ON SOME ASPECTS OF TOTALLY DUAL INTEGRAL SYSTEMS

by

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Abstract

A system of rational linear inequalities $Ax \le b$ is defined to be a totally dual integral system if for each integer vector w for which the linear program $\min\{yb: yA = w, y \ge 0\}$ has an optimal solution, the optimum can be achieved by an integer vector y. Totally dual integral systems were introduced by A.J. Hoffman and J. Edmonds and R. Giles and are closely related to combinatorial min-max theorems. This thesis deals with general properties of totally dual integral systems and with totally dual integral systems related to matching problems. The topics studied include: 1) operations that preserve total dual integrality, 2) upper bounds on the number of nonzero variables needed in integral optimal solutions to dual linear programs associated with totally dual integral systems, 3) recognition of totally dual integral systems, and 4) minimal totally dual integral systems for matching polyhedra.

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CHAPTER I

INTRODUCTION

1. Definition of total dual integrality

There are many interesting theorems in combinatorics that equate the minimum value of some function over some set and the maximum value of some other function over another set. Indeed, some of the fundamental results in graph theory are of this nature, e.g. Menger's theorem (on disjoint paths), the König-Egerváry theorem (on matchings in bipartite graphs), and the Tutte-Berge theorem (on factors of graphs). Often such min-max theorems can be phrased as "a certain linear program and its dual each have integral optimal solutions." For example, consider the König-Egerváry theorem:

(1.1) In a bipartite graph the maximum number of pairwise disjoint edges is equal to the minimum number of nodes meeting all edges.

Letting A be the edge-node incidence matrix of a bipartite graph G (i.e. A has a row for each node of G, a column for each edge of G, and $a_{ve} = 1$ if v is an end node of e and $a_{ve} = 0$ otherwise), this result is equivalent to the fact that both sides of the linear programming duality equation

$$\max\{wx\colon Ax \le 1, x \ge 0\} = \min\{y1\colon yA \ge w, y \ge 0\} \tag{1.2}$$

can be achieved by integer vectors if $w = 1$, where 1 is the vector of all 1's (see Section 2 for conventions regarding notation). In fact, a well-known "weighted" version of the König-Egerváry theorem is equivalent to the fact that for each integer vector w both sides of (1.2) can be achieved by integer vectors.

Translating combinatorial min-max theorems into linear programming results has proven to be very useful, both in unifying known min-max theorems and in obtaining new theorems.

The first major result in the application of linear programming to combinatorial min-max theorems is due to Hoffman and Kruskal [56], who, in 1956, proved that if A is a totally unimodular matrix (i.e. the determinant of every square submatrix of A is either 0,1 or -1) then, for all integer vectors b and w such that the optima exist, both sides of the equality

$$(1.3) \qquad \max\{wx: Ax \le b\} = \min\{yb: yA = w, y \ge 0\}$$

can be achieved by integer vectors. The König-Egerváry theorem is a direct consequence of this result (see Hoffman [60] for other applications). During the late fifties and sixties, Ford and Fulkerson [62], Edmonds [65, 70], and others used the Hoffman-Kruskal result and other linear programming ideas to prove combinatorial min-max theorems where the coefficient matrix of the corresponding linear system was not necessarily totally unimodular. In each case, the min-max result was obtained by showing that for a class of linear systems each side of (1.3) can be achieved by integer vectors. Hoffman [74] was the first to see that, in fact, to prove min-max theorems it suffices to show that the minimum in (1.3) always has an integer optimal solution. Indeed, it follows from a result of Hoffman [74,82] and Edmonds and Giles [77] (see Chapter 2, Section 2), that if $Ax \le b$ is a linear system with b integral and if the minimum in (1.3) can be achieved by an integer vector for each integer vector w for which the optima exist, then the maximum in (1.3) can also be achieved by an integer vector for each integer vector w for which the optima exist. This result plays an important role in a technique established by Hoffman [74], Younger [69], Lovász [76], Robertson (see Lovász [76]), Johnson [75], Edmonds and Giles [77], and Hoffman and Schwartz [78] for proving combinatorial min-max theorems (see Chapter 2, Section 10). This technique has been used quite successfully in recent years in obtaining very general min-max results by Edmonds and Giles [77], Hoffman and Schwartz [78], Frank [79], Schrijver [82a], and others. These general results include as special cases such well known theorems as the König-Egerváry theorem, Menger's theorem, the max-flow min-cut theorem, Dilworth's theorem, Fulkerson's branching theorem, the Vidyasankar-Younger theorem on longest paths, and the Lucchesi-Younger theorem on directed cuts.

Motivated by the above results, a rational linear system $Ax \le b$ is defined to be a totally dual integral system if the minimum in (1.3) can be achieved by an integer vector for each integer vector w for which the optima exist (a linear system $\{A_1x \le b_1, A_2x \ge b_2, A_3x = b_3\}$ is called totally dual integral if $\{A_1x \le b_1, -A_2x \le -b_2, A_3x \le b_3, -A_3x \le -b_3\}$ is totally dual integral). Thus, to prove a combinatorial min-max theorem it often suffices to show that a certain linear system is totally dual integral.

This thesis deals with general properties of totally dual integral systems and with totally dual integral systems related to matching problems. A summary of the results contained in the thesis is given in Section 3, following a section on notation and background material.

2. Background material.

In this section a survey of some well known results, which will be used throughout the thesis, is presented, together with notational conventions and references to background material.

Throughout the thesis, linear systems and linear spaces are assumed to be rational unless otherwise specified. Rational spaces suffice for the applications mentioned in the thesis. Moreover, two of the key results in the theory of total dual integrality (the theorem of Hoffman and Edmonds-Giles and Hilbert's Finite Basis Theorem) do not hold for real linear systems (see Giles and Pulleyblank [79] and Mandel [81]). The n-dimensional rational space will be denoted by $\mathbb{Q}^n$.

Let $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_n)$ be vectors. The expression "$a \le b$" means $a_i \le b_i$ for $i = 1,\ldots,n$ and "ab" denotes the inner product $\Sigma\{a_i b_i: i = 1,\ldots,n\}$. When matrix equations and inequalities are expressed it is assumed that the sizes of the matrices and vectors involved are compatilbe for matrix multiplication. The expression "0" will often be used to denote the vector with the value of zero in each component. Similarly, "1" will denote the vector of all ones.

a) Linear programming

A linear programming problem (or a linear program) is the problem of maximizing or minimizing a linear (objective) function subject to a set of linear inequality constraints. For a history of linear programming see Dantzig [63].

If $Ax \leq b$ is a system of linear inequalities and w is a vector, the expression $\max\{wx: Ax \leq b\}$ is used to denote both the problem of maximizing wx subject to $Ax \leq b$ and the number wx^*, where x^* is a vector such that $Ax^* \leq b$ and $wx^* \geq w\bar{x}$ for all vectors $\bar{x}$ for which $A\bar{x} \leq b$, if such a vector x^* exists (x^* is called an <u>optimal solution</u> to the linear program). A basic result in linear programming (see Dantzig [63]) is that if $P = \{x: Ax \leq b\}$ is nonempty and there exists a number α such that $w\bar{x} \leq \alpha$ for all $\bar{x} \in P$, then the number $\max\{wx: Ax \leq b\}$ exists (i.e. there exists an optimal solution to the linear program).

With each linear program $\max\{wx: Ax \leq b\}$ there is an associated dual linear program $\min\{yb: yA = w, y \geq 0\}$. J. von Neumann (see Dantzig [63]) and Gale, Kuhn, and Tucker [51] proved the following fundamental result, known as the duality theorem.

<u>Theorem 2.1:</u> Let $Ax \leq b$ be a linear system and w a vector. If $\max\{wx: Ax \leq b\}$ exists, then $\min\{yb: yA = w, y \geq 0\}$ exists and

(2.1) $$\max\{wx: Ax \leq b\} = \min\{yb: yA = w, y \geq 0\}. \qquad \square$$

The duality theorem implies similar results for linear programs in other forms. For instance, if $\max\{wx: Ax \leq b, x \geq 0\}$ exists then

(2.2) $$\max\{wx: Ax \leq b, x \geq 0\} = \min\{yb: yA \geq w, y \geq 0\}$$

holds. It should be noted that a linear system $\{Ax \leq b, x \geq 0\}$ is totally dual integral if and only if $\min\{yb: yA \geq w, y \geq 0\}$ can be achieved by an integer vector for each integral w for which it exists.

The following result, known as the complementary slackness theorem, will be used in Chapter 2.

Theorem 2.2: Let $Ax \leq b$ be a linear system of m inequalities and let w be a vector. Suppose that x^* is a member of $\{x: Ax \leq b\}$ and that y^* is a member of $\{y: yA = w, y \geq 0\}$. The vector x^* achieves the maximum in (2.1) and the vector y^* the minimum in (2.1) if and only if for all $i = 1,\ldots,m$, $y_i^* > 0$ implies that $a_i x^* = b_i$, where $a_i x \leq b_i$ is the i'th inequality in the system $Ax \leq b$. □

For a proof of this theorem and for additional material on linear programming see Dantzig [63].

b) Polyhedra

A polyhedron is a subset of $\mathbb{Q}^n$ of the form $\{x: Ax \leq b\}$, where A is an $m \times n$ matrix and b is a vector. For an account of the theory of polyhedra see Rockafellar [79] and Stoer and Witzgall [70], and the papers Bachem and Grotschel [82] and Pulleyblank [82].

If P is a polyhedron, then a linear system $Ax \leq b$ such that $P = \{x: Ax \leq b\}$ is a defining system for P. A polyhedron P is bounded if there exist vectors ℓ and u such that $\ell \leq x \leq u$ for all x in P. A hyperplane is a polyhedron of the form $\{x: \alpha x = \beta\}$, where α is a vector and β is a scalar. The intersection of finitely many hyperplanes is an affine subspace, i.e. a polyhedron of the form $\{x: Ax = b\}$, where $Ax = b$ is a system of equations.

A finite set of vectors M is an affinely independent set if the only vector $\lambda = (\lambda_m: m \in M)$ such that $\Sigma\{\lambda_m m: m \in M\} = 0$ and $\Sigma\{\lambda_m: m \in M\} = 0$ is the zero vector. The dimension of a polyhedron P (denoted by dim P) is one less than the cardinality of a maximal set of affinely independent vectors in P. A polyhedron $P \subseteq \mathbb{Q}^n$ is of <u>full dimension</u> if $\dim P = n$.

Let P be a polyhedron defined by a linear system $Ax \leq b$. A linear inequality $\alpha x \leq \beta$ is <u>valid</u> for P if each $\bar{x} \in P$ satisfies $\alpha\bar{x} \leq \beta$. A hyperplane $H = \{x: \alpha x = \beta\}$ is a <u>supporting hyperplane</u> of P if $H \cap P \neq \emptyset$ and $\alpha x \leq \beta$ is a valid inequality for P. A <u>face</u> of P is a subset which is the intersection of P with a supporting hyperplane of P. Equivalently, a face of P is a nonempty set of the form $\{x \in P: A'x = b'\}$, where $A'x \leq b'$ is a subset of the inequalities $Ax \leq b$. A face, F, of P is called a <u>facet</u> of P if $\dim F = \dim P-1$. A valid inequality $\alpha x \leq \beta$ is <u>facet-inducing</u> for P if $\{x: \alpha x = \beta\} \cap P$ is a facet of P. An inequality $\alpha x \leq \beta$ is <u>essential</u> for P if every defining system of P must include some positive scalar multiple of $\alpha x \leq \beta$. If P is of full dimension, then there exists a minimal defining system for P that is unique up to positive scalar multiples. Moreover, if P is of full dimension then there exists a one to one correspondence between the facets of P and the essential inequalities of P (each essential inequality induces a different facet of P and each facet-inducing inequality is an essential inequality for P).

Each minimal face of P (minimal with respect to inclusion) is of the form $\{x: A'x = b'\}$, for some subsystem $A'x \leq b'$ of $Ax \leq b$ (thus, each minimal face is an affine subspace). A face of P consisting of exactly one point is called a <u>vertex</u>.

The <u>convex hull</u> of a finite set of vectors, M, is the intersection of all convex sets containing M. A well known result is that the convex hull of a finite set of vectors is a polyhedron.

If w is a vector such that $\max\{wx: x \in P\}$ exists, then the maximum is achieved by each member of some face of P. The polyhedron P is an integer polyhedron if $\max\{wx: x \in P\}$ can be achieved by an integer vector for each vector w for which the maximum exists. The following well known result follows immediately from this definition.

Lemma 2.3: A polyhedron P is an integer polyhedron if and only if each minimal face of P contains an integer point. □

A polyhedral cone is a polyhedron of the form $\{x: Ax \geq 0\}$, where A is a matrix. By a theorem of Weyl [35] and Minkowski [96], a set C is a polyhedral cone if and only if there exists a finite set of vectors $\{a_1,\ldots,a_k\}$ that generate C, i.e. $C = \{\lambda_1 a_1 + \ldots + \lambda_k a_k: \lambda_i \geq 0, i = 1,\ldots,k\}$. A polyhedral cone C is pointed if $x \in C$, $x \neq 0$ implies that $-x \notin C$. If C is a pointed polyhedral cone then there exists a vector w such that $wx > 0$ for each nonzero $x \in C$.

c) Complexity

Results in complexity theory and formal definitions of polynomial-time algorithm, NP, and co-NP can be found in S. Cook [77], Garey and Johnson [79], and Karp [72,75] (see also Edmonds [65b]). Informally, a class of objects is in the class NP if there is some information which can be supplied about an object in the class so that one can verify in an amount of time polynomially bounded by the size of the object that the object is in the class. (The size of the information must be bounded by a polynomial in the size of the object.) Similarly, a class of objects is in the class co-NP if there is some information which can be supplied about an object not in the class so that one can verify in polynomial time that the object is not in the class. A polynomial-time algorithm is one which has a running time which is bounded by some polynomial in the size of the input.

Most of the complexity results given in this thesis involve linear systems. The size of a linear system $Ax \leq b$ is the number of binary digits needed to write the system in binary notation.

Two important polynomial-time algorithms, which will be used often in Chapter 2, are the ellipsoid algorithm for linear programming and Lenstra's integer programming algorithm.

Khachian [79] showed that it is possible to solve linear programming problems in polynomial time using the ellipsoid method of Shor [70] and Yudin and Nemirowskii [76] (see Akgül [81] and Bland, Goldfarb, and Todd [81] for surveys of results on the ellipsoid method). Grötschel, Lovász, and Schrijver [81,82] later showed that a modification of this algorithm can be used to solve many combinatorial optimization problems in polynomial time.

H.W. Lenstra [81] proved that for any fixed positive integer r, there exists a polynomial-time algorithm which, for any linear system $Ax \leq b$ with A a matrix of rank r, finds an integer point in $\{x: Ax \leq b\}$ or proves that no such point exists. See Grötschel, Lovász, and Schrijver [82] and Lenstra, Lenstra and Lovász [82] for results related to this algorithm of Lenstra.

d) <u>Smith and Hermite normal forms</u>

Let A be an $m \times n$ integer matrix of rank r. Hermite [51] showed that there exists a unimodular U (an $n \times n$ integer matrix is called <u>unimodular</u> if it has determinant 1 or -1) such that

(2.3) $$AU = \begin{vmatrix} L & 0 \\ B & 0 \end{vmatrix}$$

where L is an $r \times r$ lower triangular matrix, B is an $(m-r) \times r$ matrix, and the 0's are matrices of all zeros (the matrix AU is said to be in Hermite normal form). Smith [61] proved that there exist unimodular matrices V and U such that

(2.4) $$VAU = \begin{vmatrix} D & 0 \\ 0 & 0 \end{vmatrix}$$

where D is an $r \times r$ diagonal matrix and the 0's are matrices of all zeros (VAU is in Smith normal form).

There are many applications of Smith and Hermite normal forms (see, for example, Bachem and Kannan [79]). One application that will be used in Chapter 2 is the following result, which appears as an exercise in van der Waerden [40](cf. Bachem and von Randow [79]).

Theorem 2.4: Let $Ax = b$ be a system of linear equations. There exists an integer vector $\bar{x}$ such that $A\bar{x} = b$ if and only if there does not exist a vector y such that yA is integral and yb is nonintegral. □

This theorem follows directly from the result of Smith [61] mentioned above. It will be referred to as the Smith-van der Waerden theorem.

Polynomial-time algorithms for computing the Hermite and Smith normal forms of an integer matrix were found by Kannan and Bachem [79]. These algorithms will be used at several points in Chapter 2.

Let $a_1,\ldots,a_k$ be vectors. The set $L = \{\lambda_1 a_1 + \ldots + \lambda_k a_k : \lambda_i$ is an integer for $i = 1,\ldots,k\}$ is the lattice generated by $a_1,\ldots,a_k$.

A basis for a lattice L is a set of linearly independent vectors that generate L. It follows from the result of Hermite [61] that every lattice has a basis. For results on lattices see Lekkerkerker [69].

e) Graph theory

A graph G is an ordered triple (VG, EG, Ψ_G) consisting of a nonempty finite set of nodes, a finite set EG of edges, disjoint from VG, and an incidence function Ψ_G that associates with each edge an unordered pair of distinct nodes of G (loops are not permitted). Often, an edge e such that $\Psi_G(e) = (u,v)$ will be identified by the unordered pair (u,v). For an edge e with $\Psi_G(e) = (u,v)$, u and v are called the ends of e. For graph theory terminology see Bondy and Murty [76]. Special notation, which will be used in Chapter 3 is given below.

Let G be a graph. For each $v \in VG$, $\delta_G(v)$ denotes the subset of edges of G which meet v (i.e. have v as an end node) and $N_G(v)$ denotes the subset of nodes $u \in VG\setminus\{v\}$ for which there exists an edge $(v,u) \in EG$. For each $S \subseteq VG$, $\gamma_G(S)$ denotes the subset of edges of G having both ends in S. If it is clear which graph is being considered, the "G" subscripts on Ψ_G, δ_G, N_G, and γ_G will not be used.

For each $S \subseteq VG$, $G[S]$ denotes the subgraph of G induced by S, i.e. $G[S] = (S, \gamma(S), \bar{\Psi})$, where $\bar{\Psi}$ is the restriction of Ψ to $\gamma(S)$.

If $\mathcal{S}$ is a set of subsets of VG and e is an edge of G, then $\mathcal{S}(e)$ denotes the set $\{S \in \mathcal{S}: e \in \gamma(S)\}$.

If $\mathcal{H}$ is a set of subgraphs of G and e is an edge of G, then $\mathcal{H}(e)$ denotes the set $\{H \in \mathcal{H}: e \in EH\}$.

If $x = (x_i: i \in I)$ and $S \subseteq I$, where I is a finite set, then $x(S)$ denotes the sum $\Sigma\{x_i: i \in S\}$.

If β is a number, then $\lfloor\beta\rfloor$ denotes the largest integer less than or equal to β and $\lceil\beta\rceil$ denotes the smallest integer greater than or equal to β. If $x = (x_1,\ldots,x_n)$ is a vector then $\lfloor x\rfloor$ denotes the vector $(\lfloor x_1\rfloor,\ldots,\lfloor x_n\rfloor)$ and $\lceil x\rceil$ denotes the vector $(\lceil x_1\rceil,\ldots,\lceil x_n\rceil)$.

3. Summary of results.

This thesis deals with general properties of totally dual integral systems and with totally dual integral systems related to matching problems.

The theory of totally dual integral systems is developed in Chapter 2. The presentation is meant to be as complete as possible. Hence, in sections 2 and 5, proofs of key results due to Hoffman [74,82] and Edmonds and Giles [77], Giles and Pulleyblank [79], and Schrijver [81] are presented. The proofs of these results do not differ substantially from those given in the above papers and in Schrijver [80], but it is hoped that having the material collected here with standard terminology will be of benefit to the reader. Also, it is pointed out in two places how the results may be generalized from the lattice of all integer points to arbitrary lattices.

To apply results of total dual integrality to combinatorial problems it is necessary to find totally dual integral defining systems for classes of polyhedra. In the analogous situation for integer polyhedra certain operations which may be applied to a linear system, such that if the linear system defines an integer polyhedron then the derived system does as well, have been used often. Section 6 of Chapter 2 investigates whether or not these operations also preserve total dual integrality of linear systems. Among the results in that section, it is shown that adding slack variables to linear systems with integer left hand sides, setting some inequalities to equalities (to obtain a defining system for a face of a polyhedron), and applying Fourier-Motzkin elimination to eliminate a variable with 0 and ± 1 coefficients from a linear system (to obtain a defining system for a projection of the polyhedron) all preserve total dual integrality. (The fact that setting inequalities to equalities preserves total dual integrality is also given in Schrijver [82b].) It is also shown that the only nontrivial scalar multiplication of inequalities that preserves total dual integrality is multiplication by scalars of the form $1/k$ for some positive integer k.

Suppose that $Ax \leq b$ is a totally dual integral system with b integral and that w is an integer vector such that $\max\{wx: Ax \leq b\}$ exists. In Section 7 of Chapter 2 the problem of finding integer vectors x and y that achieve the optima in the equation

$$(3.1) \qquad \max\{wx: Ax \leq b\} = \min\{yb: yA = w, y \geq 0\}$$

is considered. It is shown that there exists a polynomial-time algorithm which, for any linear system $Mx \leq d$ and any vector c for which $\max\{cx: Mx \leq d\}$ exists, either finds an integer vector which achieves the maximum or shows that $Mx \leq d$ does not define an integer polyhedron. This easy result implies that an integer vector x which achieves the maximum in (3.1) can be found in polynomial time. A result of Chandrasekaran [81] is presented which shows that for integral A an integral y which achieves the minimum in (3.1) can also be found in polynomial time

Totally dual integral systems associated with combinatorial problems are often very large with respect to the size of the combinatorial objects. Thus, the above integer programming results are often not sufficient to find, in polynomial time, combinatorial objects which achieve the maximum and minimum in a min-max equality. Grötschel, Lovász, and Schrijver [81,82] have shown in various cases that the ellipsoid algorithm can be used to overcome the difficulty of having large linear systems associated with combinatorial problems. However, unlike the case of finding integer optimal solutions to primal linear programs whose solution set is a bounded polyhedron, they were not able to derive under general conditions a polynomial-time algorithm for finding integer optimal dual solutions.

One difficulty in finding such an algorithm is the problem of expressing the integer optimal dual solution. In general, there may be exponentially many dual variables associated with a linear system for a combinatorial problem. Thus, there may be integer optimal solutions to the dual linear program that cannot be described in polynomial time. A result of Cook, Fonlupt, and Schrijver [83] presented in Section 8 resolves this difficulty for totally dual integral systems. This result is that if $Ax \leq b$ is a totally dual integral system, with A an integer matrix of rank r, such that $P = \{x: Ax \leq b\}$ is of full dimension, then for each integer vector w for which $\min\{yb: yA = w, y \geq 0\}$ exists the minimum can be achieved by an integer vector with at most $2r-1$ nonzero components (it is also shown that if P is not of full dimension then there is no upper bound on the number of nonzero dual variables needed in an integer optimal solution to $\min\{yb: yA = w, y \geq 0\}$ in terms of r). This result improves several known results when specialized to particular combinatorial problems.

A basic problem in the study of any type of mathematical object is that of recognizing the object. Two easy results on recognizing linear systems that define integer polyhedra are presented in Section 9 of Chapter 2. It is shown that the class of linear systems that define integer polyhedra is in co-NP and that for any fixed positive integer r, there exists a polynomial-time algorithm which, for any linear system $Ax \leq b$ with A a matrix of rank r, determines whether or not $Ax \leq b$ defines an integer polyhedron. Cook, Lovász, and Schrijver [82] showed that the class of linear systems $Ax \leq b$ with A integral that are totally dual integral is in co-NP and that for any fixed positive integer r, there exists a polynomial-time algorithm which, for any linear system $Ax \leq b$ with A an integer matrix of rank r, determines whether or not $Ax \leq b$ is totally dual integral. These two results are also presented in Section 9.

Schrijver [81] showed that each polyhedron of full dimension is defined by a unique minimal totally dual integral system with integer left hand sides. A result given in Section 9 of Chapter 2 is that for any fixed positive integer r, there exists a polynomial-time algorithm which, for any linear system $Ax \leq b$ with A a matrix of rank r which defines a polyhedron of full dimension and any inequality $\alpha x \leq \beta$ determines whether or not $\alpha x \leq \beta$ is in the unique minimal totally dual integral defining system with integer left hand sides for P.

A well known "uncrossing" technique for proving that combinatorially described linear systems are totally dual integral is presented in Section 10.

The results on linear systems for matching problems center around the objects known as simple b-matchings. If G is a graph and $b = (b_v: v \in VG)$ is a nonnegative integer vector, a simple b-matching of G is a subset of the edges of G such that each node v of G meets at most b_v edges in the subset. A simple b-matching of G which meets each node v in exactly b_v edges is often called a b-factor of G (b-factors have been studied by Tutte[47,52], Gallai[50], Lovász[70] and others). A description of a totally dual integral defining system for the convex hull of (the incidence vectors of) the simple b-matchings of a graph follows easily from a result of Edmonds and Johnson [70]. In Chapter 3, the unique minimal totally dual integral defining system with integer left hand sides for this convex hull is characterised. This result yields a "best possible" min-max theorem for simple b-matchings. A similar result for the slightly more general "capacitated" b-matching problem is also given. This general result is then used to prove a theorem which characterizes the unique minimal totally dual integral defining system for the convex hull of the matchings of a graph due to Cunningham and Marsh [78], and to prove a theorem which characterizes the unique minimal totally dual integral defining system for the convex hull of the b-matchings of a graph due to Cook [81] and Pulleyblank [81]. The proof technique used is simpler than those used by Cook [81] and Pulleyblank [81] in that it does not require the knowledge of a complete list of the facet-inducing inequalities for the polyhedron in question. An essential result used throughout Chapter 3 is a connection between minimal totally dual integral systems and separability. This simple result is presented in Section 2 of that chapter and is used to prove a result of R. Giles on matching separability, a result of Edmonds and Giles (see Pulleyblank [82]) on matroid polyhedra, a result of Giles [75] on matroid intersection polyhedra, and a result on b-matching separability.

