Detectability of Nonlinear Systems under Singular Perturbations

Leonard P. Vu

Department of Applied Mathematics
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1
lpvu@math.uwaterloo.ca

Brian P. Ingalls

Department of Applied Mathematics
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1
bingalls@math.uwaterloo.ca

Abstract— This note addresses system stability under singular perturbations. Specifically, detectability for nonlinear systems is considered. We address a notion of detectability which has been introduced as a generalization of input-to-state stability. This work extends previous results in the literature by determining conditions under which this property, known as input-output-to-state stability, is preserved when a system is treated through singular perturbation.

I. Introduction

Detectability is a key aspect of the analysis of control systems. It is rare that full knowledge of the state is available from measurement, and so basic issues such as stability must be cast in the context of an output map. The notion of detectability allows characterization of those systems for which the output provides sufficient information to determine stability of the system, and identifies those systems in which it may be useful to employ dynamic feedback for stabilization.

The theory of detectability for *linear* systems is well established and provides a very elegant set of analytic results. Attempts to extend these ideas to *nonlinear* systems have met with less success. To begin with, there are a number of equivalent ways to define detectability for linear systems, many of which lead to distinct concepts when applied in a nonlinear setting. Moreover, most of these definitions generalize to a rather weak concept of detectability: detectability of the zero trajectory only. This weaker notion is typically referred to as *zero-detectability.*

One definition of zero-detectability which has generated some interest is the property of *Input-Output-to-State Stability* (IOSS) [4, 8, 11]. This notion provides a natural extension of the Input-to-State Stability (ISS) property [12] to systems with output maps. The ISS framework allows incorporation of the effects of disturbances on otherwise stable nonlinear systems [14].

Any new characterization of system behaviour will only prove useful if it can be incorporated into established analytic procedures. In particular, notions of stability are especially practical if they are preserved under approximations by system reduction. This note addresses such an issue. Specifically, the main result provides conditions under which the IOSS property is retained under an approximation through singular perturbation.

Singular perturbation is a standard technique for model reduction in which a separation of timescales is exploited [6, 7]. A system's dynamics are separated into "slow" and "fast" modes, each of which is treated separately while the other either acts instantaneously or is held constant. An overview of the stability features of singularly perturbed systems can be found in [7], see also [5, 6]. The behaviour of ISS systems under singular perturbations was addressed by Christofides and Teel [3]. Those authors showed that under suitable assumptions the ISS property is preserved (in a practical sense) provided the singular perturbation parameter is sufficiently small. That work is extended in this note, in which detectability of systems with outputs is addressed. The result is a direct generalization of that presented in [3].

II. Preliminaries

A. Notation

- $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$, and $I_{n\times n}$ denotes the $n\times n$ identity matrix.
- For a signal $u(t)$ defined on $[0,T)$, where T can be infinite: for each $\tau\in[0,T), u_\tau$ is a signal defined on $[0,T)$ given by

$$u_\tau(t)=\begin{cases} u(t) & t\in[0,\tau] \\ 0 & t\in(\tau,T)\end{cases} ;$$

  and for each $\rho\in[0,T), u^\rho$ is a signal defined on $[0,T)$ given by

$$u^\rho(t)=\begin{cases} 0 & t\in[0,\rho) \\ u(t) & t\in[\rho,T)\end{cases}$$

- For any measurable function $\theta:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$, $\|\theta\|$ denotes ess $\sup_{t\in\mathbb{R}_{\geq 0}}|\theta(t)|$, whereas $\|\theta\|_{[0,t]}$ denotes $\|\theta_t\|$.

B. Definitions

Definition 1 *A function $\gamma:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and is zero at zero. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded.*

Definition 2 *A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity. In this note it will be further assumed that for any class $\mathcal{KL}$ function $\beta(s, 0) \geq s$.*

In what follows it will be crucial to distinguish between the effect of the perturbation parameter on the system and the effect of other disturbances. To that end, we take a system with two separate input channels as our primary object of study. The following discussion applies to systems of the form

$$\begin{aligned} \dot{x} &= f(x, u_1, u_2) \\ y &= k(x) \end{aligned} \tag{1}$$

where the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^r$ and the inputs $u_1 \in \mathbb{R}^{m_1}$, $u_2 \in \mathbb{R}^{m_2}$. The dynamics $f : \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^n$ and the output map $k : \mathbb{R}^n \to \mathbb{R}^r$ are assumed to be locally Lipschitz and to satisfy $f(0, 0, u_2) = 0$ for all u_2 and $k(0) = 0$. The inputs are assumed measurable and essentially bounded. For ease of presentation it will be assumed that the system is *forward complete*, meaning for each initial condition $x(0)$ and each essentially bounded pair (u_1, u_2) the resulting state trajectory $x(t)$ is defined for all $t \geq 0$.

