Discussion on: “The Non-Uniform in Time Small-Gain Theorem for a Wide Class of Control Systems with Outputs”

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1 Discussion

Small-gain statements play a key role in system analysis. These results state that the interconnection of two stable systems is stable provided that the closed loop gain is less than unity. This intuitively satisfying statement can be made rigorous in a number of ways depending on the notion of stability and the family of systems under consideration. Such results were originally given in the 1960’s by Zames [6, 7], Sandberg [4] and others who addressed stability of input-output systems by the use of linear gains.

When notions of stability are generalized by allowing nonlinear gains, an extended small-gain theorem applies, provided by Mareels and Hill in [3]. Further extensions of stability concepts were supplied by Sontag [5]. His notions of input-to-state stability (ISS) and input-to-output stability (IOS) incorporate a bound on both transient and asymptotic behaviour. Jiang, Teel and Praly provided a small-gain theorem for ISS and IOS systems in [2].

The current paper by I. Karafyllis extends the results in [2] by providing a small-gain result in a very general setting. A broad definition of control system is given which encompasses continuous and discrete time systems in finite and infinite dimensional state spaces. Systems are allowed to be time-varying, and gains are not restricted to be uniform in time. The results thus provide a very satisfactory generalization of existing small-gain statements to a much broader class of systems.

The small-gain result in [2] was generalized by a different approach in [1] (reference 24 in the paper). The remainder of this note will compare and contrast these two approaches.

Karafyllis gives an abstract definition of a (time-varying) control system inspired by Kalman’s description of a topological dynamical system. This is augmented with abstract definitions of equilibrium points and forward completeness. Throughout the paper, statements are made “robustly” with respect to a class of input disturbances by defining properties to hold uniformly over all allowed disturbances. (When addressing systems defined by differential equations, such statements are sometimes made in reference to “strong” properties of corresponding differential inclusions.)

In contrast, the results in [1] are not stated with respect to any particular class of systems, but rather with respect to families of trajectories. In that paper a *trajectory* is formally defined as a quadruple $(\tau, u(\cdot), x(\cdot), y(\cdot))$ where τ is a positive extended real number, $u(\cdot)$ is an element of an arbitrary input space, and $x(\cdot)$ and $y(\cdot)$ are nonnegative real valued functions defined on $[0, \tau)$ (to be interpreted as the norm of the state and output respectively). This abstract formalism, which makes no assumptions about the interconnections between the various signals, can be used to represent the dynamics of many systems of interest, even those which are acausal. However, this concept of trajectory can only be applied to time-varing systems in a uniform manner: the definition does not allow reference to a specific initial time.

After setting the scene in terms of which systems will be addressed, the papers then turn to stability properties. Karafyllis begins by defining *non-uniform in time robust global asymptotic output stability* (RGAOS) for systems in the absence of control input. This is defined as the combination of Lagrange output stability, Lyapunov output stability and output attractivity, and is later shown (Lemma 3.4) to imply a bound on the output of the form:

$$|y(t, t_0, x_0, d)| \leq \beta(\alpha(t_0)\, |x_0|\, , t - t_0)$$

where $y(t, t_0, x_0, d)$ is the output at time t starting from x_0 at time t_0 under disturbance d, $\beta \in \mathcal{KL}$,

α is a positive continuous function, and the norms are interpreted in the appropriate spaces. (A simplified version of the notation in the paper is being used here). Extending this concept to systems with nonzero control inputs leads to the definition of *non-uniform in time Input-to-Output Stability* (IOS) which holds whenever

$$|y(t,t_0,x_0,d,u)| \leq \max\left\{\beta(\alpha(t_0)\,|x_0|\,,t-t_0),\right.$$
$$\left.\sup_{\tau\in[t_0,t]}\beta(\alpha(\tau)\rho(\gamma(\tau)\,|u(\tau)|),t-\tau)\right\}$$

is satisfied, where $y(t,t_0,x_0,d,u)$ is the output at time t starting from x_0 at time t_0 under disturbance d and control input u, $\beta \in \mathcal{KL}$, $\rho \in \mathcal{K}_\infty$ and α and γ are positive continuous functions.

Alternatively, the treatment of input to output stability given in [1] does not incorporate non-uniform in time behaviour, but does extend the standard definition of IOS [5] by defining families of trajectories as *practically input-to-output stable* (as in [2]) provided the bound

$$y(t) \leq \max\left\{\beta(x(t_0),t-t_0),\gamma_u(\|u\|_{[t_0,t]}),C\right\} \quad (1)$$

holds, where $\beta \in \mathcal{KL}$, $\gamma_u \in \mathcal{K}$, and $C \geq 0$. No norms are required on y or x since these signals represent the norm of the output and state respectively. In the paper the notation is simplified by absorbing the gain γ_u into the definition of norm on the input function space.

Karafyllis states a small-gain theorem as follows. The interconnection of two systems is defined, and this interconnection is shown to be stable (RGAOS) provided (i) each system is IOS with respect to the output of the other and (ii) the composition of the gains on the outputs is less than unity. Due to the non-uniform in time nature of the gains, this condition on the closed loop gain takes a somewhat complex form (display 3.14 in the paper).

In contrast, the main small-gain statement in [1] is made for a single "system" (i.e. family of trajectories) which is IOS with respect to its own output. That is, if

$$y(t) \leq \max\left\{\beta(x(t_0),t-t_0),\gamma_y(\|y\|_{[t_0,t]}),\right.$$
$$\left.\gamma_u(\|u\|_{[t_0,t]}),C\right\}$$

holds, and if $\gamma_y(s) < s$, then (1) holds with possibly larger gains β and γ_u and larger offset C. This result can then be applied to interconnections as discussed in the paper. Moreover, the result applies to a number of different notions of stability and detectability.

There is a clear similarity between the current paper by Karafyllis and reference [1]. Indeed, these papers have the same goal: to provide generalizations of the ISS-type small gain result of [2]. The differences arise in the choice of that generalization. In [1] the small-gain result is presented in an abstract framework which allows application to a number of systems and notions of stability. However, a major shortcoming is the inability of that framework to adequately address non-uniform in time behaviour. In contrast, Karafyllis develops results in a general setting which allows explicit description of time-varying behaviour.

The results presented in these two papers provide complementary approaches to the generalization of the ISS-type small-gain theorem of [2]. In both cases the improved results allow application of this important result to wider classes of systems, providing an aid to system analysis. Moreover, it seems that these two approaches could be successfully merged to provide a single unifying statement, each improving upon the contribution of the other.

References

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