In the final section of Chapter 3, the unique minimal totally dual integral system with integer left hand sides that defines the convex hull of the triangle-free 2-matchings of a graph is characterized.

CHAPTER II

TOTALLY DUAL INTEGRAL SYSTEMS

1. Introduction

Properties of totally dual integral systems are presented in this chapter. For completeness, the presentation includes proofs of key results due to Hoffman [74,82], Edmonds and Giles [77], Giles and Pulleybank [79], and Schrijver [81].

The relationship of totally dual integral systems and integer polyhedra is presented in Sections 2 and 5. This relationship provides motivation for the study of totally dual integral systems. Further motivation is provided in Sections 3 and 4, which contain some well known results on classes of totally dual integral systems. Results of Hoffman and Kruskal [56] and Fulkerson, Hoffman, and Oppenheim [74] on totally unimodular and balanced matrices are presented in Section 3, without proof. Fulkerson's anti-blocking theory is presented in Section 4, including proofs of most of the results (Fulkerson developed this theory before general totally dual integral systems were studied, so his papers do not use the terminology of total dual integrality nor do they make use of the results on general totally dual integral systems). Section 4 includes a generalization of a theorem of Fulkerson [71,72] to h-perfect graphs. Section 6 deals with the problem of determining what operations can be performed on a totally dual integral system such that the derived system is also totally dual integral. Integer programming problems related to totally dual integral systems are studied in Section 7. Section 8 contains results on the number of nonzero dual variables needed in an integer optimal solution to the dual linear program of a linear program with a totally dual integral constraint set. Section 9 deals with the problem of recognizing a totally dual integral system. In Section 10, a well known "uncrossing" technique for proving that combinatorially described linear systems are totally dual integral is presented. This technique is illustrated with several examples which make use of results on totally unimodular and balanced matrices.

A property of linear systems that is stronger than total dual integrality and which will not be dealt with here is box total dual integrality. A linear system $Ax \le b$ is called box totally dual integral if for all vectors c and d, the linear system $\{Ax \le b,\ c \le x \le d\}$ is totally dual integral. For results on box total dual integrality see Edmonds and Giles [77].

Another related concept that will not be dealt with here, except for an application to matroid theory given in Section 8, is that of integer rounding. A linear system is said to have the integer rounding property (Baum and Trotter [77], Chandrasekaran [81]) if

$$(1.1) \qquad \min\{yb: yA = w,\ y \ge 0,\ y \text{ integral}\} = \lceil \min\{yb: yA = w,\ y \ge 0\} \rceil$$

for each integer vector w for which the right hand side exists. One reason for not dealing with integer rounding is that Giles and Orlin [81] have shown that a linear system has the integer rounding property if and if only if a corresponding system is totally dual integral. Indeed, it follows from the definitions of total dual integrality and integer rounding that $Ax \le b$ has the integer rounding property if and only if $\{Ax - bx_o \le 0,\ x_0 \le 0\}$ is totally dual integral. Thus, results on recognition, integer programming, etc., for integer rounding follow from the corresponding results for total dual integrality. It should be noted that Giles and Orlin [81] also proved a converse result: Given a linear system $Ax \le b$, there exists a positive integer M such that $Ax \le b$ is totally dual integral if and only if $Ax \le Mb$ has the integer rounding property.

2. Integral Polyhedra

Motivation for the study of total dual integrality is provided by the relation of totally dual integral systems to integer polyhedra. This relationship is presented in this section and in Section 5. A good reference for the material presented here is Schrijver [80a].

The starting point for the study of integer polyhedra and total dual integrality is the Smith-van der Waerden Theorem (Theorem 2.4, Chapter 1). This theorem implies the following geometric result (cf. Schrijver [80]).

Theorem 2.1: Let K be an affine subspace of $\mathbb{Q}^n$ that does not contain integer points. Then K lies in a hyperplane that contains no integer points.

Proof: Since K is an affine subspace, $K = \{x: Ax = b\}$ for some integral A and b. By the Smith-van der Waerden Theorem, there exists a rational vector α such that $\alpha Ax = \alpha b$ has no integral solutions. Since K is contained in $\{x: \alpha Ax = \alpha b\}$, the result follows. □

It is quite easy to extend this result from the lattice of all integer points to arbitrary lattices.

Corollary 2.2: Let L be a lattice in $\mathbb{Q}^n$ and let K be an affine subspace of $\mathbb{Q}^n$. If K does not contain any points in L, then K lies in a hyperplane that contains no points in L.

Proof: Suppose that the dimension of L is n. Let $\{a_1,\ldots,a_n\}$ be a basis for L. Consider the linear transformation T given by the matrix with columns $a_1,\ldots,a_n$. A point $x \in \mathbb{Q}^n$ is an integer point if and only if $T(x)$ is a point in L. The pre-image of K, $T^{-1}(K)$, is an affine subspace of Q^n which does not contain any integer points. By Theorem 2.1, $T^{-1}(K)$ lies in a hyperplane H that contains no integer points. So K lies in the hyperplane T(H) and T(H) contains no points in L.

Now suppose that the dimension of L is d and $d < n$. Let $\{a_1,\ldots,a_d\}$ be a basis for L and let M be the linear space spanned by $a_1,\ldots,a_d$. By the above argument, the affine subspace $K \cap M$ lies in an affine subspace H of dimension d-1 that contains no points in L. The subspace H can be extended to a hyperplane that contains K but contains no points in L. □

The Smith-van der Waerden Theorem will be used to obtain the following fundamental theorem of Hoffman [74,82] and Edmonds and Giles [77] (this particular form of the theorem is due to Schrijver [80]).

Theorem 2.3: A polyhedron P is an integer polyhedron if and only if every supporting hyperplane of P contains integer points.

Proof. Suppose that P is an integer polyhedron. The intersection of P with a supporting hyperplane is a face and every face of P contains integer points, so every supporting hyperplane of P contains integer points. Conversely, suppose that every supporting hyperplane of P contains integer points. Suppose also that P contains a minimal face F that contains no integer points. Since F is an affine subspace, by Theorem 2.1 F lies in a hyperplane H that contains no integer points. Let $P = \{x: Ax \le b\}$, where A and b are integral. By adding a large positive integer to each component of the vector α in the proof of Theorem 2.1, it can be assumed that H is of the form

$\{x: \alpha A^1 x = \alpha b^1\}$ where $A^1 x \le b^1$ is a subsystem of $Ax \le b$ and $F = \{x: A^1 x = b^1\}$ and α is positive. Each point $\bar{x} \in P$ satisfies $A^1\bar{x} \le b^1$, so each point also satisfies $\alpha A^1\bar{x} \le \alpha b^1$. So H is a supporting hyperplane of P, a contradiction. □

This theorem is a special case of a result of Schrijver [80], which gives a geometric description of the work of Chvátal [75] related to the cutting plane algorithm of Gomory[63].

As in Corollary 2.2, Theorem 2.3 can be used to show that if L is a lattice of dimension n in $\mathbb{Q}^n$ and P is a polyhedron in $\mathbb{Q}^n$, then every face of P contains points in L if and only if every supporting hyperplane contains points in L. As examples, L can be taken to be the set of all half integer points in $\mathbb{Q}^n$ or the set of all even integer points in $\mathbb{Q}^n$.

Theorem 2.3 has several equivalent forms which are given in the following theorems.

Theorem 2.4: A polyhedron P is an integer polyhedron if and only if $\max\{wx: x \in P\}$ is an integer for each integral w for which the maximum exists.

Proof: If P is an integer polyhedron then $\max\{wx: x \in P\}$ is achieved by an integral x for each w for which the maximum exists so if w is integral then $\max\{wx: x \in P\}$ is an integer. Now suppose that $\max\{wx: x \in P\}$ is an integer for each integral w for which the maximum exists. Let H be a supporting hyperplane of P and let $H = \{x: \alpha x = \beta\}$ for some integral α. Now $\max\{\alpha x: x \in P\} = \beta$, so β is an integer. But since it can be assumed that the components of α are integers with greatest common divisor 1, β can be expressed as an integer combination of the components of α. So H contains an integer point. By Theorem 2.3, P is an integer polyhedron. □

Theorem 2.5: If $Ax \leq b$ is a totally dual integral system and b is integral, then $P = \{x: Ax \leq b\}$ is an integer polyhedron.

Proof: This follows from the duality theorem and Theorem 2.4. □

Theorem 2.6: If $Ax \leq b$ is a totally dual integral system and b is integral, then each side of the equation

$$\max\{wx: Ax \leq b\} = \min\{yb: yA = w, y \geq 0\}$$

can be achieved by an integer vector for each integral w for which the optima exist.

Proof: This is just an interpretation of Theorem 2.5. □

3. Totally unimodular and balanced matrices.

It is not clear from the definition of total dual integrality that nontrivial totally dual integral systems exist. Therefore, in this section, some familiar classes of totally dual integral systems will be studied. Further examples will be considered in Section 10.

Totally unimodular and balanced matrices are two important classes of matrices that give rise to totally dual integral systems. These classes can be used to build classes of totally dual integral systems which include systems arising from many different combinatorial problems - see Section 10.

A matrix A is totally unimodular if the determinant of every square submatrix of A is either $0,1$, or -1. Hoffman and Kruskal [56] showed the importance of total unimodularity in the study of integer polyhedra by proving the following theorem.

Theorem 3.1: An integer matrix A is totally unimodular if and only if for every integral b the polyhedron $\{x: Ax \le b, x \ge 0\}$ is an integer polyhedron. □

Thus, if A is totally unimodular then $Ax \le b$ is totally dual integral for every vector b.

A short proof of this theorem is given in Veinott and Dantzig [68], where the following result is also proven.

Theorem 3.2: Let A be an $m \times n$ integer matrix of rank m. The polyhedron $\{x: Ax = b, x \ge 0\}$ is an integer polyhedron for every integral b if and only if each $m \times m$ submatrix of A has determinant $0,1$, or -1.

The following corollary is a restatement of Theorem 3.2.

Corollary 3.3: Let A be an $m\times n$ matrix of rank m. The linear system $Ax \leq b$ is totally dual integral for every integral b if and only if each $m\times m$ submatrix of A has determinant $0,1$, or -1.

Proof: The system $Ax \leq b$ is totally dual integral for every integral b if and only if for every integral w the polyhedron $\{y: yA = w, y \geq 0\}$ is an integer polyhedron. □

For applications of total unimodularity see the papers of Baranyai [73], Hoffman [60,76,75], and Lovász [79].

It does not follow directly from the definition that there exists a method to determine whether or not a matrix is totally unimodular in time polynomial in the size of the matrix - there are too many square submatrices. However, a polynomial time recognition algorithm for totally unimodular matrices does follow from Seymour's deep characterization of regular matroids - see Seymour [80].

Much work has been done on finding simple sufficient conditions for total unimodularity as well as finding characterizations of total unimodularity - see Padberg [75] and Truemper [78] for surveys. A sufficient condition for total unimodularity due to Hoffman and Kruskal will be used in Chapter 3. The following well known result is due to Heller and Tompkins [56] (see also Egerváry [31]).

Lemma 3.4: If A is a 0-1 matrix whose rows can be partitioned into two sets A_1 and A_2 such that each column has at most one nonzero component in each part of the partition, then A is totally unimodular.

This result will be used to prove the following theorem of Hoffman and Kruskal [56].

Theorem 3.5: Let E be a finite set and F_1, F_2 families of subsets of E. For $i = 1,2$, suppose that if $C, D \in F_i$ then either $C \cap D = \emptyset$, $C \subseteq D$, or $D \subseteq C$. The incidence matrix A of the family $F_1 \cup F_2$ is totally unimodular.

Proof: Let M be a square submatrix of A and let A_i be the rows of A corresponding to sets in F_i $(i = 1,2)$. Let D be a minimal row of A_i ($i = 1$ or 2). Subtract D from each other row in A_i that is greater than D. Repeating this procedure, a matrix which statisfies the conditions in Lemma 3.4 is obtained. So the matrix obtained has determinant 0,1, or −1, which implies that M has determinant 0,1, or −1. □

Total unimodularity has been generalized to the notion of local unimodularity by Hoffman and Oppenheim [78]. For results on local unimodularity see Hoffman [79] and Truemper and Chandrasekaran [78].

By Lemma 3.4, the edge-node incidence matrix of a bipartite graph is totally unimodular. Berge [72] introduced balanced matrices as a generalization of these edge-node incidence matrices. A 0-1 matrix A is balanced if A contains no square submatrix of odd size which has exactly two ones in every column and exactly two ones in every row. Since a bipartite graph contains no odd cycles, it is clear that the edge-node incidence matrix of a bipartite graph is balanced. In fact, since the determinant of any minimal (with respect to taking submatrices) 0-1 matrix of odd size with exactly two ones in every row and column is two, every 0-1 totally unimodular matrix is balanced. The following example of Berge [72] shows that not all balanced matrices are totally unimodular.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} .$$

Many nice min-max theorems for bipartite graphs generalize to the context of balanced matrices - see Berge [72,73,80], Lovász [72], and the survey in Schrijver [79]. The importance of balanced matrices to total dual integrality lies in the following theorem of Fulkerson, Hoffman and Oppenheim [74].

<u>Theorem 3.6:</u> If A is a balanced matrix, then $Ax \leq 1$, $x \geq 0$ is totally dual integral and $Ax \geq 1$, $x \geq 0$ is totally dual integral.

Berge [72,80], Berge and Las Vergnas [70], Lovász [72], and others have given characterizations of balanced matrices, but, as yet, there is no known way to test whether a given matrix is balanced or not in polynomial time. Polynomial time recognition algorithms have been found for a class of balanced matrices that generalize the edge-node incidences matrices of trees (the so called totally balanced matrices), by Anstee and Farber [82], Hoffman, Kolin, and Sakarovitch [82] and Lubiw [82].

4. Anti-blocking and the perfect graph theorem.

The theory of blocking and anti-blocking polyhedra was developed by Fulkerson [68,70,71,72] and Lehman [79]. A brief account of the anti-blocking portion of this theory is presented in this section, since it contains several interesting results concerning total dual integrality which provide further motivation for the study of totally dual integral systems. These results include Fulkerson's pluperfect graph theorem and the perfect graph theorem of Lovász. The theory of blocking polyhedra will not be studied here, but several differences in this theory and anti-blocking theory will be pointed out.

Let M be a nonnegative matrix with no column of all zeros and let P be the polyhedron $\{x: Mx \leq 1,\ x \geq 0\}$. Define the anti-blocking polyhedron of P to be the polyhedron

$$A(P) = \{y: y \geq 0 \text{ and } yd \leq 1 \text{ for each } d \in P\}.$$

It is possible to give a more compact description of $A(P)$. Let $a_1,\ldots,a_k$ be the vertices of P and let A be the matrix with rows $a_1,\ldots,a_k$. Since every point in P can be expressed as a convex linear combination of $a_1,\ldots,a_k$, $A(p) = \{y: Ay \leq 1,\ y \geq 0\}$. Since every point in P is nonnegative, A is nonnegative. Furthermore, if m_{ij} is the largest element in column j of the matrix M, then a vector with $1/m_{ij}$ in the jth position and zero in every other position is a vertex of P. So A has no column of all zeros and the expression $A(A(P))$ is meaningful. Fulkerson [72] gave the following result and proof.

Theorem 4.1: If M and P are defined as above, then $A(A(P)) = P$.

Proof: It follows immediately from the definition of $A(P)$ that $P \subseteq A(A(P))$. Suppose that there exists $b \in A(A(P))$ such that $b \notin P$. Since $b \notin P$, there exists some row, m_i, of M such that $m_i b > 1$. But this is a contradiction, since $m_i \in A(p)$. □

Theorem 4.1, together with the alternative description of $A(P)$ given above, implies that if a row m_i is an essential row of M (that is, if $m_i x \leq 1$ is a facet inducing inequality of P), then m_i is a vertex of $A(P)$. It is not true that every vertex of $A(P)$ must be a row of M. However, Fulkerson [72] has shown that every vertex of $A(P)$ that is not an essential row of M can be obtained from an essential row of M by setting some components of the row to zero. Consequently, if every row of M is integral then every vertex of $A(P)$ is integral.

To prove the above mentioned result of Fulkerson, first note that Farkas' Lemma implies that a row m_i of M is an essential row of M if and only if it is not less than or equal to a convex linear combination of the other rows of M. So all that needs to be shown is the following theorem of Fulkerson [72].

Theorem 4.2: Let M and P be defined as above and let d be a vertex of P. If d is less than or equal to a convex linear combination of $f^1,\ldots,f^s \in P$, then d can be obtained from some $f^i (1 \leq i \leq s)$ by setting some components of f^i to zero.

Proof: Suppose that d is less than or equal to $c = \Sigma\{\alpha_i f^i : i = 1,\ldots,s\}$, where $f^1,\ldots,f^s \in P$, $\Sigma\{\alpha_i : i = 1,\ldots,s\} = 1$, and $\alpha_1,\ldots,\alpha_s \geq 0$. The result is trivial if $d = 0$. So it may be assumed that $d_1,\ldots,d_k$ are the positive components of d. Let $d' = (d_1,\ldots,d_k)$ and $c' = (c_1,\ldots,c_k)$.

Since d is a vertex of P, there exists a $k \times k$ submatrix E of M such that $Ex' = 1$ has d' as its unique solution. But $Ec' \leq 1$ and $d' \leq c'$. So $Ed' = Ec'$ and $d' = c'$. Thus d can be expressed as a convex linear combination of $\bar{f}^1,\ldots,\bar{f}^s$, where $\bar{f}^i$ is obtained from f^i by setting the components other than $f^i_1,\ldots,f^i_k$ to zero. Since d is a vertex, d is equal to $\bar{f}^i$ for some $i \in \{1,\ldots,s\}$ and the result follows. □

The theory of anti-blocking polyhedra is connected to total dual integrality by the following theorem of Fulkerson [72], which is known as the "Pluperfect graph theorem." If M is a nonnegative matrix with no zero column, call a matrix A an <u>anti-blocking matrix</u> of M if $A(P) = \{x: Ax \leq 1, x \leq 0\}$, where $P = \{x: Mx \leq 1, x \geq 0\}$.

<u>Theorem 4.3:</u> Let M be a 0-1 matrix having no zero column, let $P = \{x: Mx \leq 1, x \geq 0\}$, and let A be an anti-blocking matrix of M. The polyhedron P is an integer polyhedron if and only if $Mx \leq 1$, $x \geq 0$ is totally dual integral and $Ax \leq 1$, $x \geq 0$ is totally dual integral.

<u>Proof:</u> By Theorem 2.5, if $Mx \leq 1$, $x \geq 0$ is totally dual integral then P is an integer polyhedron (Fulkerson [72] gives a direct proof of this). Since M is integral, it follows from Theorem 4.1 and Theorem 4.2 that $A(P)$ is an integer polyhedron. Suppose that P is an integer polyhedron. Let A' consist of the essential rows of A. If $A'x \leq 1$, $x \geq 0$ is totally dual integral then $Ax \leq 1$, $x \geq 0$ is also totally dual integral. Now since each vertex of P is 0-1 valued, A' is a 0-1 matrix. Thus, to prove the theorem it suffices to show that the fact that P is an integral polyhedron implies that

$Mx \le 1$, $x \ge 0$ is totally dual integral. The following short proof of this is due to Baum and Trotter [82] and Korach [82].

It must be shown that for each nonnegative integral w, $\min\{1\cdot y\colon yM \ge w,\ y \ge 0\}$ can be achieved by an integral vector. The proof is by induction on

(4.1) $$z^* = \max\{wx\colon Mx \le 1,\ x \ge 0\} = \min\{1\cdot y\colon yM \ge w,\ y \ge 0\}.$$

Clearly z^* is nonnegative and integral. If $z^* = 0$, then the result is true since $y = 0$ attains the minimum in (4.1). Suppose that for all integral w such that $z^* \le k-1$ the result is true and that w is such that $z^* = k (k \ge 1)$. Since $z^* > 0$, there exists some row m_i of M such that $m_i x = 1$ for each vector x that achieves the maximum in (4.1) (by the complementary slackness theorem). Consider the linear programming duality equation

(4.2) $$v^* = \max\{(w-m_i)x\colon Mx \le 1,\ x \le 0\} = \min\{1\cdot y\colon yM \ge w-m_i,\ y \ge 0\}.$$

By the choice of m_i, $v^* = k-1$. So the minimum in (4.2) is achieved by an integral vector y'. Let $y''_i = y'_i+1$ and $y''_i = y'_j$ for all other j, where y_i is the variable corresponding to row m_i. Since $1\cdot y' = k-1$, $1\cdot y'' = k$ and y'' achieves the minimum in (4.1). □

A generalization of this theorem is given in Korach [82].

Lovász [72] proved the following result, which is closely related to Theorem 4.3.

Theorem 4.4: Let M be a 0-1 matrix with no zero column and let A

be an anti-blocking matrix of M. If $\min\{1\cdot y: yM \geq w, y \geq 0\}$ is achieved by an integral vector for each 0-1 valued w, then $Mx \leq 1, x \geq 0$ is totally dual integral and $Ax \leq 1, x \geq 0$ is totally dual integral.

This result is the "Perfect graph theorem." It can be proven using Theorem 4.3 and a duplication lemma of Lovász [72]. A proof of this theorem which does not use Theorem 4.3 is also given in Lovász [72]. Other proofs can be found in Chvátal [75] and Lovász [79].

Lovász' Perfect graph theorem is so named for the following reason. Berge [62] defined a perfect graph as a graph G such that for each induced subgraph H of G the maximum cardinality of an independent set in H is equal to the minimum cardinality of a family of cliques such that each node of G is in some clique of the family. Based on the properties of some known classes of perfect graphs, Berge [62] conjectured that if a graph G is perfect then its complement $\bar{G}$ is also perfect. Lovász' Perfect graph theorem proves this conjecture. Indeed, if M is the incidence matrix of maximal cliques of a perfect graph G, then $\min\{1\cdot y: yM \geq w, y \geq 0\}$ is achieved by an integral vector for each 0-1 valued w. Thus, the vertices of $\{x: Mx \leq 1, x \leq 0\}$ are precisely the incidence vectors of the independent sets of G. So the incidence matrix, A, of maximal cliques in $\bar{G}$ is an anti-blocking matrix of M. Since $\{x: Ax \leq 1, x \geq 0\}$ is totally dual integral, it follows that $\bar{G}$ is perfect.

So, by specializing M to clique-node incidence matrices, Theorem 4.3 and Theorem 4.4 give results on perfect graphs. In fact, Fulkerson [71,73] showed that no generality is lost by assuming that M is the clique-node incidence matrix of a graph.

Theorem 4.5: Let M be a 0-1 matrix with no zero column. If $P = \{x: Mx \le 1, x \ge 0\}$ is an integer polyhedron, then there exists a graph G such that the essential rows of M are precisely the incidence vectors of the maximal cliques of G.

Proof: Suppose that $P = \{x: Mx \le 1, x \ge 0\}$ is an integer polyhedron. Define a graph G as follows. The nodes of G correspond to the columns of M and nodes j and k are joined by an edge if there exists an essential row m_i of M such that $m_{ik} = m_{ij} = 1$. Clearly, each essential row of M is the incidence vector of a clique of G. Suppose that there is a clique C of G which is not contained in any clique that corresponds to an essential row of M. Let $\bar{x}_i = 1/(k-1)$ for each $i \in C$ and $\bar{x}_i = 0$ for each $i \in VG-C$, where k is the cardinality of C. The vector $\bar{x}$ is contained in P, since $m_i\bar{x} \le 1$ for each essential row m_i of M. Let $w_i = 1$ for each $i \in C$ and $w_i = 0$ for each $i \in VG-C$. Now $w\bar{x} = k/(k-1)$, but $wx \le 1$ for each integral point in P. So P is not an integer polyhedron, a contradiction. □

Fulkerson [71,73] gives a different proof of Theorem 4.5, based on the faithful graph representation theorem of Gilmore.

The above proof technique can be used to generalize Theorem 4.5 to h-perfect graphs. Sbihi and Uhry [81] and Fonlupt and Uhry [82] define an h-perfect graph as a graph G such that the following linear system defines an integer polyhedron:

$$\begin{aligned} &\Sigma\{x_v: v \in C\} \le 1 \quad \forall\, C \in \mathcal{C} \\ (4.3)\qquad &\Sigma\{x_v: v \in H\} \le (|H|-1)/2 \quad \forall\, H \in \mathcal{H} \\ &x_v \ge 0 \qquad \forall\, v \in VG \end{aligned}$$

where $\mathcal{C}$ is the set of maximal cliques of G and $H \subseteq V$ is a member of $\mathcal{H}$ if the subgraph G[H] induced by H is an odd cycle (i.e. H is an "odd hole"). Clearly, perfect graphs are h-perfect and if G is h-perfect then (4.3) defines the convex hull of the incidence vectors of the independent sets of G. The concept of h-perfection is motivated by a result of Boulala and Uhry [79] which shows that series-parallel graphs are h-perfect. The following generalization of Theorem 4.7 shows that if a system appears to come from an h-perfect graph then it actually does come from an h-perfect graph.

Theorem 4.6 Let $\mathcal{C}$ and $\mathcal{H}$ be families of subsets of V such that if $H \in \mathcal{H}$, then $|H| = 2k+1$ for some $k \geq 2$, $|H \cap C| \leq 2$ for each $C \in \mathcal{C}$, and the elements of H may be labeled $h_0, \ldots, h_{n-1}$ such that for each $i = 0, \ldots, n-1$ there exists $C \in \mathcal{C}$ with $h_i, h_{i+1} \in C$ (subscripts should be taken modulo n). If the linear system (4.3) defines an integer polyhedron P, then there exists a graph G such that the maximal cliques of G are the sets in $\mathcal{C}$ that define facet-inducing inequalities for P and the odd holes of G are the sets in $\mathcal{H}$ that define facet-inducing inequalities for P.