Definition 3 *System (1) is said to be Globally Asymptotically Stable (GAS), uniformly in (u_1, u_2) if there exists a $\mathcal{KL}$ function β so that for each $x(0) \in \mathbb{R}^n$ and each input pair (u_1, u_2),*

$$|x(t)| \leq \beta(|x(0)|, t)$$

for all $t \in [0, \infty)$.

Our central definition is the following.

Definition 4 *[11] A system of type (1) is said to be input-output-to-state stable (IOSS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2, \gamma_y \in \mathcal{K}$ such that for any initial state $x(0)$ and input pair $(u_1(\cdot), u_2(\cdot))$, the resulting state trajectory satisfies*

$$\begin{aligned} |x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\|u_1\|_{[0,t]}) \\ + \gamma_2(\|u_2\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) \end{aligned}$$

for each time $t \in [0, \infty)$.

It was shown in [8] that a system of the form (1) satisfies the IOSS property if and only if it admits an IOSS-Lyapunov function, defined as follows.

Definition 5 *A C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an IOSS-Lyapunov function for system (1) if there exist $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha$ and $\mathcal{K}$ functions $\sigma_1, \sigma_2, \sigma_3$ such that*

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)$$

and

$$\begin{aligned} \nabla V(\xi) \cdot f(\xi, \mu_1, \mu_2) \leq -\alpha(|\xi|) \\ + \sigma_1(|\mu_1|) + \sigma_2(|\mu_2|) + \sigma_3(|k(\xi)|) \end{aligned}$$

hold for all $\xi \in \mathbb{R}^n$ and all $(\mu_1, \mu_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$.

III. SINGULARLY PERTURBED SYSTEMS

In system modeling it is often the case that a mathematical description can be simplified by neglected "small" secondary effects. This process can be formalized by the use of tools from asymptotic analysis [9, 10]. When neglecting a parameter has only a direct effect on the dynamics the perturbation is referred to as *regular*. On the other hand, when the parameter appears in the model in such a way that setting it to zero introduces a singularity, this results in a *singular perturbation* and is a much more delicate procedure (see, e.g. [6]).

In the context of system (1), we introduce a perturbation parameter ε which appears as an auxiliary input (u_2). The primary input is a disturbance $\theta(\cdot)$ (appearing as input u_1). Organizing the system in the standard framework for addressing singular perturbations, we partition the state space and dynamics as follows:

$$\begin{aligned} \dot{x} &= f(x, z, \theta, \epsilon), \qquad y = k(x) \\ \epsilon\dot{z} &= g(x, z, \theta, \epsilon) \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^{n_s}$, $z \in \mathbb{R}^{n_f}$ ($n_s + n_f = n$). The assumption has been made that the output map depends only on the first n_s components of the state. The functions f, g and k are assumed to be locally Lipschitz and to satisfy $f(0, 0, 0, \varepsilon) = 0$, $g(0, 0, 0, \varepsilon) = 0$ for all ε and $k(0) = 0$.

The decomposition into two subsystems as above allows exploitation of the separation of timescales which results from ε being small.

To describe the *slow* or *reduced system* we begin by setting $\epsilon = 0$ and solving for z as a function of x. That is, setting $\epsilon = 0$ gives us:

$$0 = g(x, z_s, \theta, 0)$$

where z_s denotes a quasi-steady state for the fast variable z. The singularly perturbed system (2) is said to be in *standard form* if the algebraic equation $g(x, z_s, \theta, 0) = 0$ admits an isolated root

$$z_s = h(x, \theta) \tag{3}$$

such that h and its partial derivatives are locally Lipschitz.

Substituting z_s from (3) into (2) with $\epsilon = 0$ gives

$$\dot{x} = f(x, h(x, \theta), \theta, 0), \qquad y = k(x). \tag{4}$$

System (4) is called the *slow* or *reduced system.* This system is typically more tractable that the full system (2), and one hopes that it remains an accurate reflection of the physical system being modeled. In what follows we will address the question of when detectability of the reduced system implies detectability of the full system.

