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On the Complexity of Branch and Cut Methods for the Traveling Salesman Problem

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ABSTRACT. In this note, we give an extension of the results of Chvátal, Cook, and Hartmann [2] on the complexity of the traveling salesman problem. In particular, we show that the branch and cut method has an exponential worst-case running time, even if a separation oracle for the clique-tree polytope is available and the length of the optimal hamiltonian circuit is used as an upper bound.

RESULTS

In Chvátal, Cook, and Hartmann [2] lower bounds are given on the length of cutting-plane proofs and the Chvátal rank of polytopes associated with a number of $\mathcal{NP}$-complete graph problems. Most of these are worst-case results that hold for a particular class of graphs, such as the complete graphs K_n. In fact, many of these results hold for a much larger class of graphs, as is shown in Hartmann [4]. Here we give a construction that allows many of the results concerning the traveling salesman problem to be extended to a larger class of graphs.

Let H_i be the graph with nodes a_i, b_i, c_i, d_i, e_i, f_i, g_i, and h_i and edges (a_i, b_i), (b_i, c_i), (c_i, d_i), (d_i, e_i), (e_i, f_i), (f_i, g_i), (g_i, h_i), (h_i, a_i), (b_i, f_i), and (d_i, h_i), and let $G_k = (V_k, E_k)$ be the graph formed by taking the subgraphs $H_1, \dots, H_k$ and adding the additional edges (c_i, g_{i+1}) and (e_i, a_{i+1}) for $i = 1, \dots, k-1$, (c_k, g_1) and (e_k, a_1). Let the set E_k^* consist of the edges (b_i, f_i) and (d_i, h_i) for $i = 1, \dots, k$, (c_i, g_{i+1}) and (e_i, a_{i+1}) for $i = 1, \dots, k-1$, (c_k, g_1) and (e_k, a_1). The graphs G_k are related to a class of hypohamiltonian graphs introduced by Chvátal [1], and it follows that G_k has no hamiltonian circuit that uses all of the edges in E_k^*. Graphs obtained

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from G_k by subdividing the edge (e_k, a_1) are used in both Chvátal, Cook, and Hartmann [2] and Hartmann [4]. Here we observe that all of these proofs remain valid when an arbitrary number of edges in E_k^* are subdivided, as in Figure 1. Thus the results hold for any graph that contains a subgraph isomorphic to one of these subdivided graphs.