Proof: Suppose (4.3) defines an integer polyhedron P. Let G be the graph with node set V with two distinct nodes u,w joined by an edge if there is a set $C \in \mathcal{C}$ such that $u,w \in C$. As in the proof of Theorem 4.5, each maximal clique of G is a set in $\mathcal{C}$ that defines a facet-inducing inequality for P. Similarly, each odd hole of G is a set in $\mathcal{H}$ that defines a facet-inducing inequality for P. Also, each $C \in \mathcal{C}$ is a clique of G. It is straightforward to check that each set $H \in \mathcal{H}$ which defines a facet-inducing inequality for P is an odd hole of G (use the labelling $h_0, \ldots, h_{n-1}$ of H to show that

$G[H]$ contains a cycle which meets all of the nodes in H and note that if $G[H]$ contains an edge which is not in this cycle then $\Sigma\{x_v: v \in H\} \le (|H|-1)/2$ is implied by other valid inequalities for P), which completes the proof. □

Analogous to anti-blocking theory, Fulkerson developed a theory of blocking polyhedra. Blocking theory involves systems of the form $Bx \ge 1$, $x \ge 0$ where B is a nonnegative matrix. One of the main differences in blocking theory and anti-blocking theory is that the analogue of the result in Theorem 4.3 that $Mx \le 1, x \ge 0$ is totally dual integral if and only if $Ax \le 1$, $x \ge 0$ is totally dual integral does not hold in the blocking case. Nevertheless, some very interesting results involving blocking theory and total dual integrality can be found in Fulkerson, Hoffman, and Oppenheim [74] and Seymour [77].

5. Hilbert bases and totally dual integral defining systems.

Let S be a finite set of integer vectors in $\mathbb{Q}^n$. Suppose that S corresponds to some set of combinatorial objects (S could be the set of incidence vectors of matchings in a graph, for instance). Let $w \in \mathbb{Q}^n$ be an integer "weight" vector - the weight of an object $s \in S$ being ws. A well known method for finding a min-max relation for the maximum weight of an object in S is to find a linear system that defines the convex hull of S and then apply the duality theorem of linear programming (see Chapter 1). If this linear system is also totally dual integral, then the min-max relation can be strengthened by requiring the dual variables to take on integer values (often integer solutions to the dual linear program correspond to combinatorial objects, such as "coverings" or "cuts"). There always exists a linear system that defines the convex hull of S, but does there always exist a totally dual integral defining system for this polyhedron? A result of Giles and Pulleyblank [79] given below shows that the answer to this question is yes and, in fact, that this system can be taken to have integer left hand sides.

Let C be a polyhedral cone and let A be an integer matrix such that $C = \{x: Ax \leq 0\}$. Consider the linear program

(5.1) $\max\{wx: Ax \leq 0\}$,

where w is an integer vector. An optimal solution to the dual linear program of (5.1) corresponds to an expression of w as a nonnegative linear combination of $a_1,\ldots,a_m$ (the rows of A). So $Ax \leq 0$ is totally dual integral if and only if each integer vector in the cone C^* generated by $\{a_1,\ldots,a_m\}$ can be expressed as a nonnegative integral combination of $a_1,\dots,a_m$. Furthermore, if $\{b_1,\dots,b_t\}$ is any set of vectors that generates C^*, then $C = \{x: Bx \le 0\}$, where the rows of B are $b_1,\dots,b_t$. So there exists a totally dual integral defining system for C if and only if there exists a set of vectors $\{c_1,\dots,c_q\}$ that generates C^* such that each integral w in C^* can be expressed as a nonnegative integer combination of $c_1,\dots,c_q$. Hilbert [90] proved that such vectors always exist.

Theorem 5.1 (Hilbert's Finite Basis Theorem): If C is a polyhedral cone then there exists a finite set of integer vectors $\{h_1,\dots,h_k\}$ such that each integral $v \in C$ can be expressed as a nonnegative integer combination of $h_1,\dots,h_k$.

Proof: By the theorem of Weyl[35] and Minkowski[96], there exists a finite set of integer vectors, $\{d_1,\dots,d_j\}$, that generate C. Let H be the set of integer vectors in the bounded set $\{\lambda_1 d_1,+\dots+\lambda_j d_j: 0 \le \lambda_i \le 1, i = 1,\dots,j\}$. Let $v \in C$ be an integer vector. Since $\{d_1,\dots,d_j\}$ generates C, v can be expressed as $\gamma_1 d_1+\dots+\gamma_j d_j$ for some nonnegative rational γ_i, $i = 1,\dots,j$. The vector $v' = (\gamma_1-\lfloor\gamma_1\rfloor)d_1+\dots+(\gamma_j-\lfloor\gamma_j\rfloor)d_j$ is in H, so $v'+(\lfloor\gamma_1\rfloor d_1+\dots+\lfloor\gamma_j\rfloor d_j)$ expresses v as a nonnegative integer combination of vectors in H. Thus, H is the required set of vectors. □

The proof of Theorem 5.1 given above is contained in Giles and Pulleyblank [79]. Proofs of this theorem which do not make use of the Weyl-Minkowski Theorem can be found in Bachem [77] and Jeroslow [78].

In view of Theorem 5.1, a set of integer vectors $\{h_1,\ldots,h_k\}$ is called a Hilbert basis if each integer vector in the cone generated by $\{h_1,\ldots,h_k\}$ can be expressed as a nonnegative integer combination of $h_1,\ldots,h_k$ (so Theorem 5.1 implies that every polyhedral cone is generated by a finite Hilbert basis). Theorem 5.1 will now be used to prove a result of Giles and Pulleyblank [79].

Let $Ax \le b$ be a linear system with A integral. If F is a face of the polyhedron $P = \{x: Ax \le b\}$, then a row a_i of A is active for F if $a_i\bar{x} = b_i$ for each $\bar{x} \in F$ (if a_i is active for F then $a_i x \le b_i$ is an active inequality for F). Let C_F be the cone generated by the active rows of A for F. It follows from the complementary slackness theorem that C_F is the set of all those vectors w such that $\max\{wx: Ax \le b\}$ is achieved by each point in F. So C_F does not depend on the defining system for P. The complementary slackness theorem also implies the following lemma.

Lemma 5.2: If A is an integral matrix, then $Ax \le b$ is totally dual integral if and only if for each minimal face of $\{x: Ax \le b\}$ the set of active rows of A is a Hilbert basis. □

Giles and Pulleyblank [79] proved the following theorem.

Theorem 5.3: Let P be a polyhedron. There exists a totally dual integral system $Ax \le b$, with A integral, such that $P = \{x: Ax \le b\}$. Furthermore, if P is an integer polyhedron then b can be taken to be integral.

<u>Proof:</u> Let $A'x \le b'$ be a defining system for P. For each minimal face F of P let C_F denote the cone generated by the active rows of A' for F and let H_F be a Hilbert basis for C_F (i.e. H_F generates C_F). Also, for each $h \in H_F$ let $\alpha_h = h \cdot \bar{x}$ where $\bar{x}$ is some point in F (note that if x^1 and x^2 are any two points in F then $hx^1 = hx^2$). The inequality $hx \le \alpha_h$ is valid for P since it can be obtained by taking nonnegative multiples of the active inequalities in $A'x \le b'$ for F.

Let $Ax \le b$ be the system of inequalities $hx \le \alpha_h$ for each $h \in H_F$ and each minimal face F of P. Each facet-inducing inequality for P in $A'x \le b'$ can be obtained by taking nonnegative multiples of inequalities in $Ax \le b$ (since the inequality must be active for some minimal nonempty face of P). So $Ax \le b$ is a defining system for P. If F is a minimal face of P, then each $h \in H_F$ is active for F in A. So Lemma 5.2 implies that $Ax \le b$ is totally dual integral.

Finally, if P is an integer polyhedron then each minimal nonempty face F of P contains an integer point, which implies that α_h is integral for each $h \in H_F$. So if P is an integer polyhedron then b is integral. □

Consider again the finite set S of integer vectors. Suppose that a min-max relation for the maximum weight of an object in S has been obtained by finding a totally dual integral defining system, $Ax \le b$, for the convex hull of S, with A integral. As was mentioned earlier, the min-max relation can be strengthened by restricting the dual variables to integer values. The min-max relation can be further strenghtened if some of the inequalities in $Ax \leq b$ can be removed to obtain a smaller totally dual integral defining system for the convex hull of S. This process can be repeated until a minimal totally dual integral defining system for the convex hull of S is obtained, i.e. until a totally dual integral system is obtained such that if any inequality is removed from the system then it either no longer defines the convex hull of S or is no longer totally dual integral. If the convex hull of S is a polyhedron of full dimension, then a result of Schrijver [81] implies that such a minimal totally dual integral system is uniquely determined by S. So, if the convex hull of S is of full dimension, a "best possible" min-max relation for S can be obtained in this way (see Chapter 3).

The following theorem, given in Jeroslow [78] and Schrijver [81], is needed in order to prove the above mentioned result of Schrijver (the proof is taken from Schrijver [80a]).

Theorem 5.4: If C is a pointed polyhedral cone then C is generated by a unique minimal Hilbert basis.

Proof: Suppose that C is a pointed polyhedral cone (recall that C is pointed if $x \in C$ implies $-x \notin C$ for each $x \neq 0$) and that C is generated by distinct minimal Hilbert bases H_1, H_2. Since C is pointed, there exists a vector w such that $wc > 0$ for each nonzero $c \in C$. Let h be a vector in the symmetric difference of H_1 and H_2 with wh minimized. It can be assumed that $h \in H_1$. Since H_2 is a Hilbert basis, h can be expressed as $\lambda_1 h_1 + \ldots + \lambda_k h_k$ where $\{h_1, \ldots, h_k\} \subseteq H_2$ and λ_i is a positive integer, $i = 1, \ldots, k$. Since $wh = \lambda_1 wh_1 + \ldots + \lambda_k wh_k$ and $h \notin \{h_1, \ldots, h_k\}$, $wh_i < wh$, $i = 1, \ldots k$.

So, by the choice of h, for $i = 1,\ldots,k$ h_i is in $H_1 \cap H_2$, which implies that H_1 is not minimal, a contradiction. □

Suppose that $P = \{x: Ax \leq b\}$ is of full dimension. Let F be a minimal face of P and let C_F be the cone generated by the active rows of A for F. Suppose that there exists a nonzero $c \in C_F$ such that $-c \in C_F$. Let $\alpha = c\bar{x}$ where $\bar{x}$ is a point in F. Now $cx \leq \alpha$ and $-cx \leq -\alpha$ are valid inequalities for P. So P lies on the hyperplane $cx = \alpha$, which is impossible since P is of full dimension. So C_F is a pointed cone. This fact will be used in the proof of the following theorem of Schrijver [81].

<u>Theorem 5.5:</u> Let P be a polyhedron of full dimension. There exists a unique minimal totally dual integral system $Ax \leq b$ with A integral such that $P = \{x: Ax \leq b\}$. Furthermore, P is an integer polyhedron if and only if b is integral.

<u>Proof:</u> Proceed as in the proof of Theorem 5.3, but for each minimal nonempty face F of P let H_F be the unique minimal Hilbert basis for the pointed cone C_F. The system $Ax \leq b$ that is obtained as in the proof of Theorem 5.3 is a totally dual integral defining system for P with A integral. Suppose that $A''x \leq b''$ is a totally dual integral defining system for P with A'' integral. It will be shown that each inequality in $Ax \leq b$ must be present in the system $A''x \leq b''$ (it will then follow that $Ax \leq b$ is the unique minimal totally dual integral defining system for P with integer left hand sides).

Let F be a minimal face of P and let $h \in H_F$. Suppose that $hx \leq \alpha_h$ is not an inequality in the system $A''x \leq b''$. Consider the linear program

(5.2) $\max\{hx\colon A''x \le b''\}$.

Each vector in F is an optimal solution to this linear program, so the complementary slackness theorem implies that each optimal solution to the dual linear program of (5.2) corresponds to an expression of h has a nonnegative linear combination of the rows in A'' that are active for F. Each active row in A'' for F is in the cone C_F. Since h cannot be expressed as a nonnegative integral combination of other integer vectors in C_F, there does not exist an integer optimal solution to the dual linear program of (5.2). So $A''x \le b''$ is not totally dual integral, a contradiction.

It only remains to be shown that P is an integer polyhedron if and only if b is integral, but this follows directly from the theorem of Hoffman and Edmonds-Giles (Corollary 2.5 of this chapter) and the second part of Theorem 5.3. □

This result is the analogue of the well known theorem that there exists a unique (up to positive scalar multiples of the inequalities) minimal defining system for each full dimensional polyhedron. If P is a polyhedron of full dimension and $Ax \le b$ is the unique minimal totally dual integral defining system for P with integer left hand sides, then call $Ax \le b$ the <u>Schrijver system</u> for P. The Schrijver systems for various combinatorial polyhedra associated with matchings will be studied in Chapter 3.

It should be noted that a result for non-full dimensional polyhedra, analogous to Theorem 5.5, is also given in Schrijver [81].

6. Operations that preserve total dual integrality.

The result of Giles and Pulleyblank [79] given in the previous section shows that for any finite set of integer vectors a min-max relation for S can be obtained by finding a totally dual integral defining system for the convex hull of S. However, how can such a system be found? Consider the analogous problem of finding a linear system that defines the convex hull of S. Certain operations that can be performed on a linear system such that if the linear system defines an integer polyhedron then the derived system does as well are quite useful. Some of these operations, such as adding slack variables or taking scalar multiples of inequalities, are useful because they allow certain assumptions to be made on the form of a linear system without losing any generality. Others, such as setting inequalities to equalities or applying Fourier-Motzkin elimination, are useful because often a system of interest arises by applying the operations to a system which is known to define an integer polyhedron. The problem of determining whether or not these operations preserve total dual integrality of linear systems is dealt with in this section (cf. Cook [83]). Other aspects of finding totally dual integral defining systems are studied in Section 10 of this chapter.

The first type of operations that will be looked at are those that allow assumptions to be made on the form of a system. It may be assumed, without loss of generality, that systems are of the form $Ax \leq b$. Throughout this section let f denote a vector and β a scalar.

a) Adding slack variables.

In linear programming, linear systems are often transformed by adding "slack" variables to obtain problems which are in the correct form for the simplex algorithm. The following result shows that, with integral data, such a transformation maintains total dual integrality of a linear system.

Proposition 6.1: Let f be integral. The system $\{Ax \le b,\ fx \le \beta\}$ is totally dual integral if and only if the system $\{Ax \le b,\ fx+s = \beta,\ s \ge 0\}$ is totally dual integral.

Proof: Let D1 denote the linear program $\min\{yb+z\beta\colon yA+zf = w,\ y \ge 0,\ z \ge 0\}$ and let D2 denote the linear program $\min\{yb+z\beta\colon yA+zf = w,\ z \ge w_s,\ y \ge 0\}$. Suppose that the system $\{Ax \le b,\ fx+s = \beta,\ s \le 0\}$ is totally dual integral and let w be an integer vector such that D1 has an optimal solution. An integral optimal solution to D1 can be found by setting $w_s = 0$ in D2. Now suppose that $\{Ax \le b,\ fx \le \beta\}$ is totally dual integral and let (w,w_s) be an integral vector such that D2 has an optimal solution. An integral optimal solution to D2 is (y^*,z^*+w_s), where (y^*,z^*) is an integral optimal solution to D1 with right hand side $w' = w-w_s f$. □

This result does not hold in general for nonintegral f, e.g. $\{x_1 \le 0,\ (2/3)x_1 \le 0\}$ is totally dual integral but $\{x_1 \le 0,\ (2/3)x_1+x_2 = 0,\ x_2 \ge 0\}$ is not.

b) Splitting unrestricted variables.

Another transformation that is often used in linear programming is the replacement of variables which are not restricted to nonnegative values by the difference of two nonnegative variables. This transformation allows the assumption to be made that all variables are nonnegative.

However, it does not preserve total dual integrality of linear systems, e.g. the system $\{x_1+5x_2 \le 1, x_1+6x_2 \le 1\}$ is totally dual integral but the system obtained by replacing x_1 by $x_1' - x_1''$ and x_2 by $x_2' - x_2''$, where $x_1' \ge 0$, $x_1'' \ge 0$, $x_2' \ge 0$, $x_2'' \ge 0$, is not totally dual integral (this example also shows that the property of defining an integer polyhedron is not preserved under such a splitting operation, since $x_1' = 0$, $x_1'' = 0$, $x_2' = (1/6)$, $x_2'' = 0$ is a nonintegral vertex of the derived polyhedron). So, when working with totally dual integral systems, it cannot be assumed that all variables are nonnegative.

c) Scalar multiplication of inequalities.

It is trivial that if a system $A'x \le b'$ is obtained by multiplying some inequality in $Ax \le b$ by a positive scalar then the two systems define the same polyhedron. However, it is not possible to multiply an inequality by an arbitrary positive scalar and maintain total dual integrality of a system, e.g. $x_1 \le 0$ is totally dual integral but $2x_1 \le 0$ is not. In fact, Giles and Pulleyblank [79] have shown that for any system $Ax \le b$ there exists a positive scalar d such that $dAx \le db$ is totally dual integral. However, an inequality in a totally dual integral system can be multiplied by a scalar of the form $1/k$, where k is a positive integer, since solutions to the new dual linear program can be found by setting a single component $\bar{y}_i$ to $k\bar{y}_i$ in a solution to the original dual linear program. For linear systems with integer left hand sides that define full dimensional polyhedra, the following proposition shows that this is the only type of nontrivial scalar multiplication that is possible.

Proposition 6.2: Let A and f be integral and let d be a positive scalar. Suppose that $Ax \leq b$, $fx \leq \beta$ is totally dual integral and that $P = \{x: Ax \leq b, fx \leq \beta\}$ is of full dimension. The system $Ax \leq b$, $dfx \leq d\beta$ is also totally dual integral if and only if either $fx \leq \beta$ is not in the Schrijver system for P or d is of the form $1/k$ for some positive integer k.

Proof: The sufficiency of either condition is easily seen. To show necessity, suppose $\{Ax \leq b, dfx \leq d\beta\}$ is totally dual integral and $fx \leq \beta$ is in the Schrijver system for P. There exists a convex cone, C, generated by the active rows of some minimal face, F, of P, that contains f in its unique minimal Hilbert basis (C is pointed since P is of full dimension). Let λ be the multiplier of df in an expression of f as a nonnegative integer combination of the active rows of F in $\{A, df\}$. The multiplier λ must be nonzero since f cannot be expressed as a nonnegative integer combination of other integer vectors in C. Since $-f \notin C$ (C is pointed), λd must be less than or equal to 1. Since λd must be integral, $\lambda df = f$ (f is the first nonzero integral point on the ray $\{y_1 f: y_1 \leq 0\}$). So $d = 1/\lambda$. □

d) Unimodular transformations.

Let U be a square integral matrix such that $\det(U) = \pm 1$. A simple and well known result is that $Ax \leq b$ is totally dual integral if and only if $AUx \leq b$ is totally dual integral. (This corresponds to the result that P is an integer polyhedron if and only if $P' = \{U^{-1}x: x \in P\}$ is an integer polyhedron.) This result is used in Section 8 of this chapter in connection with Hermite normal forms.

The remaining two operations that will be looked at are the type that can be used to prove that a given system is totally dual integral by showing that it can be obtained via the operation from a system that is known to be totally dual integral (of course, this remark can be made about operations a) through d) as well, but those operations are usually not used in this way).

e) Setting inequalities to equalities.

The following proposition is given in Schrijver [82b]. For completeness a proof is given here.

Proposition 6.3: If $\{Ax \le b,\ fx \le \beta\}$ is totally dual integral, then $\{Ax \le b,\ fx = \beta\}$ is also totally dual integral.

Proof: Note that $\{Ax \le b,\ fx = \beta\}$ is totally dual integral if and only if $\{Ax \le b,\ fx \le \beta,\ -fx \le -\beta\}$ is totally dual integral. Also, if $a_1,\dots,a_k,f,-f$ are the active rows of a minimal face of $\{x: Ax \le b,$ $fx \le \beta,\ -fx \le -\beta\}$ then $a_1,\dots,a_k,f$ are the active rows of a minimal face of $\{x: Ax \le b,\ fx \le \beta\}$. So it suffices to show that if each integer vector in the cone generated by $a_1,\dots,a_k$ can be expressed as a nonnegative integer combination of $a_1,\dots,a_k$, then each integer vector in the cone generated by $a_1,\dots,a_k,-a_1$ can be expressed as a nonnegative integer combination of $a_1,\dots,a_k,-a_1$. Suppose that r is an integral vector and $r = \Sigma\{\lambda_i a_i:\ i = 1,\dots,k\} - \gamma a_1$, where $\lambda_i \ge 0$, $i = 1,\dots,k$ and $\gamma \ge 0$. Choose γ' such that $r+\gamma' a_1$ is in the cone generated by $a_1,\dots,a_k$ and such that $\gamma' a_1$ is integral and γ' is integral. By assumption, there exist nonnegative integers λ_i', $i = 1,\dots,k$, such that $r+\gamma' a_1 = \Sigma\{\lambda_i' a_i:\ i=1,\dots k\}$. Now $\Sigma\{\lambda_i' a_i:\ i=1,\dots,k\} - \gamma' a_1$ expresses r as a nonnegative integer combination of $a_1,\dots,a_k,-a_1$. □

This result corresponds to the well known result that any face of an integer polyhedron is also an integer polyhedron. It can be used to find a totally dual integral defining system for any face of a polyhedron for which such a system is known, e.g. totally dual integral defining systems for the convex hull of the perfect matchings of a graph and the convex hull of the common bases of two matroids can be derived from the totally dual integral defining systems for the matching polyhedron and the matroid intersection polyhedron respectively (cf. Pulleyblank [82] and Chapter 3).

f) Fourier-Motzkin elimination.

Let K be a set in $\mathbb{Q}^n$ and let $I \subseteq \{1,\ldots,n\}$. If $x \in \mathbb{Q}^n$, let x_I denote the vector $(x_i: i \in I)$. The projection of K onto the I coordinates is the set $\{x_I: x \in K\}$. A well known and useful result is that any projection of an integer polyhedron is again an integer polyhedron (for an example of its application see Balas and Pulleyblank [82]). To obtain a corresponding result for total dual integrality, linear systems must be considered.

If a defining system for a polyhedron is given then a defining system for any projection of the polyhedron can be found by repeated application of Fourier-Motzkin elimination (cf. Stoer and Witzgall [70]). The version of Fourier-Motzkin elimination used here is such that if the original system has integral data then the resulting system does as well (the result of Giles and Pulleyblank [79] mentioned in subsection c shows that the scaling of the resulting system must be specified). Consider the linear system

$$
(6.1) \qquad \begin{array}{lll}
a_i x - \alpha_i x_o \le b_i & & i \in I = \{1,\ldots,i_o\} \\
c_j x + \gamma_j x_o \le d_j & & j \in J = \{1,\ldots,j_o\} \\
f_k x \le g_k & & k \in K = \{1,\ldots,k_o\}
\end{array}
$$

where a_i, c_j, f_k are vectors, α_i, γ_j are positive scalars, and b_i, d_j, g_k are scalars, for $i \in I$, $j \in J$, $k \in K$. The following system is obtained by applying Fourier-Motzkin elimination to eliminate the variable x_o

$$(6.2)\qquad \begin{array}{ll} (\gamma_j a_i+\alpha_i c_j)x \le \gamma_j b_i+\alpha_i d_j & i \in I,\ j \in J \\ f_k x \le g_k & k \in K. \end{array}$$

It is not true that (6.2) must be totally dual integral if (6.1) is, even if all data is integral and the coefficients of any row of (6.2) have greatest common divisor 1. For example, $\{x_1-x_3 \le 0,\ x_2-x_3 \le 0,\ -x_2+2x_3 \le 0\}$ is totally dual integral since the coefficient matrix is unimodular but $\{2x_1-x_2 \le 0,\ x_2 \le 0\}$ is not totally dual integral. However, the result is true in a special case.

Theorem 6.4: Let $Ax \le b$ be a totally dual integral system. If each coefficient of the variable x_o is either 0,1, or -1 then the system obtained by eliminating x_o by Fourier-Motzkin elimination is also totally dual integral.

Proof: What must be shown is that if (6.1) is totally dual integral and α_i, γ_j are equal to 1 for $i \in I$, $j \in J$, then (6.2) is also totally dual integral. Let D1 denote the linear program

$$(6.3)\qquad \begin{array}{ll} \min & \Sigma\{b_i y_i' : i \in I\}+\Sigma\{d_j y_j'' : j \in J\}+\Sigma\{g_k z_k : k \in K\} \\ \text{s.t.} & \Sigma\{a_i y_i' : i \in I\}+\Sigma\{c_j y_j'' : j \in J\}+\Sigma\{f_k z_k : k \in K\} = w \\ & -\Sigma\{y_i' : i \in I\}+\Sigma\{y_j'' : j \in J\} = w_o \\ & y_i' \ge 0,\ y_j'' \ge 0,\ z_k \ge 0 \quad i \in I,\ j \in J,\ k \in K \end{array}$$

and let D2 denote the linear program

$$\text{(6.4)}\qquad \begin{aligned} \min\ & \Sigma\{(b_i+d_j)y_{ij}: i \in I,\ j \in J\}+\Sigma\{g_k z_k: k \in K\} \\ \text{s.t.}\ & \Sigma\{(a_i+c_j)y_{ij}: i \in I,\ j \in J\}+\Sigma\{f_k z_k: k \in K\} = w \\ & y_{ij} \geq 0,\ z_k \geq 0 \quad i \in I,\ j \in J,\ k \in K. \end{aligned}$$

Suppose that w is integral and D2 has an optimal solution $(\bar{y},\bar{z})$. An optimal solution to D1 with $w_o = 0$ is $\bar{z}$ together with

$$\text{(6.5)}\qquad \begin{aligned} \bar{y}'_i &= \Sigma\{\bar{y}_{ij}: j \in J\} \qquad i \in I \\ \bar{y}''_j &= \Sigma\{\bar{y}_{ij}: i \in I\} \qquad j \in J \end{aligned}$$

since any solution (y',y'',z) to D1 with $w_o = 0$ corresponds to a solution to D2 with the same objective value, by finding nonnegative y_{ij}'s such that

$$\text{(6.6)}\qquad \begin{aligned} \Sigma\{y_{ij}: j \in J\} &= y'_i \qquad i \in I \\ \Sigma\{y_{ij}: i \in I\} &= y''_j \qquad j \in J \end{aligned}$$

($w_o = 0$ implies that $\Sigma\{y'_i: i \in I\} = \Sigma\{y''_j: j \in J\}$). An integral optimal solution to D2 can be found by finding an integral optimal solution, (y',y'',z), to D1 with $w_o = 0$ and then finding a nonnegative integral y that satisfies (6.6). □

7. Integer programming.

Suppose that $Ax \le b$ is totally dual integral and that b is integral. The results of Hoffman and Edmonds-Giles given in Section 2 show that both sides of the equation

$$\max\{wx: Ax \le b\} = \min\{yb: yA = w, y \ge 0\} \tag{7.1}$$

can be achieved by integer vectors for each integer vector w for which the optima exist. The problem dealt with in this section is that of finding integer x and y that obtain the optima for a given integer vector w.