In addition to this reduced system, which is of primary interest, there is also a "fast" timescale on which the complementary subsystem acts. These dynamics can be described as follows. Let $\tau = \frac{t}{\epsilon}$ represent the fast time scale, and define $w := z - h(x, \theta)$ as a measure of the displacement of z from the position it is assumed to maintain in the reduced system. The *boundary layer* or *fast* subsystem is then described by (again with $\varepsilon = 0$)

$$\frac{\partial w}{\partial \tau} = g(x, h(x,\theta) + w, \theta, 0), \tag{5}$$

in which x and θ are constant.

Having described the system in this standard framework we are now in a position to state the main result.

IV. Main Result

There are a number of results that address stability of singular perturbed systems, beginning with the foundational work of Tikhonov (see, e.g. [5]). In particular, the maintenance of input-to-state stability under this type of perturbation was addressed by Christofides and Teel in [3]. The result which follows is a natural generalization of their work to IOSS. The technique of proof generalizes that of [3].

As mentioned in the previous section, the basic idea behind these results on singular perturbations is to provide extension of a property derived for the reduced system to the full system. Of course any such result can only be expected to hold for small values of the perturbation parameter ε. Moreover, there are typically a number of conditions on the reduced and boundary layer system which must be satisfied. In this case there are three primary assumptions:

Assumption 1 *: The perturbed system (2) is in standard form.*

Assumption 2 *: The reduced system (4) is IOSS.*

Assumption 3 *: The equilibrium $w = 0$ of the boundary layer system (5) is GAS, uniformly in $x \in \mathbb{R}^{n_s}, \theta \in \mathbb{R}^{m_1}$.*

The first assumption is required for application of the basic tools of singular perturbation theory. Assumption 2 is the main hypothesis. The last assumption ensures that the boundary layer system is sufficiently well behaved.

The main result is as follows.

Theorem 1 *: Consider the singularly perturbed system (2). Suppose Assumptions 1-3 hold and that $\theta(\cdot)$ is absolutely continuous. Define $w := z - h(x,\theta)$ and let $(\gamma_\theta, \gamma_y)$ be the Lyapunov gains on θ and y provided by Assumption 2. Then there exist functions β_x, β_w of class $\mathcal{KL}$ such that the following holds. For each pair of positive numbers (δ, d), there is an $\epsilon^* > 0$ such that if $\epsilon \in (0, \epsilon^*]$,then for all initial states $x(0), w(0)$ with $\max\{|x(0)|, |w(0)|\} \leq \delta$ the solutions of (2) satisfy*

$$|x(t)| \leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \gamma_y(\|y\|_{[0,t]}) + d \tag{6}$$

$$|w(t/\epsilon)| \leq \beta_w(|w(0)|, t/\epsilon) + d \tag{7}$$

for all $t \in [0, \infty)$ such that $\max\{\|\theta\|_{[0,t]}, \|\dot{\theta}\|_{[0,t]}, \|y\|_{[0,t]}\} \leq \delta$.

The preceding statement is rather technical but it can be interpreted easily enough, as follows. The number δ is chosen as a bound on the system data (initial conditions, input, and output). This bound can be chosen arbitrarily, and clearly the larger this bound is chosen the richer the set of trajectories which will be addressed by the Theorem. As one might expect, such an increase in the domain of the result comes at a price: the value of ε^* depends inversely on δ. Since ε^* is an upper bound on the allowed values of the perturbation parameter, it follows that enlarging the domain of trajectories has the effect of restricting the result to a tighter range of approximations.

The number d appears as a constant offset in the bounds (6) and (7). Stability conditions which appear with this type of relaxation by a constant are often referred to as *practical*, so e.g. (6) says that the full system is *practically IOSS*. This offset can be chosen arbitrarily, and clearly the smaller the value of d the stronger the result. Again, as expected, there is a trade-off involved, since ε^* is forced smaller by smaller choices of d.

A final comment on the form of the result is called for. The restriction on the initial conditions and inputs are standard for this sort of stability result. The restriction on the output is needed for the same reasons, but is in this case rather unfortunate. Specifically, it makes the resulting detectability of the full system difficult to apply when attempting to detect boundedness through output measurements (since this typically involve large values of the output). What is retained is the ability to infer local behaviour, i.e. that the system is remaining near the equilibrium. It is this feature which will be exploited in the worked example in section V.