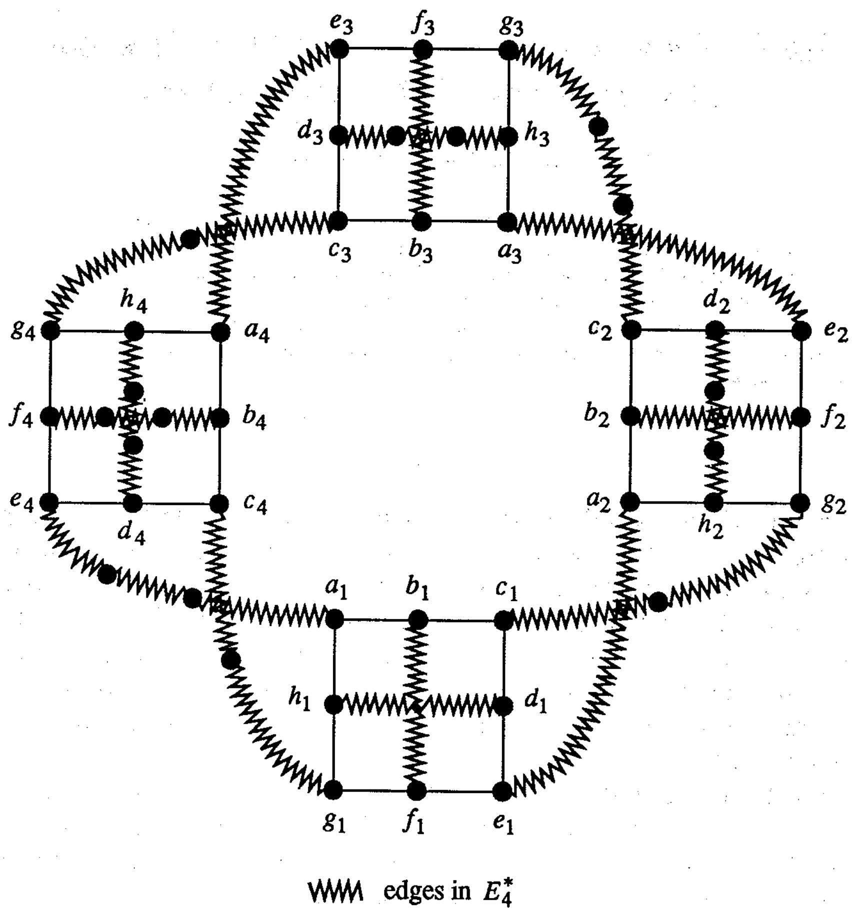

FIGURE 1. A subdivision of G_4.

Padberg and Rinaldi [5] present a branch and cut method for solving the traveling salesman problem on the complete graph K_n. After optimizing over the subtour polytope,

(1) $x(\delta(S)) \geq 2$ for all $S \subseteq V(K_n)$, $2 \leq |S| \leq n-2$,

(2) $x(\delta(v)) = 2$ for all $v \in V(K_n)$,

(3) $x_e \geq 0$ for all $e \in E(K_n)$,

they heuristically generate comb constraints (see Grötschel and Padberg [3]) that are violated by the linear programming optimal solution and solve the resulting linear program. This process is continued until an integral optimal solution is reached (the incidence vector of an optimal hamiltonian circuit) or no violated comb constraints can be identified. If an integral optimal solution is not found, then they choose one of the variables x_e to branch on, and the constraints $x_e \geq 1$ and $x_e \leq 0$ are enforced on the "up-branch" and the "down-branch," respectively. The resulting linear program is then solved for each of the branches, and violated comb inequalities are generated until further branches are required. Thus at each point the cutting planes generated are comb inequalities that are facets of the traveling salesman polytope. These comb inequalities can be obtained from (1)–(3) by taking a single nonnegative linear combination and rounding down the right-hand side. Therefore an exponential lower bound on the worst-case running time of Padberg and Rinaldi's branch and cut method follows from the theorem below.

THEOREM 1. *There are $\{0, 1, 2\}$-valued distances for which the branch and cut method requires $\Omega(2^{n/2}/n^2)$ operations to solve the traveling salesman problem for K_{8n}, even if a separation oracle for the subtour polytope is available, integer rounding is allowed, and the length of the optimal hamiltonian circuit is used as an upper bound.*

PROOF. When applied to a vector d of distances, the branch and cut method can be viewed as a proof of the validity of the inequality $d^{\mathrm{T}}x \geq l$, where l is the length of the optimal hamiltonian circuit. Such a "branch and cut" proof has the tree structure shown in Figure 2. At each node, a series of cutting planes are generated by sequentially rounding down the right-hand sides of valid inequalities with integral coefficients.

Consider a branch and cut proof of the inequality

$$2x(E\backslash E_n) + x(E_n\backslash E_n^*) \geq 4n + 1,$$

which is equivalent to the inequality

$$x(E_n) + x(E_n^*) \leq 12n - 1, \tag{4}$$

which states that G_n has no hamiltonian circuit that uses all of the edges in E_n^*. First we will argue that the branch and cut proof can be made to have a very special form without increasing the length of the proof too much. If at some point the proof branches on an edge $e \notin E_n$, then the "up-branch" in which $x_e \geq 1$ is enforced admits a one-step cutting-plane proof of the inequality (4). The same is true of the "down-branch" where $x_e \leq 0$ is enforced if $e \in E_n^*$. In either case, we will call the branch that admits the one-step cutting-plane proof the "null leaf," and the series of operations "null branching." Likewise, we can assume that there is no variable fixing per se, since this can be replaced by a branching for which either the "up-branch" or the "down-branch" admits a one-step cutting-plane proof.

