x, it follows that $\bigcap_{e\in A, e'\in B} K_{e,e'} = \varnothing$, whence $Y = \left(\bigcup_{e\in A} K_b^e \right) \cup \left(\bigcup_{e\in B} K_a^e \right)$.

To see, for example, that $\bigcup_{e\in A} K_b^e$ is open in Y, note first that $K_b^e \cup e = Y \setminus K_a^e$, so $K_b^e \cup e$ is open in Y. Thus, it suffices to show that, for each $e \in A$, there is an $e' \in A$ so that $K_b^e \cup e \subseteq K_b^{e'}$. Suppose for the edge $e^+ \in A$, there is no such e'. Then the end of e^+ in $K_a^{e^+}$ is in $\bigcap_{e\in A, e'\in B} K_{e,e'}$, a contradiction. □

Theorem 4.3. *Let (X, E) be a compact, weakly Hausdorff edge space with $X \setminus E$ totally disconnected and let Y_1 and Y_2 be distinct cycles in (X, E). Then $E \cap Y_1$ is not a subset of $E \cap Y_2$.*

Proof. For $i = 1, 2$, Lemma 4.2 shows $Y_i = \mathtt{Cl}(E \cap Y_i)$. If $E \cap Y_1 \subseteq E \cap Y_2$, then there is an edge $e \in E \cap Y_2 \setminus Y_1$ (as the Y_i are distinct) and $Y_1 = \mathtt{Cl}(E \cap Y_1) \subseteq \mathtt{Cl}(E \cap Y_2) = Y_2$. If f is any edge of Y_1, with ends a and b, a and b are in distinct components of $Y_2 \setminus \{e, f\}$. But then a and b are in distinct components on $Y_1 \setminus f$, contradicting the assumption that $Y_1 \setminus f$ is connected. □

We conclude this section with two asides relating some of this material to articles such as [1, 3, 4, 5], in which the spaces under consideration have intervals for edges. Given an edge space (X, E) having $X \setminus E$ totally disconnected, the most natural topological space arising from X in which every edge is an interval is that space X' consisting of the points of $X \setminus E$ and an open interval γ_e for each $e \in E$. Each open set U in X yields the open set $(U \setminus E) \cup \{\gamma_e \mid e \in U\}$ in X'. These open sets, together with the open subintervals of the γ_e make the basic open sets for X'.

If, in addition, (X, E) is a compact, weakly Hausdorff edge space, then X' is a compact, Hausdorff space. It is now not hard to show that if C is a cycle in (X, E) and $C \cap E$ is countable, then $(C \setminus E) \cup \{\gamma_e \mid e \in C \cap E\}$ is the homeomorphic image of a circle. The cyclic sequence of edges in C tells us how to place the intervals γ_e around the circle; the fact that C is closed (Theorem 4.2) shows that the points of $C \setminus E$ go in the right places to make the map to the circle a homeomorphism. This generalizes [6, Thm. 35], without the assumption that X is metric.

The second aside is an example. Let α be the smallest uncountable ordinal. For each ordinal $\beta < \alpha$, we include the edge e_β having ends β and $\beta + 1$. We also include an edge e_α joining α and 0. The basic open sets are: the singletons e_β, $\beta \le \alpha$; for each non-limit ordinal β, $\{e_{\beta-1}, \beta, e_\beta\}$; and, for each limit ordinal $\beta \le \alpha$ and each ordinal γ, $\{\delta \mid \gamma < \delta < \beta\} \cup \{e_\delta \mid \gamma \le \delta \le \beta\}$. The resulting edge space is compact, weakly Hausdorff and is a cycle having uncountably many edges. This cannot possibly correspond to the homeomorphic image of a circle.

5. Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. To this end, let (X, E) be a compact, weakly Hausdorff edge space with $X \setminus E$ totally disconnected and let x and y be distinct points of $X \setminus E$. Let Y be the quotient space obtained from X by identifying x and y. That is, the function $f : X \to Y$ defined by $f(t) = t$ for $t \notin \{x, y\}$ and $f(x) = f(y) = \{x, y\}$ is a continuous surjection so that if $U \subseteq X$ is open in X and either $U \cap \{x, y\} = \emptyset$ or $\{x, y\} \subseteq U$, then $f(U)$ is open in Y.

Claim 1. *Let S be a subspace of X with finitely many components so that $x, y \in S$ and some component of S is disjoint from $\{x, y\}$. Then $f(S)$ is not connected.*

Proof. The proof is standard. Let C be a component of S disjoint from $\{x, y\}$. Because there are only finitely many components, there are open sets U_1, U_2 in X so that $U_1 \cap S = C$ and $U_2 \cap S = S \setminus C$. Since $x, y \notin U_1$ and $x, y \in U_2$, both $f(U_1)$ and $f(U_2)$ are open in Y and $f(S) \cap f(U_1)$ and $f(S) \cap f(U_2)$ make a separation of $f(S)$. ∎

Let $\widehat{P}$ be a minimal connected closed subspace of X containing x and y such that $P = \widehat{P} \cap E$.

Claim 2. *P is the edge set of a cycle in* (Y, E).

Proof. In X, the deletion of any edge e of P disconnects $\widehat{P}$, producing precisely two components (see [6, Lemma 8]), one containing x and the other containing y. Thus, $f(\widehat{P} \setminus \{e\})$ is connected in Y. On the other hand, if e and e' are distinct edges of P, then $\widehat{P} \setminus \{e, e'\}$ has precisely three components. One of these is disjoint from $\{x, y\}$ and so Claim 1 shows $f(\widehat{P} \setminus \{e, e'\})$ is not connected. Therefore, P is a cycle in Y, so $P \in Z_{(Y,E)}$. ∎

Claim 3. $P \notin Z_{(X,E)}$.

Proof. Suppose P is in $Z_{(X,E)}$. Then P is the edge-disjoint union of cycles $\{C_\lambda\}$ [6, Thm. 14]. By the preceding claim, P is a cycle in (Y, E) and each of the cycles C_λ is in $Z_{(Y,E)}$ and, therefore, is the edge disjoint union of cycles $\{C_{\lambda,\mu}\}$. But then each $C_{\lambda,\mu}$ is a cycle contained in P. By Theorem 4.3, there is only one $C_{\lambda,\mu}$ and, therefore, P is a cycle in (X, E). Lemma 4.2 implies $\widehat{P} = P \cap E$. But in the proof of the preceding claim we showed $\widehat{P} \setminus e$ is not connected, a contradiction. ∎

Now let T_X and T_Y be edge sets of spanning trees $\widehat{T}_X$ and $\widehat{T}_Y$ of X and Y, respectively, with $T_Y \subseteq T_X$. Because $P \in Z_{(Y,E)} \setminus Z_{(X,E)}$, Theorem 1.3 implies T_X has at least one edge that is not in T_Y. We claim it has precisely one such edge.

If T_X has edges e, f not in T_Y, then $\widehat{T}_X \setminus \{e, f\}$ has precisely three components. By Claim 1, $\widehat{T}_Y \setminus \{e, f\}$ is not connected in Y, a contradiction.

So let e be the edge in $T_X \setminus T_Y$. By Theorem 1.3, there is a $z \in Z_{(X,E)}$ so that $P = C_{T_Y}(e) + z$. Rearranging, $C_{T_Y}(e) = P + z$. Again by Theorem 1.3, if $z'' \in Z_{(Y,E)} \setminus Z_{(X,E)}$, there is a unique z' so that $z'' = C_{T_Y}(e) + z'$, whence $z'' = P + (z + z')$, completing the proof of Theorem 1.4.

Acknowledgment

We are grateful to a referee of an earlier version of this work for suggesting an alternative proof of Claim 3 in the proof of Theorem 1.4. Its incorporation led to Theorem 4.3.

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