A. Sketch of Proof

The complete proof cannot be reported here due to space constraints. It can be found in [15]. Nevertheless, a short sketch should give some idea as to the flow of the argument and the tools used.

The proof is modeled after the argument used by Christofides and Teel in [3] to prove the analogous result for ISS (which is included as a special case of Theorem 1). As in [3], the proof can be efficiently organized around three main lemmas, which are stated below.

The first preliminary result is a generalization of Lemma 3.2 in [13], which states that if a nonlinear system is internally stable (i.e. the unforced system is asymptotically stable), then it can be made ISS through scaling the input by a function of the state. That result was generalized by Christofides and Teel in [3] by considering a system with two input channels. They showed that if the system is ISS with respect to one channel, then it can be made ISS with respect to both by scaling the second input by a function of the state. Lemma 1 below is a direct generalization of that result to the IOSS case.

Lemma 1 *: Suppose that system (1) with $u_2(t) \equiv 0$ admits an IOSS Lyapunov function with gains α_x and α_y on the state and output as in Definition 5. Then there exist functions $\alpha_{u_1}, \alpha_{u_2} \in \mathcal{K}_\infty$ and a nonincreasing continuous function $b(s) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $0 < b(s) \leq 1$ and $b(s) \equiv 1$ in a neighborhood of the origin which satisfy the* *following: with* $B(x) := \delta(|x|)I_{q\times q}$ *the system*

$$\begin{aligned}\dot{x}(t) &= f(x(t), u_1(t), B(x(t))u_2(t)) \\ y(t) &= k(x(t))\end{aligned}$$

satisfies

$$\begin{aligned}&\nabla V(\xi)\cdot f(\xi,\mu_1,B(\xi)\mu_2) \le \\ &\qquad -\alpha_x(|\xi|)+\alpha_{u_1}(|\mu_1|)+\alpha_y(|k(\xi)|)+\alpha_{u_2}(|\mu_2|)\end{aligned}$$

for all $\xi \in \mathbb{R}^n$, $\mu_1 \in \mathbb{R}^{m_1}$, $\mu_2 \in \mathbb{R}^{m_2}$.

The next result again addresses a system with two inputs that is IOSS when the second input is held to zero. It is shown that an IOSS bound on the full system can be shown to hold provided that the second input is small compared to the other data (i.e. initial condition, first input, and output).

Lemma 2 *: Assume that (1) with* $u_2(t) \equiv 0$ *admits an IOSS Lyapunov function with gain* γ_y *on output (as in Definition 5). Then there exist* $\beta \in \mathcal{KL}$, $\gamma_{u_1}, \gamma_{u_2} \in \mathcal{K}$ *and a continuous nonincreasing function* $\sigma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ *such that* $\sigma(s) \le 1$ *for all* $s \ge 0$ *satisfying the following. For each* $x(0) \in \mathbb{R}^n$ *and each pair of essentially bounded inputs* $u_1(\cdot), u_2(\cdot)$*, the solution of (1) satisfies*

$$\begin{aligned}|x(t)| \;\le\;& \beta(|x(0)|,t)+\gamma_{u_1}(\|u_1\|_{[0,t]}) \\ &+\gamma_{u_2}(\|u_2\|_{[0,t]})+\gamma_y(\|y\|_{[0,t]})\end{aligned}$$

for all $t \in [0,t^*)$ *where* t^* *is the largest time for which* $\|u_2\|_{[0,t^*]} \le \sigma(\max\{|x(0)|, \|u_1\|_{[0,t^*]}, \|y\|_{[0,t^*]}\})$.

The final preliminary result allows modification of a bound on the trajectory so that the influence of the input over an initial transient can be ignored at the expense of including an additional constant. The hypothesis of the Lemma is in line with the result of Lemma 2.

Lemma 3 *Referring to (1), suppose that there exist functions* $\beta \in \mathcal{KL}, \gamma_{u_1}, \gamma_{u_2}, \gamma_y \in \mathcal{K}$ *and a continuous nonincreasing function* $\sigma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ *with* $\sigma(\cdot) \le 1$ *such that for each* $x(0) \in \mathbb{R}^n$*, and each pair of essentially bounded inputs* $u_1(\cdot)$, $u_2(\cdot)$

$$\begin{aligned}|x(t)| \;\le\;& \beta(|x(0)|,t)+\gamma_{u_1}(\|u_1\|_{[0,t]}) \\ &+\gamma_{u_2}(\|u_2\|_{[0,t]})+\gamma_y(\|y\|_{[0,t]})\end{aligned}$$

for all $t \in [0,t')$ *where* t' *is the largest time for which* $\|u_2\|_{[0,t']} \le \sigma(\max\{|x(0)|, \|u_1\|_{[0,t']}\})$.