If at some point the proof branches on an edge $e \in E_n\backslash E_n^*$, say (a_i, h_i), then we may sequentially branch on $(a_i, b_i), \ldots, (g_i, h_i)$ on both the "up-branch" and the "down-branch," forcing $x_{(a_i, b_i)} = x_{(c_i, d_i)} = x_{(e_i, f_i)} = x_{(g_i, h_i)}$,

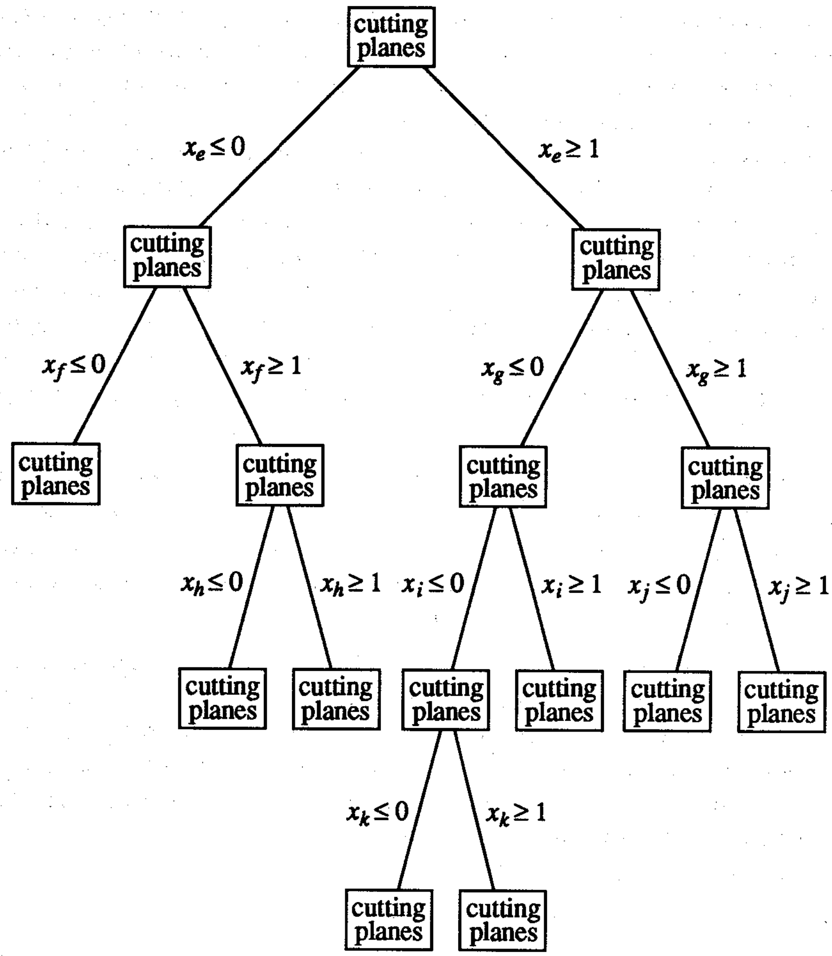

FIGURE 2. A branch and cut proof.

$x_{(b_i,c_i)} = x_{(d_i,e_i)} = x_{(f_i,g_i)} = x_{(h_i,a_i)}$, and $x_{(a_i,b_i)} = 1 - x_{(a_i,h_i)}$ to hold on both branches, since all of the other possibilities admit a one-step cutting-plane proof of the inequality (4). For example, if we have $x_{(a_i,h_i)} \ge 1$ and $x_{(a_i,b_i)} \ge 1$, then $x_{(e_{i-1},a_i)} \le 0$, and we have one of the situations described above. If on the other hand we have $x_{(a_i,h_i)} \le 0$ and $x_{(a_i,b_i)} \le 0$, then summing these inequalities, $x_{(e_{i-1},a_i)} \le 1$ and $x(\delta(v)) = 2$ for all $v \ne a_i$, we get that $x(E_n) \le 8n-1$. We will call this series of operations "branching on H_i," and count it as a single branching.

It will suffice to show that, for a branch and cut proof modified so that the only branching done is null branching and branching on $H_1, \ldots, H_n$, either (i) the tree corresponding to the proof has at least $2^{n/2}$ leaves, or (ii) the total number of cutting planes encountered on the path from the root to one of the leaves is at least $2^{n/2}/(32n^2)$. To prove this we note that the number of cutting planes required in (ii) can only decrease if all of the cutting planes are generated at the leaves. Therefore, we may assume that the proof has the structure shown in Figure 3.