The left hand side of (7.1) can be handled easily. All that is needed is the fact that $Ax \le b$ defines an integer polyhedron (which follows from the results of Section 2 since b is integral) and the following theorem.

Theorem 7.1: There exists a polynomial-time algorithm which, for any linear system $Ax \le b$ and any vector w for which the maximum in (7.1) exists, either finds an integer vector which achieves the maximum in (7.1) or shows that $Ax \le b$ does not define an integer polyhedron.

Proof: Let $Ax \le b$ be a linear system, w a vector such that the maximum in (7.1) exists, and P the polyhedron $\{x: Ax \le b\}$. A minimal face $F = \{x: A'x = b'\}$ of P will be found such that every point in F achieves the maximum in (7.1). Once F is found, the problem is reduced to solving a system of linear diophantine equations, which can be done in polynomial-time by the algorithm of Kannan and Bachem [79] for finding the Smith normal form of a matrix (other polynomial-time algorithms for solving systems of linear diophantine equations are given in Frumkin [76] and Gathen and Sieveking [76]). If the linear system $A'x = b'$ has no integer solution, then F is a face of P which does not contain an integer point, which proves that P is not an integer polyhedron. To find F in polynomial time, the ellipsoid method of Khachiyan [79] will be used repeatedly. First, find an optimal solution to $\max\{wx: Ax \leq b\}$ and let $\beta = \max\{wx: Ax \leq b\}$. Consider the linear system

$$(7.2) \qquad \begin{aligned} wx &= \beta \\ Ax &\leq b \end{aligned}$$

Any solution to (7.2) achieves the maximum in (7.1). Solve the linear program $\max\{a_1x: (7.2)\}$. where $a_1x \leq b_1$ is the first inequality in $Ax \leq b$. If this maximum is b_1, then add $a_1x = b_1$ to (7.2). Repeat this procedure until each inequality in $Ax \leq b$ has been considered. Now letting $A'x = b'$ consist of the equalities $a_ix = b_i$ that were added to (7.2), F has been found. □

When A is integral, the right hand side of (7.1) can be handled by the following result of Chandrasekaran [81] (the description of the algorithm given below is from Cook, Lovász and Schrijver [82]).

Theorem 7.2: There exists a polynomial-time algorithm which, for any linear system $Ax \leq b$ with A integral and any integer vector w for which the minimum in (7.1) exists, either finds an integer vector which achieves the minimum in (7.1) or shows that $Ax \leq b$ in not totally dual integral.

Proof: Let $Ax \leq b$ a linear system with A integral and let w be an integer vector such that the minimum in (7.1) exists. Again, the ellipsoid method will be used repeatedly. Let $a_1, \ldots, a_k$ be those rows of A such that $a_i\bar{x} = b_i$ for each vector $\bar{x}$ which achieves the maximum in (7.1), where $a_i x \le b_i$ is the corresponding inequality in $Ax \le b$ (these rows can be found by considering the linear program $\min\{a_i x\colon Ax \le b,\ wx \le z^*\}$, where $z^* = \max\{wx\colon Ax \le b\}$). By the complementary slackness theorem, each vector which achieves the minimum in (7.2) corresponds to an expression of w as a nonnegative linear combination of $a_1,\ldots,a_k$.

Let C be the cone generated by $\{a_1,\ldots,a_k\}$. Relabel a_1 through a_k if necessary so that $a_1,\ldots,a_j$ are those vectors a_i in $\{a_1,\ldots,a_k\}$ such that $-a_i \notin C$. Let λ_1 be the largest number such that $w - \lambda_1 a_1$ is in C, i.e. $\lambda_1 = \max\{z_1\colon yA = w - z_1 a_1,\ y \ge 0\}$ (since $-a_1 \notin C$, λ_1 exists). Let $w^1 = w - \lfloor\lambda_1\rfloor a_1$ and let $\lambda_2 = \max\{z_2\colon yA = w^1 - z_2 a_2\}$. Let $w^2 = w^1 - \lfloor\lambda_2\rfloor a_2$ and so forth until w^j is found. Now use the algorithm of Kannan and Bachem [79] to solve a system of linear diophantine equations to find integer $\lambda_{j+1},\ldots,\lambda_k$ such that $w^j = \lambda_{j+1}a_{j+1}+\ldots+\lambda_k a_k$, if such $\lambda_{j+1},\ldots,\lambda_k$ exist. (Note that the λ_i's may be negative.)

Suppose that no such integer $\lambda_{j+1},\ldots,\lambda_k$ exist. If $\gamma_1 a_1+\ldots+\gamma_k a_k$ is an expression of w^j as a nonnegative integer combination of $a_1,\ldots,a_k$, then $\gamma_1 = \gamma_2 = \ldots = \gamma_j = 0$, by the choice of $\lambda_1,\ldots,\lambda_j$. So w^j cannot be expressed in this way. Now w^j is in C, so $\min\{yb\colon yA = w^j,\ y \ge 0\}$ exists and the minimum can be achieved by an integer vector if and only if w^j can be expressed as a nonnegative integer combination of $a_1,\ldots,a_k$. Since w^j is integral, this shows that $Ax \le b$ is not totally dual integral.

Now suppose that integer $\lambda_{j+1},\ldots,\lambda_k$ have been found. By expressing $-a_i$ as a nonnegative linear combination of $a_{j+1},\ldots,a_k$ (excluding a_i) for $i = j+1,\ldots,k$, find positive $\gamma_{j+1},\ldots,\gamma_k$ such that $0 = \gamma_{j+1}a_{j+1}+\ldots+\gamma_k a_k$. Find an integer M, of size polynomial in the size of $Ax \le b$ and w, such that $\lambda_i+M\gamma_i$ is a nonnegative integer for $i = j+1,\ldots,k$. Since $w^j = (\lambda_{j+1}+M\gamma_{j+1})a_{j+1}+\ldots+(\lambda_k+M\gamma_k)a_k$ and $w = \lfloor\lambda_1\rfloor a_1+\ldots+\lfloor\lambda_j\rfloor a_j+w^j$, this gives an expression of w as a nonnegative integer combination of $a_1,\ldots,a_k$ (which gives an integer vector which achieves the minimum in (7.1)). □

A consequence of this result is that given a Hilbert basis $\{a_1,\ldots,a_k\}$ and an integer vector w in the cone generated by the basis, an expression of w as a nonnegative integer combination of $a_1,\ldots,a_k$ can be found in polynomial time.

In their outstanding paper on consequences of the ellipsoid method, Grötschel, Lovász, and Schrijver [81] have shown that polynomial-time algorithms for finding optimum integer dual solutions to linear programs associated with several different combinatorial problems can be derived from the ellipsoid algorithm. The linear systems involved are often very large, but the algorithms derived are polynomial in the size of the given combinatorial problem (the linear systems are given by an encoding whose size is polynomial in the size of the combintorial problem). It would be interesting to generalize their methods to obtain a result on finding optimum integer dual solutions that is similar to their general results on finding optimum integer primal solutions.

8. Bounds on the number of nonzero dual variables.

Suppose that we have a linear system $Ax \le b$ encoded in some way and we wish to consider the problem mentioned at the end of the previous section. That is, is there a polynomial-time algorithm which, for any integer vector w such that $\max\{wx\colon Ax \le b\}$ exists, either finds an integer optimal solution to the dual of this linear program or proves that $Ax \le b$ is not totally dual integral? If such an algorithm exists then it must be possible to describe the integer optimal solution in polynomial time. For example, let G be a graph and $\mathcal{C}$ the set of maximal cliques of G and suppose that the linear system

$$\text{(8.1)} \qquad \begin{aligned} \Sigma\{x_v\colon v \in C\} &\le 1 \qquad \forall \quad C \in \mathcal{C} \\ x_v &\ge 0 \qquad \forall \quad v \in VG \end{aligned}$$

is encoded by the graph G. A by-product of the polynomial-time algorithm of Grotschel, Lovász, and Schrijver [81,81a] for finding an integer optimal solution to the dual linear program of $\max\{wx\colon (8.1)\}$ for any perfect graph G and any nonnegative integer vector w (recall that G is perfect if and only if (8.1) is totally dual integral) is that for any perfect graph G and any nonnegative integer vector w there exists an integer optimal solution to the dual linear program of $\max\{wx\colon (8.1)\}$ with at most n^2+n nonzero variables, where n is the number of nodes of G. The problem considered in this section is whether or not there exists a similar polynomial bound on the number of nonzero dual variables for general totally dual integral systems.

The results presented in this section are due to Cook, Fonlupt, and Schrijver [83].

A more precise statement of the problem is now given. Let h be the least integer such that for any integer vector w for which

(8.2) $\min\{yb\colon yA = w,\ y \geq 0\}$

exists, the minimum can be achieved by an integral solution with at most h nonzero variables. The problem is to ask if, for any totally dual integral system $Ax \leq b$, where A is an integral matrix of rank n, is there an upper bound for h in terms of n?

Suppose that $\{a_1,\ldots,a_k\}$ is a Hilbert basis for a cone C of dimension n. By Lemma 5.2, the above problem is equivalent to: Is there an upper bound for h in terms of n, where h is the least integer such that each integer vector in C can be expressed as a non-negative integer combination of h vectors in $\{a_1,\ldots,a_k\}$? Carathéodory's Theorem implies that each vector w in C can be expressed as a nonnegative linear combination of n vectors in $\{a_1,\ldots,a_k\}$, so the problem dealt with here is an integer analogue of the one solved by Carathéodory (see Rockafeller [70]). It will be shown that in general there is no upper bound for h in terms of n. However, if C is pointed, the following result gives a bound for h.

<u>Theorem 8.1:</u> Let $\{a_1,\ldots,a_k\}$ be a Hilbert basis for a pointed cone C of dimension n. If w is an integer vector in C then w can be expressed as a nonnegative integer combination of $2n-1$ vectors in $\{a_1,\ldots,a_k\}$.

<u>Proof:</u> It can be assumed that 0 is not a vector in $\{a_1,\ldots,a_k\}$. Suppose that w is an integer vector in C. Let A be the matrix with rows $a_1,\ldots,a_k$. Since C is pointed, the linear program $\max\{1y\colon yA = w,\ y \geq 0\}$ has a basic optimal solution $\bar{y}$ (it has a

solution since $w \in C$ and it is easy to see that it is bounded by considering a vector d such that $dz > 0$ for each nonzero $z \in C$). Let $\lfloor \bar{y} \rfloor$ denote the vector $(\lfloor \bar{y}_1 \rfloor, \ldots, \lfloor \bar{y}_k \rfloor)$. The vector $w - \lfloor \bar{y} \rfloor A$ is an integer vector in C and so can be expressed as $\lambda_1 a_1 + \ldots + \lambda_k a_k$ where λ_i is a nonnegative integer for $i = 1, \ldots, k$. Now $\bar{y}$ has at most n nonzero components, since it is a basic solution. So $1 \cdot (\bar{y} - \lfloor \bar{y} \rfloor) < n$. Now, since $\bar{y}$ is an optimal solution and $(\lambda_i + \lfloor \bar{y}_i \rfloor: i = 1, \ldots, k)$ is a feasible solution to the linear program $\max\{1y: yA = w, y \geq 0\}$, it follows that $\Sigma\{\lambda_i + \lfloor \bar{y}_i \rfloor: i = 1, \ldots, k\} \leq 1\bar{y}$ and $\Sigma\{\lambda_i: i = 1, \ldots, k\} \leq 1 \cdot (\bar{y} - \lfloor \bar{y} \rfloor) < n$. So $(\lambda_1 + \lfloor \bar{y}_1 \rfloor) a_1 + \ldots + (\lambda_k + \lfloor \bar{y}_k \rfloor) a_k$ expresses w as a nonnegative integer combination of $a_1, \ldots, a_k$ with at most $2n-1$ nonzero multipliers. □

The above upper bound can be improved slightly if all vectors in the Hilbert basis are 0-1 valued.

Theorem 8.2: Let $\{a_1, \ldots, a_k\}$ be a Hilbert basis for a cone C of dimension $n > 1$, where a_i is 0-1 valued for $i = 1, \ldots, k$. If w is an integer vector in C then w can be expressed as a nonnegative integer combination of $2n-2$ vectors in $\{a_1, \ldots, a_k\}$.

Proof: Since a_i is 0-1 valued for $i = 1, \ldots, k$, C is pointed. Suppose that w is an integer vector in C. Find $\bar{y}$ as in the proof of Theorem 8.1. Suppose that $1 \cdot (\bar{y} - \lfloor \bar{y} \rfloor) \geq n-1$. Let A be the matrix with rows $a_{i_1}, \ldots, a_{i_n}$ where $\bar{y}_{i_1}, \ldots, \bar{y}_{i_n}$ are the nonzero variables in the solution $\bar{y}$ (it may be assumed that there are n such variables).

The only possible 0-1 vectors whose inner product with $(\bar{y}_{i_1}-\lfloor\bar{y}_{i_1}\rfloor,\ldots,\bar{y}_{i_n}-\lfloor\bar{y}_{i_n}\rfloor)$ is an integer are 0 and the vector of all 1's. But $(\bar{y}_{i_1}-\lfloor\bar{y}_{i_1}\rfloor,\ldots,\bar{y}_{i_n}-\lfloor\bar{y}_{i_n}\rfloor)A = w-(\lfloor\bar{y}_{i_1}\rfloor,\ldots,\lfloor\bar{y}_{i_n}\rfloor)A$, which is integral. So each column of A is either all zeros or all 1's which implies that n is 1, a contradiction. So $1\cdot(\bar{y}-\lfloor\bar{y}\rfloor) < n-1$, which improves the upperbound to $2n-2$. □

It is an open question whether or not the upper bound given in Theorem 8.1 can be lowered to n. (It is easy to see that it cannot be less than n.) To see that in general there is no upper bound in terms of n, consider the following example. Let $p_1,\ldots,p_k$ be distinct primes, $k \geq 2$. For $i = 1,\ldots,k$ let q_i be the product of p_1 through p_k, excluding p_i. The greatest common divisor of $q_1,\ldots,q_k$ is 1, so there exists integer λ_i, $i = 1,\ldots,k$ such that $\lambda_1 q_1+\ldots+\lambda_k q_k = 1$. For $i = 1,\ldots,k$ let $r_i = q_i$ if $\lambda_i \geq 0$ and $r_i = -q_i$ if $\lambda_i < 0$. Now $\{r_1,\ldots,r_k\}$ is a Hilbert basis for a cone of dimension 1, but, since the greatest common divisor of any proper subset of $\{r_1,\ldots,r_k\}$ is greater than 1, any expression of 1 as a nonnegative integer combination of $r_1,\ldots,r_k$ requires all k vectors.

The above example shows that in general there is no upper bound in terms of rank (A) on the number of nonzero variables needed in an integral solution that achieves (8.2), where $Ax \leq b$ is a totally dual integral system with A an integer matrix and w is an integer vector such that the minimum exists. However, as was shown in Section 5, if the polyhedron $P = \{x: Ax \leq b\}$ is of full dimension then the active rows in A of any minimal nonempty face of P generate a pointed cone. Thus, Theorem 8.1 implies the following result.

Theorem 8.3: Let $Ax \le b$ be a totally dual integral system, with A an integer matrix of rank n, such that $P = \{x: Ax \le b\}$ is of full dimension. If w is an integer vector such that (8.2) exists then (8.2) can be achieved by an integer optimal solution with at most $2n-1$ nonzero variables. □

In a similar manner, Theorem 8.2 implies that if A is a 0-1 matrix of rank $n > 1$ then the upperbound given above can be lowered to $2n-2$. A slight modification of this result is needed to improve a result given in Cunningham [81] (recall the definition of integer rounding given in Section 1).

Theorem 8.4: Let A be a 0-1 matrix with no row of zeros and b a 0-1 vector, such that $Ax \le b$ has the integer rounding property and the matrix $[A|b]$ has rank $n > 1$. If w is a nonnegative integer vector, then there exists an optimal solution to the integer program

$$(8.3) \qquad \min\{yb: yA = w,\ y \ge 0,\ y \text{ integral}\}$$

with at most $2n-2$ nonzero variables.

Proof: Suppose that w is a nonnegative integer vector. Let z^* be equal to $\min\{yb: yA = w,\ y \ge 0\}$ and let y^1 be a basic optimal solution to $\max\{1y: yb = \lceil z^* \rceil,\ yA = w,\ y \ge 0\}$. Consider the linear program

$$(8.4) \qquad \min\{yb: yA = w - \lfloor y^1 \rfloor A,\ y \ge 0\}.$$

By the choice of z^*, a lower bound for (8.4) is $z^* - \lfloor y^1 \rfloor b$. Also,

$y^1 - \lfloor y^1 \rfloor$ is a solution to (8.4) with objective value $\lceil z^* \rceil - \lfloor y^1 \rfloor b$. So, by the integer rounding property, there exists an integer solution y^2 to (8.4) with objective value $\lceil z^* \rceil - \lfloor y^1 \rfloor b$. Now $y^3 = \lfloor y^1 \rfloor + y^2$ is an optimal solution to (8.3). Since y^1 is a basic solution, $\lfloor y^1 \rfloor$ has at most n nonzero components. As in the proof of Theorem 8.2, it can be assumed that $1 \cdot (y^1 - \lfloor y^1 \rfloor) < n-1$. So y^2 has at most $n-2$ nonzero components and y^3 at most $2n-2$ nonzero components. □

The problem of Cunningham [81] mentioned above is the following. Let M be a matroid and w a nonnegative integer vector. What is the least integer k such that of all minimum cardinality families of independent sets of M having w as the sum of the incidence vectors of the sets in the family, there exists a family with at most k distinct members? This problem can be stated in terms of linear systems as follows. Let A be a matrix whose rows are the incidence vectors of independent sets of M. The problem is: What is the least k such that, for each nonnegative integer vector w, $\min\{1y: yA = w, y \geq 0, y \text{ integral}\}$ can be achieved by a vector with at most k nonzero components? Cunningham [81] showed that his algorithm for testing membership in matroid polyhedra gives an upper bound of n^4+1 for k, where n is the number of elements in M. This bound was improved to $2n$ by Schrijver (see Cunningham[81]). The matroid partition theorem of Edmonds [65b] shows that $Ax \leq 1$ has the integer rounding property, so Theorem 8.4 lowers the upper bound to $2n-2$.

It has already been mentioned that a by-product of an algorithm of Grötschel, Lovász, and Schrijver [81,81a] is that if G is a perfect graph and A is the clique-node incidence matrix of G then for each nonnegative integer w, $\min\{1 \cdot y: yA \geq w, y \geq 0\}$ can be achieved by an integer vector with at most n^2+n nonzero components, where n is the number of nodes of G. Another way of stating this is, if G is perfect and w is a nonnegative integer vector then there exists a minimum w-covering of G with at most n^2+n distinct cliques (a <u>minimum w-covering</u> is a minimum cardinality family of cliques such that each node v of G is in at least w_v cliques in the family). Since the polyhedron $\{x: Ax \leq 1, x \geq 0\}$ is of full dimension, Theorem 8.3 improves this bound to $2n-1$. In fact, Cook, Fonlupt, and Schrijver [83] showed that n is an upperbound (it is clear that the upper bound cannot be less than n).

There are other combinatorially described totally dual integral systems $Ax \leq b, x \geq 0$ such that $\min\{yb: yA \geq w, y \geq 0\}$ can be achieved by an integer vector with at most n nonzero components for each integer vector w for which the minimum exists, where n is the number of variables in $Ax \leq b, x \geq 0$. The systems arising from cross-free families (Schrijver [82a]) have this property - see Section 10. W. Cunningham observed that the algorithm for optimum matchings given in Cunningham and Marsh [78] shows that the totally dual integral matching system (see Chapter 3) has this property (which also follows from the proof of total dual integrality of matching systems given in Schrijver and Seymour [77]).

9. Recognition of totally dual integral systems.

In the previous sections properties of totally dual integral systems, integer polyhedra, and min-max relations were presented. Also, sufficient conditions for a linear system to be totally dual integral were presented in Section 3 (concerning totally unimodular and balanced matrices). However, the general problem of recognizing totally dual integral systems and systems that define integer polyhedra has not yet been dealt with. Lemma 2.3 of Chapter 1 and Lemma 5.2 of this chapter show that there exist finite algorithms to determine whether or not a given linear system is totally dual integral and whether or not it defines an integer polyhedron. The complexity of these recognition problems is studied in this section. Several easy results on integer polyhedra will be presented first, then the corresponding results for totally dual integral systems will be given.

Lemma 2.3 of Chapter 1 is very useful in obtaining results on the problem of recognizing when a linear system defines an integer polyhedron.

Proposition 9.1: The class of linear systems, $Ax \le b$, that define integer polyhedra is in co-NP.

Proof: By Lemma 2.3 of Chapter 1, if $Ax \le b$ does not define an integer polyhedron then there exists a subset, $A'x \le b'$, of the inequalities such that $A'x = b'$ has a solution but does not have an integral solution and $F = \{x: A'x = b'\}$ is a minimal face of $\{x: Ax \le b\}$. Using the algorithm of Kannan and Bachem [79], and the ellipsoid algorithm, it is possible to check in polynomial time that $A'x \le b'$ has the above properties. □

It is an open question whether or not the class of linear systems that define integer polyhedra is in NP (it is also an open question whether or not the problem of recognizing linear systems that define integer polyhedra is co-NP complete), but the following result is true.

Proposition 9.2: For any fixed positive integer r, there exists a polynomial-time algorithm which, for any linear system $Ax \le b$ with A a matrix of rank r, determines whether or not $Ax \le b$ defines an integer polyhedron.

Proof: Suppose that r is a fixed positive integer and $Ax \le b$ is a linear system with A a matrix of rank r. Each minimal nonempty face F of the polyhedron $P = \{x: Ax \le b\}$ is determined by r linearly independent rows of A in the sense that $F = \{x: A'x = b'\}$ where $A'x \le b'$ is a subsystem of r inequalities from the system $Ax \le b$ and A' is a matrix of rank r (the rank of A' can be found in polynomial time, for instance, by Edmonds' [67] version of Gaussian elimination). For each such subset of r linearly independent inequalities, check whether $A'x = b'$ has a solution and, if so, whether this solution is in the polyhedron P - this can be done in polynomial time using the ellipsoid algorithm. If $A'x = b'$ has a solution and it is in P, then $F = \{x: A'x = b'\}$ is a minimal nonempty face of P and using the algorithm of Kannan and Bachem [79] it can be checked in polynomial time whether or not F contains integer points. In this way, it can be determined in polynomial time whether or not each minimal nonempty face of P contains integer points and hence whether or not P is an integer polyhedron. □

To obtain the corresponding results for totally dual integral systems Lemma 5.2 will be used together with the integer programming results of Section 7. The following result is given in Cook, Lovász, and Schrijver [82].

Theorem 9.3: The class of linear systems $Ax \le b$ with A integral that are totally dual integral is in co-NP.

Proof: If $Ax \le b$ is not totally dual integral then, by Lemma 5.2, there exists a minimal nonempty face F of $P = \{x: Ax \le b\}$ such that the active rows in A of F do not form a Hilbert basis. Suppose that $a_1,\dots,a_k$ are the active rows of F (using the ellipsoid algorithm it can be checked in polynomial time that $a_1,\dots,a_k$ are the active rows of a minimal face of P). Since $\{a_1,\dots,a_k\}$ is not a Hilbert basis, there exists an integer vector w in $\{\lambda_1 a_1+\dots+\lambda_k a_k: 0 \le \lambda_i \le 1, i = 1,\dots,k\}$ such that w cannot be expressed as a nonnegative integer combination of $a_1,\dots,a_k$ (see the proof of Theorem 5.1). By the results of Section 7, in time polynomial in the size of w and $\{a_1,\dots,a_k\}$, either an expression of w as a nonnegative integer combination of $a_1,\dots,a_k$ can be found or it can be shown that $\{a_1,\dots,a_k\}$ is not a Hilbert basis. Since the size of w is polynomial in the size of $\{a_1,\dots,a_k\}$ it can be proven in time polynomial in the size of $\{a_1,\dots a_k\}$ that $\{a_1,\dots,a_k\}$ is not a Hilbert basis. □

It is not necessary to use Hilbert bases in the above proof, since the theorem also follows from Theorem 7.2 by noting that if $Ax \le b$ is not totally dual integral then there exists an integer vector in $\{w: yA = w, 0 \le y \le 1\}$ such that $\min\{yb: yA = w, y \ge 0\}$ exists but cannot be achieved by an integer vector. The use of Hilbert bases however corresponds more naturally to the proof of Proposition 9.1. The following theorem and its proof are given in Cook, Lovász, and

Schrijver [82].

Theorem 9.4: For any fixed positive integer r, there exists a polynomial-time algorithm which, for any linear system $Ax \le b$ with A an integer matrix of rank r, determines whether or not $Ax \le b$ is totally dual integral.