Then for each pair of positive real numbers $(\bar{\delta}, \bar{d})$*, there exist functions* $\bar{\beta} \in \mathcal{KL}$ *and* $\bar{\gamma}_y \in \mathcal{K}$ *and a positive real number* ρ^* *such that the following holds. For each* $\rho \in (0,\rho^*]$*, if* $\max\{|x(0)|, \|u_1\|_{[0,t']}, \|u_2\|_{[0,t']}\} \le \bar{\delta}$*, then the solution of (1) satisfies*

$$\begin{aligned}|x(t)| \;\le\;& \bar{\beta}(|x(0)|,t)+\gamma_{u_1}(\|u_1^{\rho}\|_{[0,t]}) \\ &+\gamma_{u_2}(\|u_2^{\rho}\|_{[0,t]})+\bar{\gamma}_y(\|y\|_{[0,t]})+\bar{d}\end{aligned}$$

for each $t \in [0,t^*)$*, where* t^* *is the maximum time such that* $\|u_2\|_{[0,t^*]} \le \sigma(\max\{|x(0)|+\bar{d}, \|u_1\|_{[0,t^*]}\})$ *holds.*

With these results in hand, we can now sketch the outline of the proof. We consider (2) in the (x, w) coordinates. In the τ time scale ($\tau = t/\epsilon$), the w dynamics are governed by

$$\begin{aligned}\frac{\partial w}{\partial \tau} &= g(x, h(x,\theta)+w, \theta, \epsilon) \\ &\qquad - \epsilon\Big[\frac{\partial h}{\partial x} f(x, h(x,\theta)+w,\theta,\epsilon)+\frac{\partial h}{\partial \theta}\dot{\theta}\Big]\end{aligned}$$

In this timescale, x, θ and $\dot{\theta}$ are treated as constants. We begin by showing that that this system is ISS with respect to ε. We next turn our attention to the slow system (4). Assumption 2 assures us that this subsystem is IOSS. Lemma 1 is used twice to extend this IOSS property to the system with two scaled inputs: the first being a displacement of z from $h(x,\theta)$, the second the perturbation parameter ε. It can be noted that the scaling used on these inputs can be explicitly bounded for early times, since $B(x(t)) \approx B(x(0))$ for small t. This leads to a practical IOSS statement which holds for a small time interval.

Next, Lemma 2 is applied to the slow system. The conclusion is that it satisfies an IOSS bound provided that the two inputs (w and ε) are sufficiently small. Then Lemma 3 is used to rewrite this IOSS bound so that the effect of the inputs over an initial time interval can be ignored (in exchange for an offset).

Finally, the timescale on which the fast system relaxes is identified. The results from Lemma 1 are used to guarantee the IOSS bound during the transient, while the results from Lemmas 2 and 3, which allow the size of the fast transient to be ignored, are used to provide an IOSS bound from that time on. Together, these constructions provide the final statement.

V. APPLICATION

We next illustrate the main result by the analysis of a circuit containing a tunnel diode (a nonlinear element).

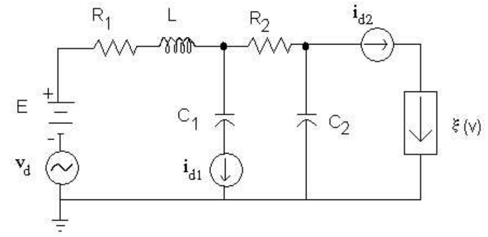

Fig. 1. Example circuit with a tunnel diode

Refer to Figure 1 (after an example in [6]). Using Kirchhoff's Current Law at nodes v_1 (atop C_2) and v_2 (atop C_1) and Kirchhoff's Voltage Law around the closed loop formed by the voltage source E and the C_1 capacitor (current i)

we get a description of the system dynamics:

$$\begin{aligned} v_2(t) &= -L\frac{di(t)}{dt} - R_1 i(t) + E + v_d(t) \\ i(t) &= C_1\frac{dv_2(t)}{dt} + \frac{v_2(t) - v_1(t)}{R_2} + i_{d1}(t) \\ \frac{v_2(t) - v_1(t)}{R_2} &= C_2\frac{dv_1(t)}{dt} + \xi(v_1(t)) + i_{d2}(t) \end{aligned}$$

where $\xi(v)$ is a function which models the i-v characteristic of a tunnel diode, as in Figure 2, and v_d, i_{d1}, and i_{d2} describe disturbances or exogenous inputs.