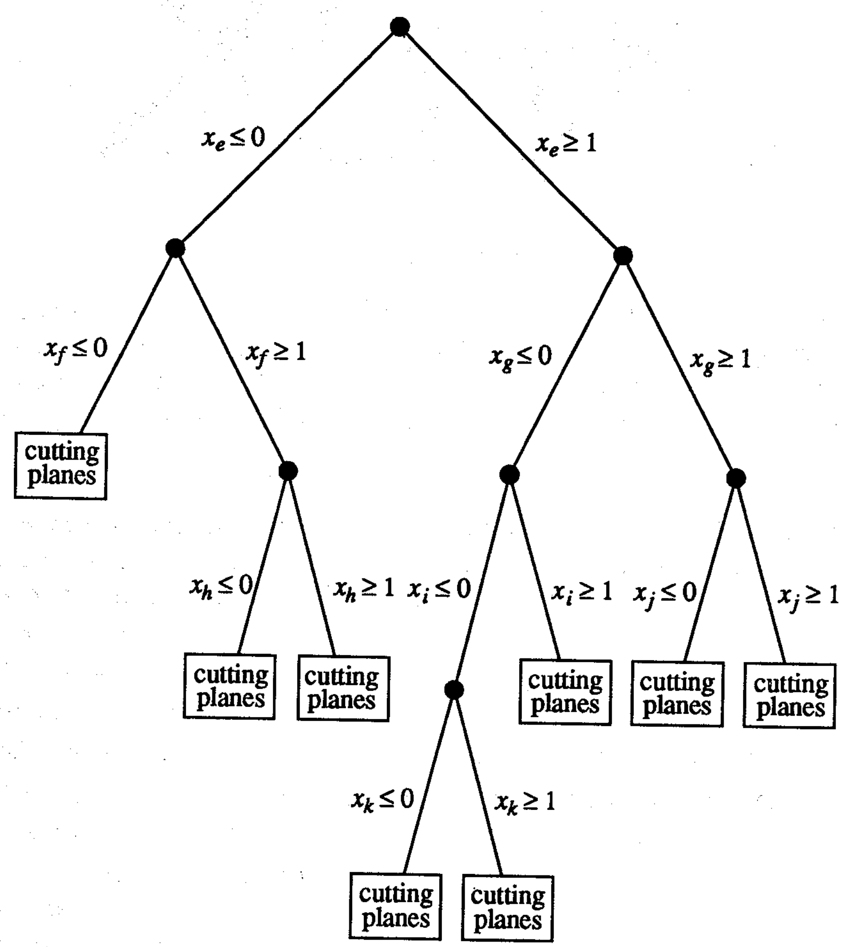

FIGURE 3. A modified branch and cut proof.

If at each *nonnull* leaf the proof has branched on at least $n/2$ of the H_i's, then the number of nonnull leaves is at least $2^{n/2}$. So we may assume that there is a nonnull leaf such that the set I of indices i for which H_i has not been branched on satisfies $|I| \geq n/2$. Observe that branching on H_i identifies a subdivision of a graph isomorphic to G_k for some $k < n$ (this is illustrated in Figure 4). This will allow us to obtain a lower bound on the length of the cutting-plane proof at this nonnull leaf in a manner similar to that of Theorem 8.3 of Chvátal, Cook, and Hartmann **[2]**.

Now for any $J \subseteq I$, the subgraph consisting of E_n^*, those edges fixed by branching on H_i for $i \notin I$, the edges (a_i, b_i), (c_i, d_i), (e_i, f_i), (g_i, h_i) for $i \in J$, and the edges (b_i, c_i), (d_i, e_i), (f_i, g_i), (h_i, a_i) for $i \in I \backslash J$,

