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1 Description of the problem

Consider the system

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

with two output maps

$$y(t) = h(x(t)), \quad w(t) = g(x(t)),$$

with state $x \in \mathbb{R}$ and controls u measurable essentially bounded functions into $\mathbb{R}^m$. Assume that the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz, and that the system is forward complete. Assume that the output maps $h : \mathbb{R}^n \to \mathbb{R}^{p_y}$ and $g : \mathbb{R}^n \to \mathbb{R}^{p_w}$ are locally Lipschitz.

The Euclidean norm in a space $\mathbb{R}^k$ is denoted simply by $|\cdot|$. If z is a function defined on a real interval containing $[0,t]$, $\|z\|_{[0,t]}$ is the sup norm of the restriction of z to $[0,t]$, that is $\|z\|_{[0,t]} = \text{ ess sup } \{|z(t)| \,:\, t \in [0,t]\}$.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ (denoted $\gamma \in \mathcal{K}$) if it is continuous, positive definite, and strictly increasing; and is of class $\mathcal{K}_\infty$ if in addition it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s,t)$ decreases to zero as $t \to \infty$.

The following definitions are given for a forward complete system with two output channels as in (1). The outputs y and w are considered as error and measurement signals, respectively.

Definition We say that the system (1) is *input-measurement to error stable* (IMES) if there exist $\beta \in \mathcal{KL}$ and γ_1, $\gamma_2 \in \mathcal{K}$ so that

$$|y(t)| \leq \max\{\beta(|x(0)|, t), \gamma_1(\|w\|_{[0,t]}), \gamma_2(\|u\|_{[0,t]})\}$$

for each solution of (1), and all $t \geq 0$.

Open Problem

Find a Lyapunov characterization of the IMES property.

2 Motivation and history of the problem

The input-measurement to error stability property was introduced as a generalizaion of input to state stability (ISS). Since its introduction in [4], the ISS property has been generalized in a number of ways. One extension is to a notion of output stability – input to output stability (IOS) – in which the magnitude of an output signal is asymptotically bounded by the input. A generalization in another direction is to a detectability notion – input-output to state stability (IOSS). In this case the size of the state is asymptotically bounded by the input and output.

In these two concepts, the outputs play distinct roles. In IOS, the output is to be kept small, e.g. an error. In IOSS, the output provides information about the size of the state, e.g. a measurement. This leads one to consider a system with two output channels – an error and a measurement. The notions of IOS and IOSS can be combined to yield (IMES), a property in which the error is asymptotically bounded by the input and a measurement. This partial detectability result is a direct generalization of IOS and IOSS (and ISS).

One of the most useful results on ISS is its characterization in terms of the existence of an appropriate Lyapunov function [5]. As the IOS and IOSS properties were introduced, they too were characterized in terms of Lyapunov functions (in [8, 9] and [3, 6, 7] respectively). A Lyapunov characterization of IMES would include both of these results, as well as the original charcterization of ISS.

3 Available results

In an attempt to determine a Lyapunov characterization for IMES, one might hope to fashion a proof along the same lines as that for the IOSS characterization given in [3]. Such an attempt has been made, with preliminary results reported in [2]. In that paper, the MES property (i.e. IMES for a system with no input)is addressed. The relation between MES and a secondary property, stability in three measures (SIT) is described, and the following (discontinuous) Lyapunov characterization for SIT is given.

Definition We say that the system (1) is *measurement to error stable* (MES) if there exist $\beta \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}$ so that

$$|y(t)| \leq \max\{\beta(|x(0)|, t), \gamma_1(\|w\|_{[0,t]})\}$$

for each solution of (1), and all $t \geq 0$.

Definition Let $\rho \in \mathcal{K}$. We say that the system (1) satisfies the *stability in three measures* (SIT) property (with gain ρ) if there exists $\beta \in \mathcal{KL}$ so that for any solution of (1), if there exists $t_1 > 0$ so that $|y(t)| > \rho(|w(t)|)$ for all $t \in [0, t_1]$, then

$$|y(t)| \leq \beta(|x(0)|, t) \qquad \forall t \in [0, t_1].$$

The MES property implies the SIT property. The converse does not hold in general, but is true under additional assumptions on the system.

Definition Let $\rho \in \mathcal{K}$. We say that a lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *lower semicontinuous SIT-Lyapunov function* for system (1) with gain ρ if

- there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|), \qquad \forall \xi \text{ so that } |h(\xi)| > \rho(|g(\xi)|),$$

- there exists $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous positive definite so that for each ξ so that $|h(\xi)| > \rho(|g(\xi)|)$,

$$\zeta \cdot v \leq -\alpha_3(V(\xi)) \qquad \forall \zeta \in \partial_D V(\xi),\ \forall v \in F(\xi). \tag{2}$$

Theorem Let a system of the form (1) and a function $\rho \in \mathcal{K}$ be given. The following are equivalent.

1. The system satisfies the SIT property with gain ρ.
2. The system admits a lower semicontinuous SIT-Lyapunov function with gain ρ.
3. The system admits a lower semicontinuous exponential decay SIT-Lyapunov function with gain ρ.

Further details are available in [2] and [1].

References

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