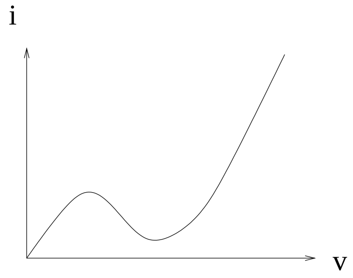

Fig. 2. Tunnel diode i-v characteristics (for positive v)

We identify an equilibrium point (v_{01}, v_{02}, i_0) and introduce a change of variables to shift it to the origin:

$$x_1 = \frac{v_1 - v_{01}}{v_{01}}, \quad x_2 = \frac{v_2 - v_{02}}{v_{02}}, \quad z = \frac{i - i_0}{i_0}.$$

The perturbation parameter, which in practice will be small, is defined as $\epsilon = \frac{L}{C_2 R_1 R_2}$. When this parameter is small the current in the loop formed by E and C_1 goes rapidly to equilibrium. Setting $a = \frac{R_1 i_0}{v_{02}}$, $k = C_2/C_1$, $b = i_0 R_2/v_{01}$ and noting that $E = v_{02} + i_0 R_1$, the above equations can be simplified to

$$\begin{aligned} \dot{x}_1 &= -x_1 - \eta(x_1) + (1+b)x_2 + \theta_1 \\ \dot{x}_2 &= k(\frac{1}{1+b})x_1 - kx_2 + k(\frac{b}{1+b})z + \theta_2 \\ \epsilon\dot{z} &= -\frac{1}{a}x_2 - z + \theta_3, \end{aligned}$$

where θ_1, θ_2 and θ_3 are the normalized disturbances, and $\eta(\cdot)$ is defined as

$$\eta(x_1) = \frac{R_2}{v_{01}}[\xi(x_1 v_{01} + v_{01}) - \xi(v_{01})].$$

In most commercial electronic devices with circuits, physical circuit board space is an extremely valuable commodity. Consequently, it is often impossible to allocate board space for voltage or current measurement points. This renders the task of circuit verification very difficult. It is advantageous to be able to determine the behaviour of some of the circuits without having to insert measurement test points at every possible location. Supposing that the circuit of Figure 1 is part of a larger electronic device and that it is only possible to have a single measurement point, we choose that value to be the voltage across the tunnel diode v_1. The resulting system is detectable, as we now show.

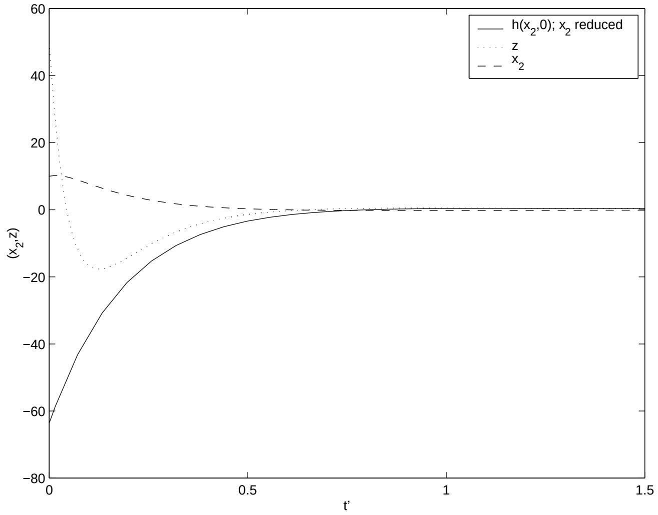

Fig. 3. z compared with the zero trajectory z_s with x_2 for comparison

Normalizing the measurement, the output can be taken as $y = x_1 = (v_1 - v_{01})/v_{01}$. Using standard perturbation techniques, we divide the system up into a slow subsystem $x = (x_1, x_2)$ and a fast subsystem through the transformation $w = z - h(x, \theta)$. It can be shown that the subsystems satisfy Assumptions 1-3 (see [15] for details). If we assume that the disturbances are absolutely continuous then, by Theorem 1, we have that there exist $\mathcal{KL}$ functions β_x, β_w and a $\mathcal{K}$ function $\bar{\gamma}_y$ such that for each pair of positive numbers (δ, d), there is an $\epsilon^* > 0$ such that if $\max\{|x(0)|, |w(0)|, \|\theta\|, \|\dot{\theta}\|\} \leq \delta$ and $\epsilon \in (0, \epsilon^*]$ then