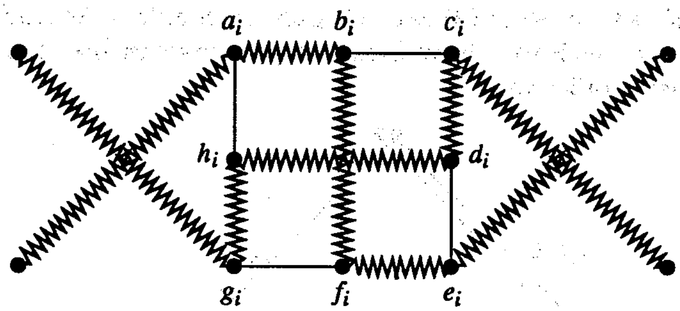

FIGURE 4. The result of branching on H_i.

consists of two circuits of length $4n$. Therefore the incidence vector of this subgraph satisfies (2)–(3) and all but one of the subtour elimination constraints (1). The number of such "necessary" subtour elimination constraints will be $2^{|I|} \geq 2^{n/2}$. Consider a cutting-plane proof of (4) from the system

$$a_i^{\mathrm{T}} x \leq b_i \quad \text{for } i = 1, \ldots, m$$

consisting of (1)–(3) and the inequalities enforced by the branchings on the path to the nonnull leaf. This proof will consist of additional inequalities $a_{m+1}^{\mathrm{T}} x \leq b_{m+1}, \ldots, a_{m+M}^{\mathrm{T}} x \leq b_{m+M}$ with integral coefficients and nonnegative numbers y_{ij} for $i = m+1, \ldots, m+M$ and $j = 1, \ldots, i-1$, such that $a_i = \sum_{j=1}^{i-1} y_{ij} a_j$ and $b_i \geq \lfloor \sum_{j=1}^{i-1} y_{ij} b_j \rfloor$ for $i = m+1, \ldots, m+M$. (The last inequality will be $x(E_n) + x(E_n^*) \leq 12n - 1$.) We have shown that at least $2^{n/2}$ of the y_{ij}'s must be positive. On the other hand, Carathéodory's theorem allows us to assume that for each i, at most $\binom{8n}{2} + 1 \leq 32n^2$ of the y_{ij}'s are positive. Comparing these two bounds we see that the cutting-plane proof at this nonnull leaf must have length at least $2^{n/2}/(32n^2)$. □

Padberg and Rinaldi also describe an improved version of this branch and cut method that generates violated clique tree inequalities (see Grötschel and Padberg [3]). Using this improved algorithm, they are able to solve large real-world (Euclidean) traveling salesman problems to optimality. However, the theorem below shows that this more powerful branch and cut method still behaves exponentially in the worst case.

THEOREM 2. *There are* $\{0, 1, 2\}$*-valued distances for which the branch and cut method requires* $\Omega(2^{n/9}/n^2)$ *operations to solve the traveling salesman problem for* K_{8n}, *even if a separation oracle for the clique-tree polytope is available, integer rounding is allowed, and the length of the optimal hamiltonian circuit is used as an upper bound.*

PROOF. We will use the same vector d of distances as in the proof of Theorem 1, and argue that for a modified branch and cut proof of $d^{\mathrm{T}} x \geq l$, either (i) the tree corresponding to the proof has at least $2^{n/9}$ leaves, (ii) the total number of clique-tree inequalities generated on the path from the root to one of the leaves is at least $2^{n/9}/(8n)$, or (iii) the total number of cutting planes encountered on the path from the root to one of the leaves is at least $2^{n/9}/(32n^2)$. As in the proof of Theorem 1, we may assume that there is a nonnull leaf such that the set I of indices i for which H_i has not been branched on satisfies $|I| \geq 8n/9$. We may also assume that the number of clique-tree inequalities generated on the path from the root to this nonnull leaf is at most $2^{n/9}/(8n)$. We now make use of claim (8.13) of Chvátal, Cook, and Hartmann **[2]**, which states that every clique-tree inequality admits a cutting-plane proof from (1)–(3) that uses at most $8n2^{2n/3}$ of the $2^{|I|}$ "necessary" subtour elimination constraints. Therefore the cutting-plane proof at the nonnull leaf must use at least $2^{|I|-7n/9} \geq 2^{n/9}$ of the "necessary" subtour elimination constraints, and the argument used in the proof of Theorem 1 shows that this cutting-plane proof must have length at least $2^{n/9}/(32n^2)$. □

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