Proof: The proof is split into three stages. First, the problem is reduced to determining whether or not a given set of integer vectors $\{a_1,\ldots,a_m\}$ which generate a cone of dimension r is a Hilbert basis. Next, the problem is reduced to the case that the cone generated by $\{a_1,\ldots,a_m\}$ is pointed and of full dimension. Finally, it is shown that Lenstra's integer programming algorithm [81] can be used to test in polynomial time whether or not $\{a_1,\ldots,a_m\}$ is a Hilbert basis.

Suppose that r is a fixed positive integer and that $Ax \le b$ is a linear system with A an integer matrix of rank r. By Lemma 5.2, it must be determined whether or not the active rows of each minimal nonempty face of $P = \{x: Ax \le b\}$ form a Hilbert basis. As in the proof of Proposition 9.2, the minimal nonempty faces of P can be found by considering each subset of r linearly independent rows of A. Once a minimal nonempty face F of P is found, the active rows in A of F can be found by finding a vector in F and checking which inequalities in $Ax \le b$ hold as an equality for that vector (since F is a minimal nonempty face, if an inequality in $Ax \le b$ holds as an equality for one vector in F then it holds as an equality for every vector in F). Since r is fixed, there are only polynomially many subsets of r rows of A to consider, so it suffices to prove that there exists a polynomial-time algorithm to test whether or not a a given set of integer vectors $\{a_1,\ldots,a_m\}$ is a Hilbert basis, where $\{a_1,\ldots,a_m\}$ generates a cone of dimension r (since there are r linearly independent vectors in $\{a_1,\ldots,a_m\}$).

Suppose that $\{a_1,\ldots,a_m\}$ is a set of integer vectors which generates a cone $C \subseteq \mathbb{R}^n$ of dimension r. Relabel the vectors in $\{a_1,\ldots,a_m\}$ if necessary so that $a_1,\ldots,a_k$ are those vectors a_i in $\{a_1,\ldots,a_m\}$ such that $-a_i \in C$. The vectors $a_1,\ldots,a_k$ generate a linear space F which is the unique minimal nonempty face of C. Let d be the dimension of F. Select d linearly independent vectors $v_1,\ldots,v_d$ from $\{a_1,\ldots,a_k\}$ and select $r-d$ vectors $v_{d+1},\ldots,v_r$ from $\{a_{k+1},\ldots,a_m\}$ such that $v_1,\ldots,v_r$ are linearly independent. Let V be the matrix with rows $v_1,\ldots,v_r$. Using the algorithm of Kannan and Bachem [79], find a unimodular matrix U such that VU is in Hermite normal form. The last n-d components of each vector in $\{v_1U,\ldots,v_dU\}$ are zero, so for each $i = 1,\ldots,k$ the last $n-d$ components of a_iU are zero. Similarly, the last n-r components of each vector in $\{a_{k+1}U,\ldots,a_mU\}$ are zero. Since $\{a_1,\ldots,a_m\}$ is a Hilbert basis if and only if $\{a_1U,\ldots,a_mU\}$ is a Hilbert basis, it can be assumed that $a_i = a_iU$ for $i = 1,\ldots,m$.

If w is a vector in F and $\gamma_1a_1+\ldots+\gamma_ma_m$ is an expression of w as a nonnegative linear combination of $a_1,\ldots,a_m$, then $\gamma_{k+1} = \gamma_{k+2} = \ldots = \gamma_m = 0$. Thus, if $\{a_1,\ldots,a_m\}$ is a Hilbert basis for C, then $\{a_1,\ldots,a_k\}$ is a Hilbert basis for F. For $i = k+1,\ldots,n$ let a_i' be the vector consisting of the components $d+1$ through r of a_i. Since the last $n-d$ components of each vector in $\{a_1,\ldots,a_k\}$ are zero, if $\{a_1,\ldots,a_m\}$ is a Hilbert basis then $\{a'_{k+1},\ldots,a'_m\}$ is a Hilbert basis. Conversely, it is easy to see that if $\{a_1,\ldots,a_k\}$ is a Hilbert basis for F and $\{a'_{k+1},\ldots,a'_m\}$ is a Hilbert basis, then $\{a_1,\ldots,a_m\}$ is a Hilbert basis. It follows that $\{a_1,\ldots,a_m\}$ is a Hilbert basis if and only if $\{a_1,\ldots,a_k\}$ is a Hilbert basis and $\{a'_{k+1},\ldots,a'_m\}$ is a Hilbert basis.

Since F is a linear space and F is the cone generated by $\{a_1,\ldots,a_k\}$, this set is a Hilbert basis for F if and only if each integer vector in F can be expressed as an integer combination of $a_1,\ldots,a_k$ (which can be checked in polynomial time using the algorithm of Kannan and Bachem [79]). As in the proof of Theorem 7.2, this can be seen as follows. The condition is clearly necessary. For the converse, suppose that the condition holds and that w is an integer vector in F. It must be shown that w can be expressed as a nonnegative integer combination of $a_1,\ldots,a_k$. Let $\lambda_1 a_1+\ldots+\lambda_k a_k$ be an expression of w as an integer combination of $a_1,\ldots,a_k$. By expressing $-a_i$ as a nonnegative linear combination of $a_1,\ldots,a_k$ (excluding a_i) for $i = 1,\ldots,k$ find positive $\gamma_1,\ldots,\gamma_k$ such that $0 = \gamma_1 a_1+\ldots+\gamma_k a_k$. There exists an integer M such that $(\lambda_1+M\gamma_1)a_1+\ldots+(\lambda_k+M\gamma_k)a_k$ is an expression of w as a nonnegative integer combination of $a_1,\ldots,a_k$.

Since $\{a'_{k+1},\ldots,a'_m\}$ generates a pointed cone of full dimension (which is less than or equal to r), it can be assumed that $\{a_1,\ldots,a_m\}$ generates a pointed cone of full dimension.

Now for the final stage of the proof. The set $\{a_1,\ldots,a_m\}$ is a

Hilbert basis for C if and only if the only integer vector in

(9.1) $$C_0 = \{w \in C: w-a_i \notin C,\ i = 1,\ldots,m\}$$

is the zero vector. This can be seen as follows. Suppose that $\{a_1,\ldots,a_m\}$ is a Hilbert basis for C and that w is an integer vector in C_0. Since w is in C, $w = \lambda_1 a_1+\ldots+\lambda_m a_m$ for some nonnegative integers $\lambda_1,\ldots,\lambda_m$. But $w-a_i \notin C$ for $i = 1,\ldots,m$, so $\lambda_i = 0$ for $i = 1,\ldots,m$ and $w = 0$. Conversely, suppose that the only integer vector in C_0 is 0 and that w is an integer vector in C. Let λ_1 be the greatest rational such that $w-\lambda_1 a_1$ is in C (λ_1 exists, since it has been assumed that C is pointed) and let $w^1 = w-\lfloor\lambda_1\rfloor a_1$. Let λ_2 be the greatest rational such that $w^1-\lambda_2 a_2$ is in C and let $w^2 = w^1-\lfloor\lambda_2\rfloor a_2$. Continue in this manner until w^m is found. Now w^m is an integer vector in C_0, so $w^m = 0$. So $\lfloor\lambda_1\rfloor a_1+\ldots+\lfloor\lambda_m\rfloor a_m$ is an expression of w as a nonnegative integer combination of $a_1,\ldots,a_m$.

The cone C is a polyhedron of full dimension. The facet-inducing inequalities for C can be found as follows. For each subset of $r-1$ linearly independent vectors from $a_1,\ldots,a_m$ find the hyperplane H which contains these vectors and the origin. If each vector in $a_1,\ldots,a_m$ lies in one of the closed half spaces determined by H, then $H \cap C$ is a facet of C. In this way (since r is fixed), vectors $b_1,\ldots,b_t$ such that $C = \{x: b_i x \geq 0,\ i = 1,\ldots,t\}$ can be found in polynomial time.

It follows from the definition of C_0 that

$$C_0 = \{w \in C\text{: for each } i = 1,\ldots,m \text{ there exists } j,\ 1 \le j \le t, \text{ such that } b_j w < b_j a_i\}. \tag{9.2}$$

Thus, if Φ denotes the collection of all functions ϕ from $\{1,\ldots,m\}$ to $\{1,\ldots,t\}$, then

$$C_0 = \bigcup_{\phi\in\Phi} \{x\colon b_j x \ge 0 \text{ for } j = 1,\ldots,t \text{ and } b_{\phi(i)} x < b_{\phi(i)} a_i \text{ for } i = 1,\ldots,m\}. \tag{9.3}$$

Using (9.3) a concise description of C_0 will now be found.

Let $w \in C_0$. By (9.3), there exists a function ϕ in Φ such that w is contained in

$$P = \{x\colon b_i x \ge 0 \text{ for } i = 1,\ldots,t \text{ and } b_{\phi(i)} a_i < b_{\phi(i)} a_i \quad i=1,\ldots,m\}. \tag{9.4}$$

Since C_0 is bounded, P is bounded. So the polyhedron $\bar{P}$ obtained by replacing each "<" by "≤" in (9.4) is bounded (P is nonempty since $w \in P$, so $\bar{P}$ is the closure of P). There exists an $\varepsilon > 0$ such that $(1+\varepsilon)w$ is contained in P (and, so, also in $\bar{P}$). Now since $0 \in \bar{P}$, there exists linearly independent vertices $z_1,\ldots,z_r$ of $\bar{P}$ such that $(1+\varepsilon)w$ can be expressed as a convex combination of 0 and $z_1,\ldots,z_r$. So w can be expressed as a convex combination of 0 and $z_1,\ldots,z_r$ with the multiplier of 0 being positive. Since $0,z_1,\ldots,z_r$ are affinely independent, this expression of w as a convex combination of $0,z_1,\ldots,z_r$ is unique. Thus, $w \in$ convex hull $\{0,z_1,\ldots,z_r\}\backslash$convex hull $\{z_1,\ldots,z_r\}$. This observation can be used as follows.

Let Z be the sets of all vectors z determined by r linearly independent equations from

$$(9.5)\qquad \begin{array}{ll} b_j z = 0 & j = 1,\dots,t \\ b_j z = b_j a_i & j = 1,\dots,t; \quad i = 1,\dots,m \end{array}$$

such that z also satisfies $b_j z \geq 0$, $j = 1,\dots,t$. Let S be the set of all subsets $\{z_1,\dots,z_r\}$ of Z such that $z_1,\dots,z_r$ are linearly independent and there exists a function $\phi \in \Phi$ such that for every $k = 1,\dots,r$ we have $b_{\phi(i)} z_k \leq b_{\phi(i)} a_i$ (since r is fixed, both Z and S can be found in polynomial time). For each $\{z_1,\dots,z_r\}$ in S, let $\sigma(z_1,\dots,z_r)$ = convex hull $\{0,z_1,\dots,z_1\}\backslash$convex hull$\{z_1,\dots,z_r\}$.

<u>Claim</u>: $C_0 = \bigcup_{\{z_1,\dots,z_r\} \in S} \sigma(z_1,\dots,z_r)$.

Once this claim has been established the proof will be complete, since the integer programming algorithm of Lenstra [81] can be used to test for each $\{z_1,\dots,z_r\} \in S$ whether or not 0 is the only integer vector in $\sigma(z_1,\dots,z_r)$ in polynomial time. Indeed, since $z_1,\dots,z_r$ are linearly independent, $z_1,\dots,z_r,0$ are affinely independent. Thus, if H is a hyperplane which contains $\{z_1,\dots,z_r\}$, then 0 is not contained in H (recall that the dimension of the space we are working in is r). Let $H = \{x\colon gx = p\}$ be such a hyperplane. It follows that $\sigma(z_1,\dots,z_r)$ = convex hull$\{0,z_1,\dots,z_r\}\backslash H$. It may be assumed that g is integral and that p is a postive integer. Each vector in $\sigma(z_1,\dots,z_r)$ satisfies $gx < p$, so each integer point in $\sigma(z_1,\dots,z_r)$ satisfies $gx \leq p-1$. Letting $Bx \leq f$ be a defining system for the convex hull of $\{0,z_1,\dots,z_1\}$, it only needs to be checked that the only integer vector in $T = \{x: Bx \leq f, gx \leq p-1\}$ is 0. One way to do this in polynomial time, using Lenstra's integer programming algorithm, is to solve the integer programs $\max\{x_i: x \in T, x \text{ integer}\}$ and $\min\{x_i: x \in T, x \text{ integer}\}$ for $i = 1,\ldots,r$. If 0 is the optimal solution for each of these linear programs, then the only integer vector in T is 0.

The "$\subseteq$" inclusion in the claim has already been shown. To see the "$\supseteq$" inclusion, let $w \in \sigma(z_1,\ldots,z_r)$ for some $\{z_1,\ldots,z_r\} \in S$. By definition of Z, w is in C. Since the multiplier of 0 in the expression of w as a convex combination of $0,z_1,\ldots,z_r$ is positive, there exists an $\varepsilon > 0$ such that $(1+\varepsilon)w$ is in $\sigma(z_1,\ldots,z_r)$. By the definition of S, there exists a function $\phi \in \Phi$ such that $b_{\phi(i)}z_k \leq b_{\phi(i)}a_i$ for $i = 1,\ldots,m$ and $k = 1,\ldots,r$. Suppose that $b_{\phi(i)}a_i \leq 0$ for some i, $1 \leq i \leq m$. Since $b_jz_k \geq 0$ for all $j = 1,\ldots,m$ and all $k = 1,\ldots,r$, $b_{\phi(i)}a_i = 0$. So $b_{\phi(i)}z_k = 0$ for $k = 1,\ldots,r$, which implies that $b_{\phi(i)} = 0$, since $z_1,\ldots,z_r$ are linearly independent. Since this is impossible, it must be the case that $b_{\phi(i)}a_i > 0$ for all $i = 1,\ldots,m$. Thus, $(1+\varepsilon)b_{\phi(i)}w \leq b_{\phi(i)}a_i$ for $i = 1,\ldots,m$ (which follows from the fact that $(1+\varepsilon)w \in \sigma(z_1,\ldots,z_r)$) implies that $b_{\phi(i)}w < b_{\phi(i)}a_i$ for $i = 1,\ldots,m$. So w is in C_0, which completes the proof of the claim. □

An immediate consequence of this theorem is that for any fixed r, there exists a polynomial-time algorithm to test whether or not a set of integer vectors $\{h_1,\ldots,h_k\}$ is a Hilbert basis if the dimension of the cone generated by $\{h_1,\ldots,h_k\}$ is less than or equal to r. Also, it follows from the second stage of the above proof that if the cone generated by a set of integer vectors $\{h_1,\ldots,h_k\}$ is a linear space, then it is possible to test in polynomial time whether or not $\{h_1,\ldots,h_k\}$ is a Hilbert basis (no assumption on the dimension of the linear space is needed).

It should be noted that Chandrasekaran and Shirali [82] obtained a polynomial-time algorithm to test whether or not $\{Ax = 0, x \geq 0\}$ is totally dual integral, where A is an integral matrix of fixed rank.

As in the case of integer polyhedra, it is an open question whether or not the class of totally dual integral linear systems is in NP. One possible application of an affirmative answer to either this question or the corresponding question for integer polyhedra would be that for any fixed polynomial f the class of perfect graphs with less than or equal to $f(n)$ maximal cliques, where n is the number of nodes in the graph, is in NP (for appropriate choices of f, classes of perfect graphs that contain many well known classes can be obtained in this way). See Burlet [81], Burlet and Fonlupt [82], Burlet and Uhry [82], Cameron [82], and Chvátal [81] for work on recognizing perfect graphs.

A problem related to the recognition of totally dual integral systems will now be considered.

If P is a polyhedron of full dimension then a theorem of Schrijver [81] presented in Section 5 implies that there is a "best possible" min-max theorem for the maximum weight of a vector in P where the minimum is taken over all subsets a certain set of integer vectors, such that the subset satisfies some condition (such as a covering condition). Is it possible to test whether or not a given integer vector is in the set? In other words, given a polyhedron P of full dimension, is a given inequality in the Schrijver system for P (recall that the Schrijver system is the unique minimal totally dual integral system with integer left hand sides that defines P)? Once again, if P is defined by a linear system $Ax \le b$ with A a matrix of bounded rank, then this recognition question can be settled in polynomial time.

Theorem 9.5: Let r be a fixed positive integer. There exists a polynomial-time algorithm which, for any linear system $Ax \le b$ with A a matrix of rank r which defines a polyhedron P of full dimension and any inequality $\alpha x \le \beta$, determines whether or not $\alpha x \le \beta$ is in the Schrijver system for P.

Proof: Suppose that r is a fixed positive integer, that $Ax \le b$ defines a polyhedron P of full dimension in $\mathbb{Q}^n$ and that $\alpha x \le \beta$ is an inequality. By multiplying the system by a positive constant if necessary, it can be assumed that A is integral. Also, it can be assumed that α is integral since otherwise $\alpha x \le \beta$ is not in the Schrijver system for P.

If $\max\{\alpha x\colon Ax \le b\}$ is not equal to β, then clearly $\alpha x \le \beta$ is not in the Schrijver system for P (this can be checked in polynomial time with the ellipsoid algorithm). Suppose that this maximum is equal to β. Usng the ellipsoid algorithm, find the active rows in A for some minimal face F of P such that $\max\{\alpha x\colon Ax \le b\}$ is achieved by each vector in F. Let $a_1,\ldots,a_n$ be the active rows for F and let C be the pointed cone generated by $a_1,\ldots,a_m$. Now

$\alpha x \leq \beta$ is in the Schrijver system for P if and only if α is in the unique minimal Hilbert basis for C.

Let $v_1,\ldots,v_r$ be r linearly independent vectors chosen from $a_1,\ldots,a_m$ and let V be the matrix with rows $v_1,\ldots,v_r$. Using the algorithm of Kannan and Bachem [79], find a unimodular matrix U such that VU is in Hermite normal form. As in the proof of Theorem 9.4 the last $n-r$ components of each vector in $\{a_1U,\ldots,a_mU\}$ are zero. Since α is in the unique minimal Hilbert basis for C if and only if αU is in the unique minimal Hilbert basis for the cone generated by $\{a_1U,\ldots,a_mU\}$, it can be assumed that $a_i = a_iU$ for $i = i,\ldots,m$ and $\alpha = \alpha U$. Now since the last $n-r$ components of each vector in $\{a_1U,\ldots,a_mU\}$ are zero, it can be assumed that C is of full dimension.

Let w be a vector such that $wz > 0$ for each nonzero z in C (such a vector w can be found in polynomial time). Let $P^1 = \{z \in C: wz \leq w\alpha\}$ and let $P^2 = \{\alpha-z: z \in P^1\}$. Now α is in the unique minimal Hilbert basis for C if and only if the only integer vectors in $P^1 \cap P^2$ are 0 and α. This can be seen as follows. If z^1 is an integer vector in $P^1 \cap P^2$ other than 0 and α, then z^1 is in C and there exists an integer vector z^2 in C such that $z^1 = \alpha - z^2$. So $z^1 + z^2$ is an expression of α as a nonnegative integer combination of other integer vectors in C Conversely, if α is not in the unique minimal Hilbert basis for C then there exist integer vectors z^1 and z^2 in $P^1 \backslash \{0,\alpha\}$ such that $\alpha = z^1 + z^2$. So z^1 and z^2 are integer vectors in $P^1 \cap P^2$ other than 0 and α.

Since r is fixed, with Lenstra's integer programming algorithm it can be checked in polynomial time whether or not the only integer vectors in $P^1 \cap P^2$ are 0 and α. □

It should be noted that, even for linear systems $Ax \leq b$ with A a matrix of fixed rank, the number of inequalities in the Schrijver system for the polyhedron defined by $Ax \leq b$ is not bounded by a polynomial in the size of $Ax \leq b$.

Edmonds, Lovász, and Pulleyblank [82] have shown that given a "proper" encoding of a polyhedron P and an optimization oracle for P, it can be determined in polynomial time whether or not a given inequality $\alpha x \leq \beta$ is facet inducing for P (see also Grötschel, Lovász, and Schrijver [81,82]). Again, it would be interesting if a similar result for minimal totally dual integral defining systems with integer left hand sides could be obtained.

10. Proving total dual integrality with cross-free families.

To this point, very little has been mentioned on the application of total dual integrality to proving combinatorial min-max theorems. Of course, some classic min-max results, such as the König-Egerváry Theorem, follow from the result of Hoffman and Kruskal [56] on total unimodularity given in Section 3, but the use of total unimodularity is somemwhat limited. In this final section of this chapter a well known technique for proving that combinatorially described linear systems are totally dual integral is presented. This technique has been quite successful in proving many combinatorial results which do not follow directly from the Hoffman-Kruskal Theorem. It involves restricting the set of nonzero dual variables by "uncrossing" them and showing that if no dual variables "cross" then the existence of an integer optimal solution to the dual linear program is implied by the Hoffman-Kruskal Theorem. The technique has evolved over a number of years. The idea of using "uncrossing" to prove min-max theorems appears in Younger [69]. Combining "uncrossing" with the Hoffman-Kruskal Theorem was done by Robertson (see Lovász [76]) and Johnson [75] (cf. Edmonds [70]). Bringing total dual integrality into this scheme for proving min-max results is due to Hoffman [74], Edmonds and Giles [77], and Hoffman and Schwartz [78] and developing a general framework for proving theorems in this way is due to Schrijver [82a].

Some of the many interesting general min-max theorems that have been proven by the above mentioned technique are given in Edmonds and Giles [79], Frank [79,81a], Frank and Tardos [81], Gröflin and Hoffman [82], Hassin [78,82], Hoffman [76a,78], Hoffman and Schwartz [78], Lawler [82], Lawler and Martel [82], and Schrijver [82,83] (see Schrijver [82b] for the interrelations of these models). Algorithmic aspects of these models are dealt with in Frank [81,82], Cunningham and Frank [82], Fujishige [78], Lucchesi [76], Schösleben [80], and Zimmerman [82].

Let V be a finite set and let $\mathcal{F}$ be a family of subsets of V. For each $f \in \mathcal{F}$ let $a_f x \le b_f$ be a linear inequality, where x is a vector. The linear system

$$(10.1) \qquad \begin{aligned} &a_f x \le b_f \quad \forall f \in \mathcal{F} \\ &x \ge 0 \end{aligned}$$

is totally dual integral if for each integer vector w such that

$$(10.2) \qquad \begin{aligned} &\min \Sigma\{y_f b_f : f \in \mathcal{F}\} \\ &\text{s.t. } \Sigma\{y_f a_f : f \in \mathcal{F}\} \ge w \\ &\qquad y_f \ge 0 \quad \forall f \in \mathcal{F} \end{aligned}$$

exists, the minimum can be achieved by an integer vector. It is a simple observation that if for some $\mathcal{F}' \subseteq \mathcal{F}$ there exists an optimal solution $\bar{y}$ to (10.2) with $\bar{y}_f = 0$ for each $f \in \mathcal{F}\backslash\mathcal{F}'$, then any vector y' which achieves the minimum in

$$(10.3) \qquad \begin{aligned} &\min \Sigma\{y_f b_f : f \in \mathcal{F}'\} \\ &\text{s.t. } \Sigma\{y_f a_f : f \in \mathcal{F}'\} \ge w \\ &\qquad y_f \ge 0 \quad \forall f \in \mathcal{F}' \end{aligned}$$

can be extended to an optimal solution to (10.2) by setting $y^*_f = 0$ for each $f \in \mathcal{F}\backslash\mathcal{F}'$. This observation suggests a method for proving that (10.1) is totally dual integral: For each integral w such that (10.2) exists, find a family $\mathcal{F}' \subseteq \mathcal{F}$ as above such that (10.3) can be achieved by an integer vector. Such a program will now be described.

The program involves "uncrossing" and will be illustrated with several examples once a lemma which sets up the technique is proven. (The lemma is implicit in the papers cited above.)

Let $\mathcal{B}$ be some binary relation on $\mathcal{F}$ and say that $A,B \in \mathcal{F}$ form a <u>$\mathcal{B}$-intersection</u> if $A \nsubseteq B$, $B \nsubseteq A$, and (A,B) satisfies $\mathcal{B}$ ($\mathcal{B}$ could be that $A \cap B \neq \emptyset$, for instance). Consider the following conditions on (10.1):

(I) If $A,B \in \mathcal{F}$ form a $\mathcal{B}$-intersection then

(i) $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$

(ii) $b_{A\cap B} + b_{A\cup B} \leq b_A + b_B$

(iii) $a_{A\cap B} + a_{A\cup B} \geq a_A + a_B$. ((iii) is equivalent to

$$\text{if } \bar{y} \geq 0 \text{ then } \Sigma\{y'_f a_f: f \in \mathcal{F}\} \geq \Sigma\{\bar{y}_f a_f: f \in \mathcal{F}\},$$

where $y'_A = \bar{y}_A - \varepsilon$, $y'_B = \bar{y}_B - \varepsilon$, $y'_{A\cap B} = \bar{y}_{A\cap B} + \varepsilon$, $y'_{A\cup B} = \bar{y}_{A\cup B} + \varepsilon$,

$y'_D = \bar{y}_D$ for all D in $\mathcal{F}\backslash\{A,B,A\cap B,A\cup B\}$, and $\varepsilon = \min\{\bar{y}_A,\bar{y}_B\}$.)

(II) If $\mathcal{F}' \subseteq \mathcal{F}$ and no two sets in $\mathcal{F}'$ form a $\mathcal{B}$-intersection, then (10.3) has an integer optimal solution for each integer vector w for which it has an optimal solution. (For example, the coefficient matrix of (10.3) may be totally unimodular.)

<u>Lemma 10.1</u>: If $\mathcal{F},\mathcal{B}$, and (10.1) satisfy (I) and (II), then (10.1) is totally dual integral.