$$\begin{aligned} |x(t)| &\leq \beta_x(|x(0)|, t) + \gamma_\theta(\|\theta\|_{[0,t]}) + \bar{\gamma}_y(\|y\|_{[0,t]}) + d \\ |w(t/\epsilon)| &\leq \beta_w(|w(0)|, t/\epsilon) + d \end{aligned}$$

for all $t \in [0, \infty)$ which satisfy $\|y\|_{[0,t]} \leq \delta$.

The result can be illustrated by choosing particular parameter values and inputs. Set $R_1 = 400\Omega, R_2 = 600\Omega, C_1 = 10\mu\text{F}, C_2 = 20\mu\text{F}$, $L = 200\text{mH}$, $E = 1\text{V}$ and suppose the tunnel diode i-v characteristics are described by

$$\xi(v) = \begin{cases} 125v^3 - 80v^2 + 15v \text{ mA} & v \geq 0 \\ 0 & v < 0 \end{cases}$$

This choice of component values gives a value of $\epsilon = 0.04$. The point $(v_{01}, v_{02}, i_0) = (0.1283\text{V}, 0.7247\text{V}, 0.6883\text{mA})$ is an equilibrium of the resulting system, which we can normalize against.

The root (there is only one in this case) for the boundary layer system is $z_s = h(x, \theta) = -\frac{1}{a}x_2 + \theta_3$. Figure 3 shows a boundary layer trajectory z converging towards its root (in the absence of disturbances). The figure also shows x_2 converging to the origin with a change of behavior around $t = 0.2$ as z starts to increase after having dipped to a local minimum.

With the above component values, the boundary layer variable w is defined by $w := z - h(x, \theta) = z + 2.63227x_2 - \theta_3$.

Applying the theorem with $\delta = 1$ and $d = 0.1$ we have that for all $\max\{|x(0)|, |w(0)|\} \leq 1$ there exists an ϵ^* such that for all $\epsilon < \epsilon^*$ the states are bounded by

$$\begin{aligned} |x(t)| &\leq |x(0)|e^{-5t} + 0.428\|\theta\|_{[0,t]} + 1.17\|y\|_{[0,t]} + 0.1 \\ |w(t/\epsilon)| &\leq \beta_w(|w(0)|, t/\epsilon) + 0.1 \end{aligned}$$

for all $t \in [0, \infty)$ which satisfy $\|y\|_{[0,t]} \leq 1$.

The proof does not provide a construction of the $\mathcal{KL}$ function $\beta_w(s,t)$. At the least, we can be sure that it dominates $se^{-t/\epsilon}$, which is the response of the boundary layer system.

Figure 4 shows some sample runs with disturbances $\theta_1 = \sin(t)$, $\theta_2 = \cos(t)$, and $\theta_3 = 0$. Parameter values are as above, so the perturbation parameter is $\varepsilon = 0.04$. The size of the state is shown together with the bound provided by the measurement output.

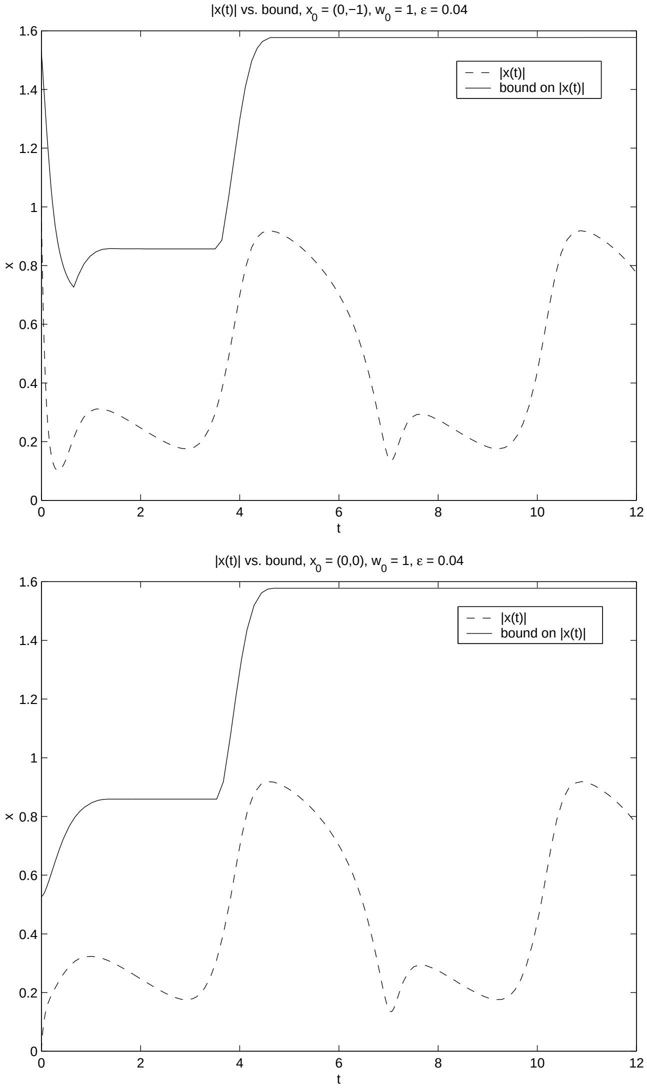

Fig. 4. Comparing the size of x trajectories with upper bound for different initial conditions

VI. Conclusion

It has been shown that the property of detectability, when characterized by the IOSS bound, is well-behaved with respect to singular perturbations. This result can be put to good use in the modeling of systems with negligible or "parasitic" dynamics. It is an assurance that detectability results derived for a reduced model will still apply to the full system (under the appropriate conditions).

There is room for improvement in this result: the construction of ε^* in the proof does not allows for a precise calculation, and so it can be difficult to estimate the accuracy of approximation which is required for the result to hold. Nevertheless, even a general existence result is useful, since it is an assurance that the system performance (in this case detectability) can be ameliorated by decreasing the effect of negligible parameters.

References

[1] D. Angeli and E.D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations," *Systems & Control Letters*, Vol. 38, pp. 209-217, 1999.

[2] D. Angeli, B. Ingalls, E. D. Sontag and Y. Wang, "Separation principles for input-output and integral-input to state stability," *SIAM J. Control Optim*, Vol. 43, pp. 256-276, 2004.

[3] P. D. Christofides and A. R. Teel, "Singular Perturbations and Input-to-State Stability," *IEEE Trans. Automatic Control*, Vol. 41, no. 11, 1996.

[4] J. Hespanha and A. Morse, "Certainty equivalence implies detectability," *Systems Control Lett.*, Vol. 36, pp. 1-13, 1999.

[5] H. K. Khalil, *Nonlinear Systems 3rd Ed.,* Prentice Hall, 2002.

[6] P. Kokotović, H. K. Khalil and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design,* Academic Press, 1986.

[7] P.V. Kokotović, R.E. O'Malley Jr. and P. Sannuti, "Singular Perturbations and Order Reduction in Control Theory," *Automatica*, Vol. 12, pp.123-132, 1976.

[8] M. Krichman, E.D. Sontag and Y. Wang, "Input-Output-to-State Stability," *SIAM J. Control Optim.*, Vol. 39, pp.1874-1928, 2001.

[9] A. H. Nayfeh, *Perturbation Methods,* Wiley-Interscience; Wiley Classics edition, 2000.

[10] J.A. Murdock, *Perturbations: Theory and Methods,* Wiley, 1991.

[11] E.D. Sontag and Y. Wang, "Output-to-state stability and detectability of nonlinear systems," *Systems & Control Letters*, Vol. 29, pp. 279-290, 1997.

[12] E.D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automatic Control*, Vol. 34, pp.435-443, 1989.

[13] E.D. Sontag, "Further facts about input to state stabilization," *IEEE Trans. Automatic Control*, Vol. 35, pp.473-476, 1990.

[14] E.D. Sontag, "The ISS philosophy as a unifying framework for stability-like behavior," in *Nonlinear Control in the Year 2000 (Volume 2)* (Lecture Notes in Control and Information Sciences, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, eds.), Springer-Verlag, Berlin, 2000, pp. 443-468.

[15] L.P. Vu, "Detectability of Singularly Perturbed Systems," Master's thesis, University of Waterloo, Waterloo, 2004