<u>Proof:</u> Let w be an integer vector such that (10.2) exists. For any solution y to (10.2) let $g(y) = \Sigma\{|f|\cdot|V\backslash f|y_f: f \in \mathcal{F}\}$. Let $\bar{y}$ be an optimal solution to (10.2) and let t be the least common denominator of $\{y_f: f \in \mathcal{F}\}$. Let $\mathcal{F}' = \{f \in \mathcal{F}: y_f > 0\}$. Suppose that there exists $A,B \in \mathcal{F}'$ that form a $\mathcal{B}$-intersection. By I-(i), $A\cup B \in \mathcal{F}$ and $A\cap B \in \mathcal{F}$. Define y' as in I-(iii). By I-(ii) and I-(iii), y' is an optimal

solution to (10.2). It can be checked that $g(y') \leq g(\bar{y})-1/t$, and that t is still a common denominator of $\{y'_f: f \in F\}$. Repeating this procedure, a solution can be found such that no B-intersections occur in F'. Now, by II, there exists an integer optimal solution to (10.3) So, by the observation made above, (10.2) can be achieved by an integer vector. □

Of course, a similar result holds for systems of the form

$$(10.1)' \qquad \begin{array}{l} a_f x \geq b_f \quad \forall f \in F. \\ x \geq 0 \end{array}$$

Indeed, letting (I)' be (I) with the "≤" sign in I-(ii) replaced by "≥" and with the "≥" sign in "$\Sigma\{y'_f a_f: f \in F\} \geq \Sigma\{\bar{y}_f a_f: f \in F\}$" replaced by "≤", it follows that if F, B, and (10.1)' satisfy (I)' and (II) then (10.1)' is totally dual integral.

Condition I-(ii) indicates that submodular functions fall naturally into theorems proven using Lemma 10.1. To illustrate this, the first example that will be considered is the Matroid Intersection Theorem of Edmonds [70].

Theorem 10.2: Let $M_i = (E, I_i)$ be a matroid for $i = 1,2$ and let $r(S) = \min\{r_1(S), r_2(S)\}$ for each $S \subseteq E$, where r_i is the rank function of M_i. If $w = (w_e: e \in E)$ is a nonnegative integer vector then the maximum value of $\Sigma\{w_e: e \in S\}$ where $S \in I_1 \cap I_2$ is equal to the minimum value of $r(A_1)+\ldots+r(A_k)$ where $A_1, \ldots, A_k$ are subsets of E(repetition is permitted) such that each $e \in E$ is in at least w_e of these subsets.

Proof: Letting $\mathcal{F}$ be the family of all subsets of E, $b_S = r(S)$ for all $S \subseteq E$, and a_S the 0-1 incidence vector of S for all $S \subseteq E$, the theorem is equivalent (by Corollary 2.6) to the total dual integrality of (10.1). For $A,B \subseteq E$, let $\mathcal{B}$ be the property that $A \cap B \neq \emptyset$ and for some $i \in \{1,2\}$, $r(A) = r_i(A)$ and $r(B) = r_i(B)$. It is easy to check that (10.1) satisfies (I). Now suppose that no two sets in $\mathcal{F}' \subseteq \mathcal{F}$ $\mathcal{B}$-intersect and let $\mathcal{F}_i = \{S \subseteq \mathcal{F}' : r(S) = r_i(S)\}$ for $i = 1,2$. By Theorem 3.5, the incidence matrix of the family $\mathcal{F}_1 \cup \mathcal{F}_2$ is totally unimodular, which implies that the incidence matrix of (10.3) is totally unimodular. So (10.1) satisfies (II) and the result follows from Lemma 10.1. □

Trivially, the above also shows the following well known extension. If for $i = 1,2$, $\mathcal{F}_i$ is a family of subsets of E such that if $A,B \in \mathcal{F}_i$ and $A \cap B \neq \emptyset$ then $A \cap B \in \mathcal{F}_i$ and $A \cup B \in \mathcal{F}_i$ and r_i is a rational valued function on $\mathcal{F}_i$ which is submodular on intersecting pairs then the system

$$\Sigma\{x_e : e \in S\} \leq r_1(s) \quad \forall\, S \in \mathcal{F}_1$$

$$(10.4) \qquad \Sigma\{x_e : e \in S\} \leq r_2(s) \quad \forall\, S \in \mathcal{F}_2$$

$$x_e \geq 0 \quad \forall\, e \in E$$

is totally dual integral.

Several examples involving directed graphs will now be considered. A directed graph is a graph where each edge has one end specified as the <u>head</u> and the other specified as the <u>tail</u>. An edge enters a set S if its head is in S and its tail is not in S.

Let G be a directed graph and let $r \in VG$. A set $B \subseteq EG$ is called a <u>branching</u> rooted at r if B forms a spanning tree and every node in $V\setminus\{r\}$ is the head of eactly one edge in B. Alternatively, a branching can be described as a minimal set of edges such that each non-empty $S \subseteq VG\setminus\{r\}$ is entered by at least one edge in the set. If $w = (w_e : e \in EG)$ is a vector, say that the "weight" of a branching B is $\Sigma\{w_e : e \in B\}$. Fulkerson [74] proved the following min-max relation.

<u>Theorem 10.3:</u> Let G be a directed graph and let $r \in VG$. Given a nonnegative integral weight vector $w = (w_e : e \in EG)$ the minimum weight of a branching rooted at r is equal to the maximum number of nonempty subsets of $VG\setminus\{r\}$, with repetition allowed, such that every edge e enters no more than w_e of these subsets.

<u>Proof:</u> Letting $\mathcal{F}$ be the set of all nonempty subsets of $VG\setminus\{r\}$, $b_S=1$ for all $S \in \mathcal{F}$, and $a_S = (a_{S_e} : e \in EG)$, where $a_{S_e} = 1$ if e enters S and $a_{S_e} = 0$ otherwise, for all $S \in \mathcal{F}$; all that needs to be shown is that (10.1)' is totally dual integral. For $A,B \in \mathcal{F}$, let $\mathcal{B}$ be the property that $A \cap B \neq \emptyset$. Again, it is easy to check that (10.1)' satisfies (I)'. Suppose that no two sets in $\mathcal{F}' \subseteq \mathcal{F}$ $\mathcal{B}$-intersect and let M be the matrix with rows $(a_S : S \in \mathcal{F}')$. Since $b_S = 1$ for all $S \in \mathcal{F}$, to show (II) it suffices, by Theorem 3.6, to show that M is balanced. Suppose that M is not balanced. There must exist $e_0,\ldots,e_{k-1} \in EG$ and $A_0,\ldots,A_{k-1} \in \mathcal{F}'$ such that for $i = 0,\ldots,k-1$ e_i enters A_i and A_{i+1}, but does not enter A_j for $j \neq i, i+1$ (all subscripts

should be taken modulo k), where $k \geq 3$ is odd. Since no $\mathcal{B}$-intersections occur in $\mathcal{F}'$, either $A_0 \subset A_1$ or $A_1 \subset A_0$. Since the ordering is not important, it can be assumed that $A_1 \subset A_0$. But this implies that $A_j \subset A_{j-1}$ for $j = 2,\ldots,k$, a contradiction. So M is balanced and (II) is satisfied. □

Actually, the matrix M given above is totally unimodular (cf. Schrijver [83]), but the proof that M is balanced is quite simple and illustrates the use of Theorem 3.6. Again, the above proof also shows the extension due to Frank [79] that, for any directed graph G, if $\mathcal{F}$ is a family of subsets VG such that if $A,B \in \mathcal{F}$ and $A \cap B \neq \emptyset$, then $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$, then the linear system

$$(10.5)\qquad \begin{aligned} &\Sigma\{x_e : e \text{ enters } S\} \geq 1 \quad \forall\, S \in \mathcal{F} \\ &\qquad x_e \geq 0 \quad \forall e \in EG \end{aligned}$$

is totally dual integral. (This is a special case of Frank's result, since he allows the right hand side of (10.5) to be any supermodular function.) This extension includes a theorem of Fulkerson [68] on the minimum length of a directed path between two specified nodes in a directed graph. A related result on the length of a longest directed path in an acylic directed graph is given in Vidyasankar and Younger [75].

As a final example of the use of the "uncrossing" technique, a proof of the Lucchesi-Younger Theorem, on directed cuts, will be given. A directed cut of a directed graph G is a nonempty set of edges that is the set of edges entering some set $S \subseteq VG$ such that no edge enters $VG \backslash S$.

The following result is due to Lucchesi and Younger [78].

Theorem 10.4: Let G be a directed graph. The maximum number of disjoint directed cuts of G is equal to the minimum cardinality of a set of edges that meets every directed cut of G.

Proof: It may be assumed that there is a path, not necessarily directed, between each pair of nodes of G. Let $\mathcal{F}$ be the set of all those subsets S of VG such that no edges enters $VG\setminus S$ and at least one edge enters S, let $b_S = 1$ for all $S \in \mathcal{F}$, and let $a_S = (a_{S_e} : e \in EG)$ where $a_{S_e} = 1$ if e enters S and $a_{S_e} = 0$ otherwise, for all $S \in \mathcal{F}$. Again, all that needs to be shown is that (10.1)' is totally dual integral. Let $\mathcal{B}$ be the property that $A \cap B \neq \emptyset$ and $A \cup B \neq V$ for $A, B \in \mathcal{F}$. As before, it is easily checked that (10.1)' satisfies (I)'. Now suppose that no two sets in $\mathcal{F}' \subseteq \mathcal{F}$ $\mathcal{B}$-intersect and let M be the matrix with rows $(a_S : S \in \mathcal{F}')$. It again suffices to show that M is balanced. Suppose that there exist $a_0, \ldots, a_{k-1} \in EG$ and $A_0, \ldots, A_{k-1} \in \mathcal{F}'$ as in the preceding proof. For each $i = 0, \ldots, k-1$, either $A_i \subset A_{i+1}$ or $A_i \supset A_{i+1}$. Since k is odd, there exists an i such that $A_i \subset A_{i+1} \subset A_{i+2}$ or $A_i \supset A_{i+1} \supset A_{i+2}$. The ordering of the sets is not important, so it may be assumed that $A_0 \supset A_1 \supset A_2$. Since it cannot be the case that $A_2 \supset A_3 \supset \ldots \supset A_{k-1} \supset A_0$, it may be assumed that $A_2 \subset A_3$. Since the tail of a_1 must be in A_3, $V\setminus A_3 \subseteq A_1 \setminus A_2$. But then for all $j = 3, \ldots, k-1$, $A_j \subseteq A_1 \setminus A_2$ or $V \setminus A_j \subseteq A_1 \setminus A_2$, a contradiction since a_{k-1} must enter only A_0 and A_{k-1}. □

As in the proof of Theorem 10.3, it can be shown that the matrix M given above is totally unimodular (see Edmonds and Giles [77]).

For a general framework for proving min-max results using the "uncrossing" technique see Schrijver [82a].

It should be noted that there is class of min-max theorems for directed graphs related to the above min-max results which involve "packing", such as the theorem of Edmonds [72] on the maximum number of disjoint branchings in a directed graph, and which do not seem to fit into the "uncrossing" scheme given above. For other results of this type see Feofiloff and Younger [82], Frank [79], Hoffman and Schwartz [77], and Schrijver [82,82b,83].

CHAPTER III

MINIMAL SYSTEMS FOR MATCHING PROBLEMS

1. Introduction.

A matching in a graph G is a subset of the edges such that each node of G is met by at most one edge in the subset. Fundamental results in matching theory were proven by Tutte [47,52,54]. Tutte's results provide a min-max relation for the cardinality of a largest matching in a graph (see Berge [58]). Edmonds [65] described a linear system which defines the convex hull of the matchings of a graph. In this chapter, "best possible" min-max relations for various matching problems are obtained by finding minimal totally dual integral systems for the relevant convex hulls as outlined in Section 5 of Chapter 2.

In Section 2, a connection between minimal totally dual integral systems and a type of separability is presented, together with a description of the unique minimal totally dual integral defining system for the convex hull of the matchings of a graph due to Cunningham and Marsh [78]. A result of Cook [81] and Pulleyblank [81] which describes the minimal totally dual integral system for b-matchings is presented in Section 3. Sections 4 and 5 contain a description of the minimal totally dual integral systems for simple b-matchings and capacitated b-matchings. The minimal totally dual integral system for triangle-free 2-matchings is descibed in Section 6.

For completeness, a short proof, due to Schrijver [81a], of a result that characterizes a totally dual integral defining system for the convex hull of the matchings of a graph is presented in Section 2. Also, detailed proofs of results that characterize totally dual integral defining systems for b-matchings, simple b-matchings, and capacitated b-matchings are given, based on the characterization for matchings and constructions of Tutte [52,54] (see Schrijver [82c]).

Two variations of matchings not dealt with here are matchings on bidirected graphs and T-joins. For min-max results on matchings on bidirected graphs see Edmonds and Johnson [70], Green-Krótki [80], and Schrijver [82c]. For min-max results on T-joins and T-cuts see Edmonds and Johnson [73], Korach [82], Lovász [75], and Seymour [77,79,81].

2. Matchings and separability.

Let G be a graph. A matching M of G will be identified with its incidence vector $x = (x_e: e \in EG)$, where $x_e = 1$ if $e \in M$ and $x_e = 0$ if $e \in EG \setminus M$. The fundamental result in the study of polyhedral aspects of matching theory was proven by Edmonds [65]:

Theorem 2.1: Let G be a graph. A defining system for the convex hull of the matchings of G is

$$(2.1) \qquad \begin{array}{ll} x_e \geq 0 & \forall e \in EG \\ x(\delta(v)) \leq 1 & \forall v \in VG \\ x(\delta(S)) \leq \lfloor \frac{1}{2}|S| \rfloor & \forall S \subseteq VG,\ |S| \text{ odd.} \end{array} \qquad \square$$

Edmonds [65] proved this result by means of a polynomial-time algorithm for the weighted matching problem, known as the blossom algorithm (the weighted matching problem is: for a given graph G and weight vector $w = (w_e: e \in EG)$, maximize wx over all matchings of G). Other proofs of this result are given in Balinski [72], Hoffman and Oppenheim [78], Lovász [79], Schrijver [81a], and Seymour [79].

Cunningham and Marsh [78] proved the following result which, together with Corollary 2.5 of Chapter 2, implies the result of Edmonds given above.

Theorem 2.2: For each graph G, the linear system (2.1) is totally dual integral.

To prove this theorem it suffices to show that for each graph G and each nonnegative integer vector w the linear program

$$\text{(2.2)}\qquad \begin{aligned} \min\ & \Sigma\{y_v : v \in VG\} + \Sigma\{\tfrac{1}{2}(|S|-1)Y_S : S \in \mathcal{S}\} \\ \text{s.t.}\ & y(\Psi(e)) + Y(\mathcal{S}(e)) \geq w_e \quad \forall\, e \in EG \\ & y_v \geq 0 \quad \forall\, v \in VG \\ & Y_S \geq 0 \quad \forall\, S \in \mathcal{S} \end{aligned}$$

where $\mathcal{S} = \{S \subseteq VG: |S| \geq 3$ and $|S|$ is odd$\}$, has an integer optimal solution. The following lemma is a well-known result that follows from Edmonds' blossom algorithm. (The construction given here is due to Pulleyblank [73] and Schrijver and Seymour [77] and will be used in Section 3.)

<u>Lemma 2.3:</u> For each graph G and each vector w, there exists an optimal solution (y,Y) to (2.2) such that if $A,B \in \mathcal{S}$ with $A \cap B \neq \emptyset$ and $A \nsubseteq B \nsubseteq A$, then either $Y_A = 0$ or $Y_B = 0$.

<u>Proof:</u> Let G be a graph and w a vector. Let (y,Y) be an optimal solution to (2.2). Suppose that there exist sets A and B in $\mathcal{S}$ with $Y_A \geq Y_B > 0$, $A \cap B \neq \emptyset$, and $A \nsubseteq B \nsubseteq A$. If $|A \cap B|$ is odd, let $\bar{Y}_{A\cap B} = Y_{A\cap B}+Y_B$, $\bar{Y}_{A\cup B} = Y_{A\cup B}+Y_B$, $\bar{Y}_A = Y_A - Y_B$, $\bar{Y}_B = 0$, $\bar{Y}_S = Y_S$ for all S in $\mathcal{S}\setminus\{A,B,A\cup B,A\cap B\}$, and $\bar{y} = y$. If $|A\cap B|$ is even, let $\bar{Y}_{A\setminus B} = Y_{A\setminus B}+Y_B$, $\bar{Y}_{B\setminus A} = Y_{B\setminus A}+Y_B$, $\bar{Y}_A = Y_A - Y_B$, $\bar{Y}_B = 0$, and $\bar{Y}_S = Y_S$ for all S in $\mathcal{S}\setminus\{A,B,A\setminus B,B\setminus A\}$ and let $\bar{y}_v = y_v + Y_B$ for all $v \in A\cap B$ and $\bar{y}_v = y_v$ for all $v \in VG\setminus(A\cap B)$. In either case, $(\bar{y},\bar{Y})$ is an optimal solution to (2.2). Also, following Schrijver and Seymour [77], in either case

$$\text{(2.3)}\qquad \Sigma\{\bar{Y}_S|S|(|VG\setminus S|+1): S \in \mathcal{S}\} \leq \Sigma\{Y_S|S|(|VG\setminus S|+1): S \in \mathcal{S}\} - \frac{1}{t}$$

where t is the least common denominator of $\{Y_S: S \in \mathcal{S}\}$. Repeating this procedure, an optimal solution can be found such that no such sets A and B exist. ☐

A family of sets $\mathcal{F}$ is called laminar if $A,B \in \mathcal{F}$ and $A \cap B \neq \emptyset$ implies that either $A \subseteq B$ or $B \subseteq A$. Thus, the above lemma states that there always exists an optimal solution (y,Y) to (2.2) such that $\{S \in \mathcal{S}: Y_S > 0\}$ is laminar.

The following short proof of Theorem 2.2 is due to Schrijver [81a]. It should be noted that the proof makes uses of Theorem 3.1.

Proof of Theorem 2.2 (Schrijver [81a]): Suppose that it is not true that for each graph G and each nonnegative integer vector w the linear program (2.2) has an integer optimal solution. Let G and w be a counterexample with $|VG|+|EG|+\Sigma\{w_e: e \in EG\}$ as small as possible.

By the choice of G and w, G is connected, $w_e \geq 1$ for all $e \in EG$ and for each $v \in VG$ there exists a maximum weight matching of G which does not meet v. Thus, if (y,Y) is any optimal solution to (2.2) then $y_v = 0$ for each $v \in VG$ (by the complementary slackness theorem). By Lemma 2.3 it can be assumed that $\{S \in \mathcal{S}: Y_S > 0\}$ is laminar. Let A be a maximal set in $\{S \in \mathcal{S}: Y_S$ is not integer$\}$ (by assumption, Y is nonintegral) with respect to inclusion and let $A_1,\ldots,A_k$ be the maximal sets of $\{S \in \mathcal{S}: Y_S > 0$ and $S \subset A\}$ with respect to inclusion. Let $\varepsilon = Y_A - \lfloor Y_A \rfloor$ and let $\bar{Y}_A = Y_A - \varepsilon$, $Y_{A_i} = Y_{A_i} + \varepsilon$ for $i = 1,\ldots,k$, and $\bar{Y}_S = Y_S$ for all S in $\mathcal{S}\backslash\{A,A_1,\ldots,A_k\}$. Now $(y,\bar{Y})$ is a feasible solution to (2.2) with objective value less than that of (y,Y), a contradiction to the optimality of (y,Y). ☐

Other proofs of Theorem 2.2 are given in Hoffman and Oppenheim [78], Schrijver [82c], and Schrijver and Seymour [77]. The result also follows from the observation that Edmonds' blossom algorithm finds, for any graph G and any nonnegative integer vector w, an optimal solution to (2.2) such that Y is integral and y is half-integral.

A matching M of G is called <u>perfect</u> if each node in VG is met by an edge in M. If G is connected, then G is called <u>hypomatchable</u> if for each $v \in VG$ the graph obtained by deleting v from G has a perfect matching. Let V' be the set of nodes $v \in VG$ such that either $|N(v)| \geq 3$ <u>or</u> $|N(v)| = 2$ and $\gamma(N(v)) = \emptyset$ <u>or</u> $|N(v)| = 1$ and v is a node of a two node connected component of G. The following result was proven by Pulleyblank and Edmonds [74].

<u>Theorem 2.4:</u> The unique minimal defining system (unique up to positive scalar multiplication of the inequalities) for the convex hull of the matchings of G is

$$
\begin{aligned}
& x_e \geq 0 \quad \forall\, e \in EG \\
(2.4) \qquad & x(\delta(v)) \leq 1 \ \forall\, v \in V' \\
& x(\gamma(S)) \leq \lfloor \tfrac{1}{2}|S| \rfloor \ \forall\, S \subseteq VG,\ |S| \geq 3,\ G[S] \text{ hypomatchable} \\
& \qquad\qquad \text{with no cutnode.}
\end{aligned}
$$
□

A short proof of this result due to L. Lovász can be found in Cornuéjols and Pulleyblank [80a].

Cunningham and Marsh [78] showed that the linear system (2.4) is totally dual integral, which immediately implies the following result.

<u>Theorem 2.5:</u> The unique minimal totally dual integral system with integer left hand sides that defines the convex hull of the matchings of G is (2.4). □

The content of this result of Cunningham and Marsh [78] may be stated in terms of separability of graphs. Let k be the cardinality of a largest matching of G. Let E_1 and E_2 be subsets of EG with $E_1 \cup E_2 = EG$ and $E_i \neq \emptyset$, $i = 1,2$. Let k_i be the cardinality of a largest matching of G contained in E_i, $i = 1,2$. If $k_1+k_2 = k$, then (E_1,E_2) is a <u>matching separation</u> of G. Call G <u>matching separable</u> if there exists a matching separation of G and <u>matching nonseparable</u> otherwise. R. Giles proved the following result.

<u>Theorem 2.6:</u> A graph G is matching nonseparable if and only if either G is isomorphic to $K_{1,n}$ for some n or G is hypomatchable with no cutnode.

In order to show the connection of this theorem to the result of Cunningham and Marsh and to obtain a new proof of this theorem, a general result on separability will be proven.

Let E be a finite set and let I be a finite set of nonnegative integer vectors $a = (a_e : e \in E)$. Call (E,I) a <u>general independence system</u> if $0 \in I$ and for each $a \in I$ and nonnegative integer $b \leq a$ it is the case that $b \in I$ (so (E,I) is an independence system if each $a \in I$ is 0-1 valued). Suppose that (E,I) has the above property. The rank, $r(A)$, of a set $A \subseteq E$ is the maximum value of $x(A)$ over all vectors $x \in I$. A set $A \subseteq E$ is <u>closed</u> if for each $e \in E \backslash A$ $r(A \cup \{e\}) > r(A)$. A <u>separation</u> of a set $A \subseteq E$ is a pair of nonempty subsets A_1, A_2 of A such that $A_1 \uplus A_2 = A$ and $r(A_1)+r(A_2) = r(A)$. If there exists a separation of $A \subseteq E$ then A is <u>separable</u> (otherwise A is non-separable). Let $C(I)$ denote the convex hull of I.

<u>Lemma 2.7:</u> Let (E,I) be a general independence system and suppose that the linear system

$$(2.5) \qquad \begin{aligned} x(A) &\le r(A) \ \forall\, A \subseteq E,\ A \ne \emptyset \\ x_e &\ge 0 \quad \forall\, e \in E, \end{aligned}$$

is a totally dual integral defining system for $C(I)$. An inequality $x(A) \le r(A)$ is in the Schrijver system for $C(I)$ if and only if $A \ne \emptyset$ is a nonseparable closed set.

<u>Proof:</u> If $x(A) \le r(A)$ is in the Schrijver system for $C(I)$ then clearly the two conditions hold. Conversely, suppose that $A \ne \emptyset$ is a nonseparable closed set. By assumption, for each integral w the linear program

$$\min \Sigma\{r(A)Y_A\colon A \subseteq E,\ A \ne \emptyset\}$$

$$(2.6) \qquad \text{subject to} \quad \Sigma\{Y_A\colon A \subseteq E,\ e \in A\} \ge w_e \quad \forall e \in E$$

$$Y_A \ge 0 \quad \forall\, A \subseteq E,\ A \ne \emptyset$$

has an integral optimal solution. To show that $x(A) \le r(A)$ is in the Schrijver system for $C(I)$ it suffices to show that for some integer vector w the linear program (2.6) has no integer optimal solution with $Y_A = 0$. Let $w_e = 1$ for each $e \in A$ and let $w_e = 0$ for each $e \in EG\backslash A$. An optimal solution to (2.6) is $\bar{Y}_A = 1$ and $\bar{Y}_B = 0$ for all other $B \subseteq E$, $B \ne \emptyset$, with objective value $r(A)$. Any integer optimal solution to (2.6) with $Y_A = 0$ corresponds to a collection of nonempty sets $A_1,\ldots,A_j \subseteq E$ with $A_i \ne A$, $i = 1,\ldots,j$, $A \subseteq A_1 \cup \ldots \cup A_j$, and $r(A_1)+\ldots+r(A_j) = r(A)$. Since A is closed, j must be at least 2. Since A is nonseparable, j must be equal to 1. So there does not exist such a solution. □

This lemma combined with the Cunningham and Marsh result gives a new proof of the characterization of matching nonseparable graphs given in Theorem 2.5. Conversely, using Lemma 2.7 and Theorem 2.2, Theorem 2.6 implies the result of Cunningham and Marsh. It will be shown later that the Cunningham and Marsh result follows from a result proven in Section 5 of this chapter.

Edmonds [71] proved that if $M' = (E', I')$ is a matroid on the ground set E', with independent sets I' and rank function r', then a totally dual integral defining system for $P(M')$, the convex hull of the independent sets of M' is

$$\begin{aligned} x(A) &\le r'(A) \quad \forall A \subseteq E', \quad A \neq \emptyset \\ x_e &\ge 0 \qquad \forall e \in E'. \end{aligned} \tag{2.7}$$

Thus, Lemma 2.6 implies that the Schrijver system for $P(M')$ is (2.7) with an inequality $x(A) \le r'(A)$ only for those $A \subseteq E'$, $A \neq \emptyset$ that are closed and nonseparable in the matroidal sense (note that Cunningham [73] has given an algorithm which may be used to test in polynomial time whether or not A has this property). This result of Edmonds can be found in Pulleyblank [82]. A similar result of Giles [75] for matroid intersection polyhedra follows in the same manner from Lemma 2.7 using a theorem of Edmonds [70] (Theorem 10.2 of Chapter 2). Lemma 2.7 also gives a characterization of the Schrijver system for the convex hull of the independent sets of the "perfect independence systems" of Euler [81].

3. b-Matchings.

Let G be a graph and $b = \{b_v : v \in VG\}$ a positive integer vector. A b-matching of G is a nonnegative integer vector $x = (x_e : e \in EG)$ such that $x(\delta(v)) \leq b_v$ for each $v \in VG$ (so a b-matching with $b_v = 1$ for each $v \in VG$ is a matching). If $w = (w_e : e \in EG)$ is a vector of edge weights then the b-matching problem for G is to find a b-matching of G which maximizes wx. A construction of Tutte [54] reduces b-matching problems to matching problems. Replace each node $v \in VG$ by the new nodes $v_1, v_2, \ldots, v_{b_v}$ and replace each edge $(u,v) \in EG$ by the new edges (u_i, v_j) with $w_{(u_i,v_j)} = w_{(u,v)}$, $i = 1,\ldots,b_u$, $j = 1,\ldots,b_v$. A matching in the new graph corresponds to a b-matching in the original graph and vice versa. As stated in Schrijver [82c], applying Theorem 2.2 to the new graph implies the following result, which is an easy consequence of Edmonds' b-matching algorithm [65] (cf. Pulleyblank [73,80]) and has also been proven by Hoffman and Oppenheim [78] and Schrijver and Seymour [77].

Theorem 3.1: A totally dual integral defining system for the convex hull of the b-matchings of G is

$$\begin{aligned} & x_e \geq 0 && \forall\ e \in EG \\ (3.1)\qquad & x(\delta(v)) \leq b_v && \forall\ v \in VG \\ & x(\gamma(S)) \leq \lfloor \tfrac{1}{2} b(S) \rfloor && \forall\ S \subseteq VG,\ |S| \geq 3. \end{aligned}$$

Proof: It must be shown that for each integral w, the linear program

$$\begin{aligned} & \min \Sigma\{b_v \bar{y}_v : v \in VG\} + \Sigma\{\lfloor \tfrac{1}{2} b(S) \rfloor \bar{Y}_S : S \in \mathcal{T}\} \\ (3.2)\qquad & \text{s.t } \bar{y}(\Psi(e)) + \bar{Y}(\mathcal{T}(e)) \geq w_e\ \forall\ e \in EG \\ & \bar{y}_v \geq 0\ \forall\ v \in VG \\ & \bar{Y}_S \geq 0\ \forall\ S \in \mathcal{T}, \end{aligned}$$

where $\mathcal{T} = \{S \subseteq VG: |S| \geq 3\}$, has an integer optimal solution. Let w be a nonnegative integer vector and let G' be the graph obtained from G by splitting the nodes as described above.

Let (y,Y) be an integer optimal solution to (2.2) relative to G' and $\mathcal{S} = \{S \subseteq VG': |S| \geq 3\}$ (such a solution exists by Theorem 2.2). Using the construction given in the proof of Lemma 2.3 it may be assumed that $\mathcal{S}' = \{S \in \mathcal{S}: Y_S > 0\}$ is laminar. Furthermore, it may be assumed that for each $q \in VG'$, $\Sigma\{Y_S: q \in S$ and $S \in \mathcal{S}\} = \max\{0, w_{(q,p)} - y_q - y_p$ for each $(q,p) \in \delta(q)\}$.

Suppose that for some $v \in VG$, it is not the case that $y_{v_1} = y_{v_j}$ for all $j = 1,\ldots,b_v$. Let $i \in \{1,\ldots,b_v\}$ be such that $y_{v_i} = \min\{y_{v_j}: 1 \leq j \leq b_v\}$ and let $j \in \{1,\ldots,b_v\}$ be such that $y_{v_i} < y_{v_j}$. Let $\bar{S} = \{S \in \mathcal{S}': v_i, v_j \in S\}$. Suppose that there exists a set $T \in \mathcal{S}'$ such that $v_j \in T$ and $v_i \notin T$. There must exist a node q adjacent to v_j such that $q \in T$ and $w_{(v_j,q)} > Y(\bar{S}((v_j,q)) + y_{v_j} + y_q$. Now since $y_{v_i} < y_{v_j}$ and since $w_{(v_i,q)} = w_{(v_j,q)}$, there must also exist a set $S \in \mathcal{S}'$ such that $v_i \in S$, $v_j \notin S$, and $q \in S$, a contradiction, since $\mathcal{S}'$ is laminar. So no such set T exists. Setting y_{v_j} equal to y_{v_i}, $Y_{S \cup \{v_j\}}$ equal to Y_S for each $S \in \mathcal{S}$ with $v_i \in S$, and $Y_S = 0$ for each $S \in \mathcal{S}$ with $v_i \in S$ and $v_j \notin S$, a new optimal solution (y,Y) is obtained with $\mathcal{S}'$ remaining laminar By repeating this procedure, it may be assumed that for all $v \in VG$,

$y_{v_1} = y_{v_j}$ for all $j \in \{1,\ldots,b_v\}$.

Using the above argument, it may also be assumed that for each $S \in \mathcal{S}'$ and each $v \in VG$, if $v_i \in S$ for some $i \in \{1,\ldots,b_v\}$, then $v_j \in S$ for all $j \in \{1,\ldots,b_v\}$.

For each $S \in \mathcal{T}$ let $\bar{Y}_S = Y_{S'}$, where $S' = \cup\{\{v_1,\ldots,v_{b_v}\}: v \in S\}$ and let $\bar{y}_v = y_{v_1}$ for each $v \in VG$. Since (y,Y) is an optimal solution to (2.2), $(\bar{y},\bar{Y})$ is an optimal solution to (3.2). □

The b-matching analogue of a hypomatchable graph is a b-critical graph. If G is connected, then G is <u>b-critical</u> if for each $v \in VG$ there exists a b-matching $\bar{x}$ of G such that $\bar{x}(\delta(v)) = b_v - 1$ and $\bar{x}(\delta(u)) = b_u$ for each $u \in VG\setminus\{v\}$. Let V'' be those nodes v in VG such that $b(N(v)) \geq b_v + 2$ <u>or</u> $b(N(v)) = b_v + 1$ and $\gamma(N(v)) = \emptyset$ <u>or</u> v belongs to a two node connected component of G and $b_v = b_u$, where v is the other node of the component. Pulleyblank [73] proved the following characterization of the unique minimal defining system for the convex hull of the b-matchings of G.

<u>Theorem 3.2:</u> The unique minimal defining system (unique up to positive scalar multiples of the inequalities) for the convex hull of the b-matchings of G is

$$
\text{(3.3)} \qquad
\begin{aligned}
& x_e \geq 0 && \forall\ e \in EG \\
& x(\delta(v)) \leq b_v && \forall\ v \in V'' \\
& x(\gamma(S)) \leq \lfloor \tfrac{1}{2} b(S) \rfloor && \forall\ S \subseteq VG,\ |S| \geq 3, \\
& && G[S] \text{ b-critical with no cutnode } v \text{ having } b_v = 1.
\end{aligned}
$$

□

Unlike the matching case, (3.3) is not totally dual integral. This can be seen by considering a triangle with $b_v = 2$ for each node v and weight $w_e = 1$ for each edge e. Such a triangle is an example of a b-bicritical graph. If G is connected, then G is <u>b-bicritical</u> if for each $v \in VG$ there exists a b-matching $\bar{x}$ of G such that $\bar{x}(\delta(v)) = b_v-2$ and $\bar{x}(\delta(u)) = b_u$ for each $u \in VG\backslash\{v\}$. Cook [81] and Pulleyblank [81] independently characterized the Schrijver system for the convex hull of the b-matchings of G.

<u>Theorem 3.3:</u> The unique minimal totally dual integral system with integer left hand sides that defines the convex hull of the b-matchings of G is (3.3) with the additional inequalities

$$x(\gamma(S)) \le \tfrac{1}{2}b(S) \quad \forall\, S \subseteq VG,\ |S| \ge 3,\ G[S] \text{ b-bicritical and each node } v \in VG\backslash S \text{ which is adjacent to a node in } S \text{ has } b_v \ge 2. \qquad (3.4)$$

□

Since $b_v \ge 2$ for each $v \in VG$ if G is b-bicritical, Theorem 3.3 implies the theorem of Cunningham and Marsh [78] (Theorem 2.5 of this chapter). It will be indicated later how Theorem 3.3 follows from a result of Section 5.

With b-matching separability defined analogously to matching separability, Lemma 2.7 and Theorem 3.3 imply the following result.

<u>Theorem 3.4:</u> A graph G is b-matching nonseparable if and only if either G is $K_{1,n}$ for some n and either $n \le 1$ or $b(N(v)) \ge b_v+1$ where v is the node of degree n <u>or</u> G is b-critical with no cut node v having $b_v = 1$ <u>or</u> G is b-bicritical. □

To close this section, a fundamental theorem on b-matchings due to Tutte [47,52] will be presented. This theorem will be used later in this chapter. A b-matching $\bar{x}$ of G is called perfect if $\bar{x}(\delta(v)) = b_v$ for each $v \in VG$. If $S \subseteq VG$, let

(3.5) $$C^\circ(S) = \{v \in VG\backslash S\colon G[\{v\}] \text{ is a connected component of } G[VG\backslash S]\}$$

and let

(3.6) $$C^1(S) = \{R \subseteq VG\backslash S\colon |R| \geq 2,\ b(R) \text{ is odd and } G[R] \text{ is a connected component of } G[VG\backslash S]\}.$$

Tutte's b-matching theorem is as follows:

<u>Theorem 3.5</u>: A graph G has a perfect b-matching if and only if for each $S \subseteq VG$, $b(S) \geq b(C^\circ(S)) + |C^1(S)|$. □

This result follows in a straightforward way from Theorem 3.1 by setting $w_e = 1$ for each $e \in EG$.

4. Simple b-matchings.

Throughout this section, let G be a graph, possibly with multiple edges and $b = (b_v: v \in VG)$ a nonnegative integer vector. A simple b-matching of G is a subset M of EG such that each node $v \in VG$ meets at most b_v edges in M. A perfect simple b-matching of G (i.e. a simple b-matching which meets each node $v \in VG$ in exactly b_v edges) is often called a "b-factor". A simple b-matching will be identified with its 0-1 incidence vector $x = (x_j: j \in EG)$. Given a vector $w = (w_e: e \in EG)$ of edge weights, the simple b-matching problem is to maximize wx over all simple b-matchings of G. Tutte [54] described the following construction which reduces a simple b-matching problem to a b-matching problem (cf. Schrijver [82c]). For each edge $e = (u,v)$ of G add vertices u_e and v_e to VG and replace e by the edges (u,u_e), (u_e,v_e), (v_e,v) (although G may have multiple edges, for simplicity edges will still be referred to as unordered pairs of vertices). Also, for each $e \in EG$ let $b_{u_e} = b_{v_e} = 1$ and let $w_{(u,u_e)} = w_{(u_e,v_e)} = w_{(v_e,v)} = w_e$. The maximum weight of a b-matching in the new graph is exactly $\Sigma\{w_e: e \in EG\}$ greater than the maximum weight of a simple b-matching of G. As in Schrijver [82c], this construction, together with Theorem 3.1, implies the following theorem, which follows from a result of Edmonds and Johnson [70].

Theorem 4.1: A totally dual integral defining system for the convex hull of the simple b-matchings of G is

$$0 \le x_e \le 1 \qquad \forall e \in EG$$

(4.1) $$x(\delta(v)) \le b_v \qquad \forall v \in VG$$

$$x(\gamma(S))+x(J) \le \lfloor\tfrac{1}{2}(b(S)+|J|)\rfloor \quad \forall S \subseteq VG,\ J \subseteq \delta(S)$$

<u>Proof</u>: It suffices to show that for each graph G and each nonnegative integer vector w the linear program

$$\min \Sigma\{z_e : e \in EG\}+\Sigma\{b_v y_v : v \in VG\}+\Sigma\{\lfloor\tfrac{1}{2}(b(S)+|J|)\rfloor Y_{(S,J)} : S \subseteq VG,\ J \subseteq \delta(S)\}$$

(4.2)
$$\text{s.t. } z_e+y(\Psi(e))+\Sigma\{Y_{(S,J)} : S \subseteq VG,\ J \subseteq \delta(S),\ e \in \gamma(S)\cup J\} \ge w_e \quad \forall e \in EG$$

$$z_e \ge 0 \ \forall e \in EG$$

$$y_v \ge 0 \ \forall v \in VG$$

$$Y_{(S,J)} \ge 0 \ \forall S \subseteq VG,\ J \subseteq \delta(S)$$

has an integer optimal solution. Suppose that this is not true and let G and w be a counterexample with $\Sigma\{w_e : e \in EG\}$ as small as possible. Let G' be the graph obtained by the above construction of Tutte and let $(\bar{y},\bar{Y})$ be an integer optimal solution to (3.2) relative to G' and the vector obtained from w by the construction. Let k be the optimal objective value of (4.2).

If $u \in VG$ and $e = (u,v) \in EG$ then it may be assumed that $u \in S$ for each $S \subseteq VG'$ with $\bar{Y}_S > 0$ and $u_e \in S$. Furthermore, if $e=(u,v) \in EG$ and there exists a set $S \subseteq VG'$ with $\bar{Y}_S > 0$, $u_e \in S$ and $v_e \notin S$, then it may be assumed that $\bar{y}_{v_e} > 0$.

Suppose that there exists a set $S \subseteq VG'$ with $\bar{Y}_S > 0$. Let $w'_e = w_e - 1$ for all $e = (u,v) \in EG$ such that either $u_e \in S$ or $v_e \in S$ (or both). Let $\bar{Y}_S{}' = \bar{Y}_S - 1$ and $\bar{Y}'_T = \bar{Y}_T$ for all $T \in \mathcal{T}\backslash\{S\}$ and let $\bar{y}'_{v_e} = \bar{y}_{v_e} - 1$ for each edge $e = (u,v) \in EG$ with $u_e \in S$ and $v_e \notin S$ and let $\bar{y}'_q = \bar{y}_q$ for all other $q \in VG'$. Also, let $\bar{S} = \{v \in VG: v \in S\}$ and let $\bar{J} = \{e=(u,v) \in EG: u_e \in S$ and $v_e \notin S\}$. There exists, by assumption, an integer optimal solution (z,y,Y) to (4.2) relative to G and w'. Now $(\bar{y}',\bar{Y}')$ is an optimal solution to (3.2) relative to G' and w' with objective value

$$\Sigma\{w_e: e \in EG\}+k-\lfloor\tfrac{1}{2}(b(\bar{S})+|\bar{J}|)\rfloor-|\{e = (u,v) \in EG: \text{either } u_e \in S \text{ or } v_e \in S\}|. \tag{4.3}$$

Since $\Sigma\{w'_e: e \in EG\} = \Sigma\{w_e: e \in EG\}-|\{e = (u,v) \in EG:$ either $u_e \in S$ or $v_e \in S\}|$, (z,y,Y) gives objective value $k - \lfloor\frac{1}{2}b(\bar{S})+|\bar{J}|\rfloor$. So (z,y,Y') is an integer optimal solution to (4.2) relative to G and w, where $Y'_{(\bar{S},\bar{J}')} = Y_{(\bar{S},\bar{J}')}+1$ $(\bar{J}' = \bar{J}\cap\delta(\bar{S})$, since it may be the case that $\bar{J}$ contains exactly one edge not in $\delta(\bar{S}))$ and $Y'_{(A,B)} = Y_{(A,B)}$ for all other $A \subseteq VG$, $B \subseteq \delta(A)$, a contradiction. So it may be assumed that no such set S exists.

Suppose that there exists a node $v \in VG$ with $\bar{y}_v > 0$. It may be assumed that for each $e = (u,v) \in \delta_G(v)$ with $w_e > 0$, we have $\bar{y}_{u_e} > 0$. Let $w'_e = \max\{0,w_e-1\}$ for each $e \in \delta_G(v)$ and $w'_e = w_e$ for all other edges $e \in EG$ and let (z,y,Y) be an integer optimal solution to (4.2) relative to G and w'. Let $\bar{y}'_v = \bar{y}_v - 1$, $\bar{y}'_{u_e} = \bar{y}_{u_e} - 1$ for each $e = (u,v)$ in $\delta_G(v)$ with $w_e > 0$, and $\bar{y}'_q = \bar{y}_q$ for all other $q \in VG'$. Since $(\bar{y}', \bar{Y})$ is an optimal solution to (3.2) relative to G' and w', (z,y,Y) gives objective value $k - b_v$ for (4.2). So (z,y',Y) is an integer optimal solution to (4.2), where $y'_v = y_v + 1$ and $y'_q = y_q$ for all $q \in VG \backslash \{v\}$, a contradiction. So it may be assumed that no such node exists.

Clearly, $\Sigma\{w_e : e \in EG\} > 0$. So there exists an edge $e = (u,v) \in EG$ with $\bar{y}_{u_e} > 0$ and $\bar{y}_{v_e} > 0$. Let $w'_e = w_e - 1$ and $w'_f = w_f$ for all $f \in EG \backslash \{e\}$. An integer optimal solution to (4.2) relative to G and w can be found by taking an integer optimal solution (z,y,Y) to (4.2) relative to G and w' and increasing the value of z_e by 1, a contradiction. □

If H is a subgraph of G, then for each $v \in VH$ let $b_v^H = \min\{b_v, d_H(v)\}$. The largest simple b-matching of G is of cardinality at most $\lfloor \frac{1}{2} b^G(VG) \rfloor$. Let $\mathcal{H}$ be the set of all connected subgraphs of G with $|VH| \geq 3$. Theorem 4.1 implies that

$$(4.4) \qquad \begin{array}{ll} 0 \leq x_e \leq 1 & \forall\ e \in EG \\ x(\delta_G(v)) \leq b_v & \forall\ v \in VG \\ x(EH) \leq \lfloor \frac{1}{2} b^H(VH) \rfloor & \forall\ H \in \mathcal{H}. \end{array}$$

is a totally dual integral defining system for $S(G,b)$, the convex hull of the simple b-matchings of G. By a series of results in this section and in Section 5, the unique minimal subset of these inequalities which is a totally dual integral defining system for $S(G,b)$ will be characterized, i.e. the Schrijver system for $S(G,b)$ will be obtained.

As in the previous sections, critical graphs play an important role here. If G is connected, then G is simple b-critical if $|VG| \geq 3$ and for each $v \in VG$ there exists a simple b-matching of G of cardinality $\lfloor\frac{1}{2}b^G(VG)\rfloor$ which contains exactly b_v^G-1 edges which meet v. Define simple b-separability in a way analogous to matching separability.

Lemma 4.2: Let H be a connected subgraph of G with $|VH| \geq 3$. If the inequality

$$(4.5) \qquad x(EH) \leq \lfloor\tfrac{1}{2}b^H(VH)\rfloor$$

is not $x(\delta(v)) \leq b_v$ for some $v \in VG$, then it is essential for $S(G,b)$ only if H is simple b-critical and simple b-nonseparable and there does not exist an edge $(u,v) \in EG\backslash EH$ with $u,v \in VH$ and $d_H(u) \geq b_v$ and $d_H(v) \geq b_v$.

Proof: Suppose that the inequality (4.5) is essential for $S(G,b)$ and that it is not $x(\delta(v)) \leq b_v$ for some $v \in VG$. Clearly, H is simple b-nonseparable and there does not exist an edge $(u,v) \in EG\backslash EH$ with $d_H(u) \geq b_u$ and $d_H(v) \geq b_v$. Let M be the set of simple b-matchings of G for which (4.5) holds as an equality. Since (4.5) is not $x(\delta(v)) \leq b_v$ for some $v \in VG$, for each $v \in VG$ there exist a simple b-matching in M which contains at most b_v-1 edges which meet v. Similarly, since (4.5) is not $x_e \leq 1$ for some $e \in EG$, for each $e \in EG$ there exists a simple b-matching in M which does not contain e. So for each $v \in VH$ there exists a simple b-matching of H of cardinality $\lfloor \frac{1}{2} b^H(VH) \rfloor$ which contains exactly $b_v^H - 1$ edges which meet v. So H is simple b-critical. □

To prove that an inequality in (4.4) is essential for $S(G,b)$, it suffices to find a vector $\bar{x}$ which does not satisfy that inequality but does satisfy each of the other inequalities in

$$(4.6) \quad \begin{array}{ll} \text{(i)} & 0 \le x_e \le 1 \quad \forall\, e \in EG \\ \text{(ii)} & x(\delta(v)) \le b_v \quad \forall\, v \in VG \\ \text{(iii)} & x(EH) \le \lfloor \frac{1}{2} b^H(VH) \rfloor \text{ for each simple b-critical subgraph } H \text{ of } G. \end{array}$$

This technique will be used to characterize which inequalities in (4.6)-(i) and (4.6)-(ii) are essential for $S(G,b)$.

Lemma 4.3: For each $e \in EG$, $x_e \ge 0$ is an essential inequality for $S(G,b)$.

Proof: Let $e \in EG$ and let $x_e = -1$ and $x_f = 0$ for each $f \in EG \backslash \{e\}$. The vector x does not satisfy $x_e \ge 0$ but it does satisfy each of the other inequalities in (4.6). □

Lemma 4.4: Let $e \in EG$. The inequality $x_e \le 1$ is essential for $S(G,b)$ if and only if e does not meet a node v with $b_v = 1$ and $d_G(v) > 1$.

Proof: If e meets a node v with $b_v = 1$ and $d_G(v) > 1$, then $x(\delta(v)) \le 1$ implies that $x_e \le 1$, which implies that $x_e \le 1$ is not an essential inequality for $S(G,b)$. Suppose that e does not meet such a node v. Let $x_e = 2$ and $x_f = 0$ for each $f \in EG\backslash\{e\}$. The vector x does not satisfy $x_e \leq 1$ but it does satisfy each of the other inequalities in (4.6). □

Lemma 4.5: Let $v \in VG$ and let $b'_u = \min\{b_u, |\delta(u) \cap \delta(v)|\}$ for each node u in $N(v)$. The inequality $x(\delta(v)) \leq b_v$ is essential for $S(G,b)$ if and only if one of the following conditions hold:

(4.7) a) v belongs to a connected component of G containing exactly one other node u and b) u has the property that $b_v = b_u$ and c) if $d_G(v) \leq b_v$ then $b_v = 1$.

(4.8) a) $b'(N(v)) = b_v+1$ and b) there is no edge $(v_1,v_2) \in \gamma(N(v))$ such that $b_{v_1} = b'_{v_1}$ and $b_{v_2} = b'_{v_2}$.

(4.9) $b'(N(v)) \geq b_v+2$

Proof: Suppose that $b'(N(v)) \leq b_v$ and that (4.7) does not hold. The inequality $x(\delta(v)) \leq b_v$ is implied by the inequalities $x_e \leq 1$ for each $e \in \delta(v)$ and $x(\delta(u)) \leq b_u$ for each $u \in N(v)$. So $x(\delta(v)) \leq b_v$ is not an essential inequality for $S(G,b)$. Now suppose that (4.7) holds. Let $x_e = 1$ for each $e \in \delta(v)$ (if $d_G(v) = 1$ let $x_e = 2$ for $e \in \delta(v)$) and let $x_e = 0$ for each $e \in EG\backslash\delta(v)$. The vector x does not satisfy $x(\delta(v)) \leq b_v$ but it does satisfy each of the other inequalities in (4.6). So $x(\delta(v)) \leq b_v$ is essential for $S(G,b)$.

Suppose that $b'(N(v)) = b_v+1$ and that (4.8) does not hold. Let $(v_1,v_2) \in \gamma(N(v))$ be an edge such that $b_{v_1} = b'_{v_1}$ and $b_{v_2} = b'_{v_2}$. Now $x(\delta(v))+x_{(v_1,v_2)} \le b_v$ is a valid inequality for $S(G,b)$ which implies $x(\delta(v)) \le b_v$. So $x(\delta(v)) \le b_v$ is not an essential inequality for $S(G,b)$. Now suppose that (4.8) holds. For each $u \in N(v)$ select b'_u edges from $|\delta(u) \cap \delta(v)|$ and let J be the collection of these edges. Let $x_e = 1$ for each $e \in J$ and let $x_e = 0$ for each $e \in EG\backslash J$. The vector x does not satisfy $x(\delta(v)) \le b_v$ but it does satisfy each of the other inequalities in (4.6). So $x(\delta(v)) \le b_v$ is essential for $S(G,b)$.

Suppose that $b'(N(v)) \ge b_v+2$. Let J be a collection of b_v+2 edges in $\delta(v)$ such that $|J \cap \delta(u)| \le b'_u$ for each $u \in N(v)$. Let $x_e = b_v/(b_v+1)$ for each $e \in J$ and let $x_e = 0$ for each $e \in EG\backslash J$. The vector x does not satisfy $x(\delta(v)) \le b_v$ but it does satisfy all other inequalities in (4.6)-(i) and (4.6)-(ii). Let $ax \le \alpha$ be an inequality in (4.6)-(iii). Let S be a proper subset of J. If $|S| \le b_v$, then $x(S) \le |S|$ and S is a simple b-matching of G. If $|S| = b_v+1$, then $x(S) = b_v$ and S contains a simple b-matching of cardinality b_v. So a_e must be equal to 1 for each $e \in J$, and consequently α must be at least $\lfloor \frac{1}{2}(b_v+(b_v+2)) \rfloor = b_v+1$. Since $x(J) < b_v+1$, x satisfies $ax \le \alpha$. So x satisfies all other inequalities in (4.6). It follows that $x(\delta(v)) \le b_v$ is essential for $S(G,b)$. □

The above lemmas imply that a defining system for $S(G,b)$ is

(4.10)

(i) $x_e \geq 0 \quad \forall e \in EG$

(ii) $x_e \leq 1 \quad \forall e \in EG$ such that e does not meet a node v with $b_v = 1$ and $d_G(v) > 1$.

(iii) $x(\delta(v)) \leq b_v \quad \forall v \quad VG$ for which either (4.7), (4.8) or (4.9) holds for v.

(iv) $x(EH) \leq \lfloor \frac{1}{2} b^H(VH) \rfloor$ for all simple b-critical simple b-nonseparable subgraphs H of G such that there does not exist an edge $(u,v) \in EG \backslash EH$ with $u,v \in VH$ and $d_H(u) \geq b_u$ and $d_H(v) \geq b_v$.

This linear system is not in general totally dual integral (consider the complete graph on four nodes with $b_i = 2$ for each node i and each edge e receiving weight 1). Thus the Schrijver system for $S(G,b)$ is not identical to the minimal defining system for $S(G,b)$ scaled so that the left hand sides are 0-1 valued.

Remark 4.6: Using a generalization of Lovász' [72a] ear-decomposition of factor-critical graphs (see also Cornuéjols and Pulleyblank [81]), it can be proven that (4.10) is the unique minimal defining system for $S(G,b)$, up to positive scalar multiples of the inequalities. This result will appear in Cook and Pulleyblank [83], but it will not be used here.

To obtain the Schrijver system for $S(G,b)$ the notion of a bicritical graph is needed. The graph G is simple b-bicritical if it is connected, $|VG| \geq 3$, $b^G(VG)$ is even, and for each $v \in VG$ there exists a b-matching of G of cardinality $\frac{1}{2}b^G(VG)-1$ which contains exactly b^G_v-2 edges which meet v. The complete graph on four nodes with $b_i = 2$ for each node i is an example of a simple b-bicritical graph.

A variation of Tutte's b-matching Theorem (Theorem 3.5) will be used to show the relation of simple b-bicritical graphs to totally dual integral systems. The transformation of Tutte [54] mentioned earlier reduces any simple b-matching problem to a b-matching problem. So, as presented in Schrijver [82c], to determine if G has a perfect simple b-matching, this transformation can be applied to G to obtain a new graph G' and then Tutte's b-matching Theorem can be applied to G'. Suppose that $S \subseteq VG$ and $T \subseteq VG\backslash S$. Let

$$(4.11) \qquad Q(S,T) = \Sigma\{b_v - d_{G[VG\backslash S]}(v): v \in T\}$$

and let $G^T(VG\backslash S)$ be the graph obtained from $G[VG\backslash S]$ by taking each node $v \in T$ and splitting it into $d_{G[VG\backslash S]}(v)$ nodes, each with $b_i = 1$ (i.e. replace v by nodes $v_1,\ldots,v_k$, where $k = d_{G[VG\backslash S]}(v)$, and replace the edges (u_1,v), (u_2,v),...,(u_k,v) by the edges (u_i,v_i), $i = 1,\ldots,k$, and let $b_{v_i} = 1$, $i = 1,\ldots,k$). Let $\mathcal{D}_1(S,T)$ denote the set of odd connected components of $G^T(VG\backslash S)$ which contain at least two nodes. (A connected component G_i of $G^T(VG\backslash S)$ is odd if $b(VG_i)$ is odd.) Notice that each connected component of $G^T(VG\backslash S)$ corresponds to a subgraph of $G[VG\backslash S]$. Using Tutte's b-matching Theorem, the following result can be proven.

Theorem 4.7: There exists a perfect simple b-matching of G if and only if $\forall S \subseteq VG$ and $\forall T \subseteq VG\backslash S$: $b(S) \geq Q(S,T)+|\mathcal{D}_1(S,T)|$.

Proof: Suppose that G has a perfect simple b-matching M and let $S \subseteq VG$ and $T \subseteq VG\backslash S$. Let $M' = M \cap \gamma(VG\backslash S)$. Since M' corresponds to a simple b-matching of $G^T(VG\backslash S)$ of cardinality $|M'|$, we have $b(VG^T(VG\backslash S)) - 2|M'| \geq |\mathcal{D}_1(S,T)|$. Since $b(VG^T(VG\backslash S)) = b(VG\backslash S) - Q(S,T)$, this implies that $b(VG\backslash S) - 2|M'|$ must be at least $Q(S,T) + |\mathcal{D}_1(S,T)|$. Now since M is a perfect simple b-matching of G, $b(S) \geq b(VG\backslash S) - 2|M'| \geq Q(S,T) + |\mathcal{D}_1(S,T)|$.

Conversely, suppose that G does not have a perfect simple b-matching. Let G' be the graph obtained from G by replacing each edge $e = (u,v) \in EG$ by the edges (u,u_e), (u_e,v_e), (v_e,v) and adding u_e and v_e to VG with $b_{u_e} = b_{v_e} = 1$. Since G does not have a perfect simple b-matching, G' does not have a perfect b-matching. So, by Tutte's b-matching Theorem, there exists a set $X \subseteq VG'$ such that $b(X) < b(\cup\{R: R \in \mathcal{C}^o(X)\}) + |\mathcal{C}^1(X)|$. Let X be such a subset of VG' and let $S = \{v \in VG: v \in X\}$. It may be assumed that for each edge $e \in (u,v) \in EG$, if $u \in S$ and $v \notin S$ then $v_e \in X$ and $u_e \notin X$ It may also be assumed that for each edge $e = (u,v) \in EG$, if $u \in S$ and $v \in S$ then neither u_e nor v_e is in X. Furthermore, it may be assumed that for each edge $e = (u,v) \in EG$ if $u \notin S$ and $v \notin S$ then $u_e \in X$ only if $u_f \in X$ for each edge $f = (u,t)$ such that $t \in VG\backslash S$ Let $T = \{v \in VG\backslash S: v \in \mathcal{C}^o(X)\}$, i.e. T is the set of nodes $v \in VG\backslash S$ that are isolated in $G'[VG'\backslash X]$. Since $b(X) < b(\mathcal{C}^o(X)) + |\mathcal{C}^1(X)|$, we have $b(S) < Q(S,T) + |\mathcal{D}_1(S,T)|$. □

Corollary 4.8: If G is simple b-bicritical then G has a perfect simple b^G-matching.

Proof: Suppose G is simple b-bicritical. Let $S \subseteq VG$ and $T \subseteq VG\backslash S$. By Theorem 4.7, it suffices to show that $b^G(S) \geq Q^G(S,T)+|\mathcal{D}_1^G(S,T)|$. Suppose that $S \neq \emptyset$ and let $v \in S$. Let $b'_v = b^G_v-2$ and $b'_u = b^G_u$ for all $u \in VG\backslash\{v\}$. Since G has a perfect simple b'-matching, $b'(S) \geq Q'(S,T)+|\mathcal{D}'_1(S,T)|$. Now $b^G(S) = b'(S)+2$, $Q'(S,T) = Q^G(S,T)$, and $\mathcal{D}'_1(S,T) = \mathcal{D}^G_1(S,T)$, so $b^G(S) \geq Q^G(S,T)+|\mathcal{D}^G_1(S,T)|$. Suppose that $S = \emptyset$. If $T = VG$ then $Q^G(S,T) \leq 0$ and $|\mathcal{D}^G_1(S,T)| = 0$, which implies that $b^G(S) \geq Q^G(S,T)+|\mathcal{D}^G_1(S,T)|$. Suppose that $VG\backslash T \neq \emptyset$ and let $v \in VG\backslash T$. Define b' as above. Again, Theorem 4.7 implies that $b'(S) \geq Q'(S,T)+|\mathcal{D}'_1(S,T)|$. Since $b'(S) = b^G(S)$, $Q'(S,T) = Q^G(S,T)$, and $\mathcal{D}'_1(S,T) = \mathcal{D}^G_1(S,T)$, $b^G(S) \geq Q^G(S,T)+|\mathcal{D}^G_1(S,T)|$. □

Theorem 4.7 will be used in Section 5 to prove the following lemma.

Lemma 4.9: Let H be a connected subgraph of G with $|VH| \geq 3$. The inequality $x(EH) \leq \lfloor \frac{1}{2} b^H(VH) \rfloor$ is in the Schrijver system for $S(G,b)$ only if either it is in (4.6)-(ii) or H is a simple b-critical simple b-nonseparable graph or a simple b-bicritical simple b-nonseparable graph. □

Using Corollary 4.8 and Lemma 4.9, the Schrijver system for $S(G,b)$ may now be characterized.

Theorem 4.10: The unique minimal totally dual integral system with integer left hand sides which defines the convex hull of the simple b-matchings of G is (4.10) together with

(4.12) $x(EH) \leq \lfloor \frac{1}{2} b^H(VH) \rfloor$ for each connected component of G which is a simple b-nonseparable simple b-bicritical graph.

<u>Proof:</u> Since (4.4) is a totally dual integral defining system for $S(G,b)$, an inequality is in the Schrijver system for $S(G,b)$ only if it is in (4.4). It follows from the proofs of Lemma 4.4 and Lemma 4.5 that an inequality in (4.6)-(i) or (4.6)-(ii) is in the Schrijver system for $S(G,b)$ only if it is in (4.10)-(i), (4.10)-(ii), or (4.10)-(iii). Furthermore, by Lemma 4.3, Lemma 4.4, and Lemma 4.5, each inequality in (4.10)-(i), (4.10)-(ii), and (4.10)-(iii) is essential for $S(G,b)$ and hence is in the Schrijver system for $S(G,b)$.

Let H be a connected subgraph of G with $|VH| \geq 3$. Suppose that $b^H(VH)$ is odd. By Lemma 4.9,

(4.13) $x(EH) \leq \lfloor \frac{1}{2} b^H(VH) \rfloor$

is in the Schrijver system for $S(G,b)$ only if H is simple b-critical simple b-nonseparable. If there exists an edge $(u,v) \in EG \setminus EH$ with $d_H(u) \geq b_u$ and $d_H(v) \geq b_v$, then $x(EH)+x_{(u,v)} \leq \lfloor \frac{1}{2} b^H(VH) \rfloor$ is a valid inequality for $S(G,b)$. So (4.13) is in the Schrijver system for $S(G,b)$ only if it is in (4.10)-(iv). Conversely, Lemma 2.7 implies that if (4.13) is in (4.10)-(iv) then (4.13) is in the Schrijver system for $S(G,b)$.

Now suppose that $b^H(VH)$ is even. By Lemma 4.9, (4.13) is in the Schrijver system for $S(G,b)$ only if H is simple b-bicritical simple b-nonseparable so suppose that H is simple b-bicritical simple b-nonseparable. Corollary 4.8 implies that H has a perfect simple b-matching. It follows that if there exists a node $v \in VH$ for which $d_H(v) \lneq b_v$, then $(e, EH\setminus\{e\})$ is a simple b-separation of H, where e is an edge in EH that meets v. So $d_H(v) > b_v$ for each $v \in VH$. If H is not a connected component of G then there exists an edge $e \in EG\setminus EH$ which meets a node in VH. If e is such an edge then $x(EH)+x_e \leq \lfloor \frac{1}{2} b^H(VH) \rfloor$ is a valid inequality for $S(G,b)$. So (4.13) is in the Schrijver system for $S(G,b)$ only if it is in (4.12). Conversely, Lemma 2.7 implies that each inequality in (4.12) is in the Schrijver system for $S(G,b)$. □

5. Capacitated b-matchings.

Capacitated b-matchings are a generalization of simple b-matchings and b-matchings. Let G be a graph, possibly with multiple edges, $b = (b_v : v \in VG)$ a nonnegative integer vector, and $c = (c_e : e \in EG)$ a positive integer vector of edge capacities. A c-capacitated b-matching ((b,c)-matching) of G is a b-matching x such that $x_e \le c_e$ for each $e \in EG$. If $c_e = 1$ for all $e \in EG$ then a (b,c)-matching of G is a simple b-matching of G. If $\beta = \max\{b_v : v \in VG\}$ and $c_e = \beta$ for all $e \in EG$ then x is a (b,c)-matching of G if and only if x is a b-matching of G.

It is easy to reduce a capacitated b-matching to a simple b-matching by replacing each edge $e \in EG$ with c_e edges each of which has the same end nodes as e. In fact, Tutte's construction for reducing simple b-matching problems to b-matching problems can be used to reduce a capacitated b-matching problem immediately to a b-matching problem: For each edge $e = (u,v)$ of G add vertices u_e and v_e to VG and replace e by the edges (u,u_e), (u_e,v_e), (v_e,v). For each $e \in EG$ let $b_{u_e} = b_{v_e} = c_e$ and $w_{(u,u_e)} = w_{(u_e,v_e)} = w_{(v_e,v)} = w_e$. The maximum weight of b-matching in the new graph is exactly $\Sigma\{w_e c_e : e \in EG\}$ greater than the maximum weight of a (b,c)-matching of G. Using this construction, the following result can be obtained from Theorem 3.1 in the same way that Theorem 4.1 was obtained from it.

Theorem 5.1: A totally dual integral defining system for the convex hull of the (b,c)-matchings of G is

$$
(5.1)\qquad
\begin{aligned}
&0 \le x_e \le c_e && \forall e \in EG\\
&x(\delta(v)) \le b_v && \forall v \in VG\\
&x(\delta(S))+x(J) \le \lfloor \tfrac{1}{2}(b(S)+c(J))\rfloor && \forall S \subseteq VG,\ J \subseteq \delta(S).
\end{aligned}
$$

Proof: Follow the proof of Theorem 4.1, using the above construction of Tutte. □

If H is a subgraph of G, let $b_v^{(H,c)} = \min\{b_v, \Sigma\{c_e : e \in EH \cap \delta(v)\}$ for each $v \in VH$. The largest (b,c)-matching of G is of cardinality at most $\lfloor\frac{1}{2}b^{(G,c)}(VG)\rfloor$. Let $P(G,b,c)$ denote the convex hull of the (b,c)-matchings of G.

Corollary 5.2: A totally dual integral defining system for $P(G,b,c)$ is

$$
(5.2)\qquad
\begin{aligned}
&\text{(i)} && 0 \le x_e \le c_e \quad \forall e \in EG\\
&\text{(ii)} && x(\delta(v)) \le b_v \quad \forall v \in VG\\
&\text{(iii)} && x(EH) \le \lfloor\tfrac{1}{2}b^{(H,c)}(VH)\rfloor \text{ for all connected subgraphs } H \text{ of } G \text{ with } |VH| \ge 3.
\end{aligned}
$$

□

If G is connected then G is (b,c)-critical if $|VG| \ge 3$ and for each $v \in VG$ there exists a (b,c)-matching $\bar{x}$ of G such that $\bar{x}(\delta(v)) = b_v^{(G,c)}-1$ and $\bar{x}(\delta(u)) = b_u^{(G,c)}$ for each $u \in VG\backslash\{v\}$. Let (b,c)-separability be defined in a way analogous to matching separability. Using the technique used in proving Lemma 4.2 the following result can be proven.

Lemma 5.3: Let H be a connected subgraph of G with $|VH| \ge 3$.

If the inequality

$$(5.3)\qquad x(EH) \le \lfloor \tfrac{1}{2} b^{(H,c)}(VH) \rfloor$$

is not in (5.2)-(ii) then it is essential for $P(G,b,c)$ only if H is (b,c)-critical and (b,c)-nonseparable and there does not exist any edge $(u,v) \in EG \backslash EH$ with $v,u \in VH$ and with $b_u^{(H,c)} = b_u$ and $b_v^{(H,c)} = b_v$. □

Let G' be the graph obtained from G by replacing each edge $e \in EG$ with edges $e_1,\ldots,e_{c_e}$, each of which has the same end nodes as e (i.e. $\Psi(e_i) = \Psi(e)$ for $i = 1,\ldots,c_e$). Each subgraph H of G corresponds to a subgraph H' of G' ($VH' = VH$ and $EH' = \cup\{\{e_1,\ldots,e_{c_e}\}: e \in EH\}$). The following lemma will be used to characterize which inequalities in (5.2)-(ii) are essential for $P(G,b,c)$.

Lemma 5.4: Let H be a subgraph of G. If $x(EH') \le \alpha$ is an essential inequality for $S(G,b)$ then $x(EH) \le \alpha$ is an essential inequality for $P(G,b,c)$.

Proof: Suppose that $x(EH') \le \alpha$ is essential for $S(G,b)$. Let M' be a collection of $|EG'|$ affinely independent simple b-matchings of G' for which $x(EH') \le \alpha$ holds as an equality. For each simple b-matching x' in M', let $\bar{x}$ be the (b,c)-matching of G obtained by letting $\bar{x}_e = \Sigma\{x'_{e_i} : i = 1,\ldots,c_e\}$ for each $e \in EG$. Consider the set of (b,c)-matchings $M = \{\bar{x}: x' \in M'\}$. Each (b,c)-matching in M satisfies $x(EH) \le \alpha$ with equality. Furthermore, since M' is an affinely independent set of vectors, M contains $|EG|$ affinely independent vectors. So $x(EH) \le \alpha$ is essential for $P(G,b,c)$. □

The inequalities in (5.2)-(i) and (5.2)-(ii) that are essential for $P(G,b,c)$ are characterized in the following three lemmas.

<u>Lemma 5.5:</u> For each $e \in EG$, $x_e \geq 0$ is an essential inequality for $P(G,b,c)$.

<u>Proof:</u> Let $e \in EG$ and let $x_e = -1$ and $x_f = 0$ for each $f \in EG\backslash\{e\}$. The vector x does not satisfy $x_e \geq 0$ but it does satisfy each of the other inequalities in (5.2). □

<u>Lemma 5.6:</u> Let $e \in EG$. The inequality $x_e \leq c_e$ is essential for $P(G,b,c)$ if and only if e does not meet a node $v \in VG$ with $b_v < c_e$ or one with $b_v = c_e$ and $d_G(v) \geq 2$.

<u>Proof:</u> Clearly, if e meets a node $v \in VG$ with $b_v < c_e$ or one with $b_v = c_e$ and $d_G(v) \geq 2$, then $x_e \leq c_e$ is not essential for $P(G,b,c)$. Suppose that e does not meet such a node v. Let $x_e = c_e+1$ and $x_f = 0$ for each $f \in EG\backslash\{e\}$. The vector x does not satisfy $x_e \leq c_e$, but it does satisfy all other inequalities in (5.2). □

<u>Lemma 5.7:</u> Let $v \in VG$ and let $b'_u = \min\{b_u, \Sigma\{c_e : e \in \delta(u) \cap \delta(v)\}$ for each node u in $N(v)$. The inequality $x(\delta(v)) \leq b_v$ is essential for $P(G,b,c)$ if and only if one of the following conditions holds.

(5.4) $b'(N(v)) = b_v$ and v is in a two node connected component of G with $b_u = b_v$, where u is the other node of the component, and if $\Sigma\{c_e : e \in \delta(v)\} = b_v$ then $d_G(v) = 1$.

(5.5) $b'(N(v)) = b_v+1$ and there does not exist an edge $(v_1,v_2) \in \gamma(N(v))$ with $b'_{v_1} = b_{v_1}$ and $b'_{v_2} = b_{v_2}$.

(5.6) $b'(N(v)) \geq b_v+2$.

Proof: As in the proof of Lemma 4.5, it is easy to check that if $x(\delta(v)) \leq b_v$ is essential for $P(G,b,c)$ then one of the three conditions holds. If (5.5) or (5.6) holds then Lemma 4.5 and Lemma 5.4 together imply that $x(\delta(v)) \leq b_v$ is essential for $P(G,b,c)$. Suppose that (5.4) holds. If $\Sigma\{c_e: e \in \delta(v)\} > b_v$, let $x_e = c_e$ for each $e \in \delta(v)$ and let $x_e = 0$ for each $e \in EG\backslash\delta(v)$. If $\Sigma\{c_e: e \in \delta(v)\} = b_v$, let $x_e = c_e+1$ for the edge e which meets v and $x_f = 0$ for each $f \in EG\backslash\{e\}$. In either case, x satisfies each inequality in (5.2) other than $x(\delta(v)) \leq b_v$. □

These lemmas imply that the following linear system defines $P(G,b,c)$.

(5.8)

(i) $x_e \geq 0$ $\forall e \in EG$

(ii) $x_e \leq c_e$ $\forall e \in EG$ such that e does not meet a node $v \in VG$ with $b_v < c_e$ or one with $b_v = c_e$ and $d_G(v) \geq 2$.

(iii) $x(\delta(v)) \leq b_v$ $\forall v \in VG$ for which either (5.4), (5.5), or (5.6) holds.

(iv) $x(EH) \leq \lfloor \frac{1}{2} b^{(H,c)}(VH) \rfloor$ for all (b,c)-critical, (b,c)-nonseparable subgraphs H of G such that there does not exist an edge $(u,v) \in EG\backslash EH$ with $v,u \in VH$ and with $b_u^{(H,c)} = b_u$ and $b_v^{(H,c)} = b_v$.

The example given in the previous section shows that this linear system is not in general totally dual integral.

Remark 5.8: Lemma 5.4 and the result mentioned in Remark 4.6 together imply that (5.8) is the unique minimal defining system (unique up to positive scalar multiples of the inequalities) for $P(G,b,c)$. Again, this result will appear in Cook and Pulleyblank [83], but will not be used here. □

If G is connected and $|VG| \geq 3$, then G is (b,c)-bicritical if for each $v \in VG$ there exists a (b,c)-matching x of G such that $x(\delta(v)) = b_v^{(G,c)}-2$ and $x(\delta(u)) = b_u^{(G,c)}$ for each $u \in VG\backslash\{v\}$. To obtain a totally dual integral defining system for $P(G,b,c)$ it suffices to add an inequality $x(EH) \leq \lfloor\frac{1}{2}b^{(H,c)}(VH)\rfloor$ for each (b,c)-bicritical, (b,c)-nonseparable subgraph H of G. To see this, another variation of Tutte's b-matching Theorem is needed. If H is a subgraph of G let $d_H^c(v) = \Sigma\{c_e : e \in \delta_H(v)\}$ for each $v \in VH$. Suppose that $S \subseteq VG$ and $T \subseteq VG\backslash S$. Let

(5.9) $$Q_c(S,T) = \Sigma\{b_v - d^c_{G[VG\backslash S]}(v) : v \in T\}$$

and let $G_c^T(VG\backslash S)$ be the graph obtained from $G[VG\backslash S]$ by taking each node $v \in T$ and performing the following three operations:

(i) replace v by the nodes $\{v_e : e \in \delta(v)\}$

(ii) replace each edge $e = (u,v) \in \delta(v)$ by the edge $e' = (u,v_e)$

(iii) let $b_{v_e} = c_e$ for each $e \in \delta(v)$.

Let $\mathcal{D}_1(S,T,c)$ denote the set of odd connected components of $G_c^T(VG\backslash S)$ which contain at least two nodes.

Theorem 5.9: There exists a perfect (b,c)-matching of G if and only if $\forall S \subseteq VG$ and $T \subseteq VG\backslash S$: $b(S) \geq Q_c(S,T)+|\mathcal{D}_1(S,T,c)|$.

Proof: Let G' be the graph obtained from G by replacing each edge $e \in EG$ with edges $e_1,\ldots,e_{c_e}$ each of which has the same end nodes as e. The graph G' has a perfect simple b-matching if and only if G has a perfect (b,c)-matching. Applying Theorem 4.7 to G' gives the result stated above. □

Note that if c_e is sufficiently large for each $e \in EG$ then x is a (b,c)-matching of G if and only if it is a b-matching of G and if for some $S \subseteq VG$ and some $v \in T \subseteq VG\backslash S$ it is the case that $d_{G[VG\backslash S]}(v) > 0$, then $b(S) \geq Q_c(S,T)+|\mathcal{D}_1(S,T,c)|$. Thus, Theorem 5.9 reduces to Tutte's b-matching Theorem in this case.

An immediate consequence of Corollary 4.8 is the following result, which may also be proven directly from Theorem 5.9.

Lemma 5.10: If G is (b,c)-bicritical, then G has a perfect (b,c)-matching. □

This result and the following lemmas will be used to characterize the Schrijver system for $P(G,b,c)$.

Lemma 5.11: Let H be a connected subgraph of G with $b^{(H,c)}(VH)$ odd and $|VH| \geq 3$. The inequality $x(EH) \leq \lfloor \frac{1}{2} b^{(H,c)}(VH) \rfloor$ is in the Schrijver system for $P(G,b,c)$ only if either it is in (5.2)-(ii) or H is a (b,c)-critical, (b,c)-nonseparable graph.

Proof: Suppose that $x(EH) \leq \lfloor \frac{1}{2} b^{(H,c)}(VH) \rfloor$ is not in (5.2)-(ii) and that H is not a (b,c)-critical, (b,c)-nonseparable graph. If H is (b,c)-separable then Lemma 2.7 implies that the inequality is not in the Schrijver system for $P(G,b,c)$. Suppose that H is not (b,c)-critical. Let $u \in VG$ be a node for which there does not exist a perfect (b',c)-matching of H where $b'_u = b_u^{(H,c)}-1$ and $b'_v = b_v^{(H,c)}$ for each $v \in VH\setminus\{u\}$. By Theorem 5.9 there exists $S \subseteq VH$ and $T \subseteq VH\setminus S$ such that

$$(5.10) \qquad b'(S) < Q'_c(S,T)+|\mathcal{D}'_1(S,T,c)|$$

where $Q'_c(S,T)$ and $\mathcal{D}'_1(S,T,c)$ are $Q_c(S,T)$ and $\mathcal{D}_1(S,T,c)$ with respect to b' and H. Let S and T be such subsets of VH (it may be assumed that if $v \in VH\setminus S$ and $d_{H[VH\setminus S]}(v) = 0$ then $v \in T$).

Let $\mathcal{H}$ be the set of all connected subgraphs $\bar{H}$ of G with $|V\bar{H}| \geq 2$. Corollary 5.2 implies that for each integer vector w, the linear program

$$\min \Sigma\{c_e z_e: e \in EG\}+\Sigma\{b_v y_v: v \in VG\}+\Sigma\{\lfloor\tfrac{1}{2}b^{(\bar{H},c)}(V\bar{H})\rfloor Y_{\bar{H}}: \bar{H} \in \mathcal{H}\}$$

$$\begin{aligned} \text{s.t.} \quad & z_e+y(\Psi(e))+Y(\mathcal{H}(e)) \geq w_e \quad \forall e \in EG \\ & z_e \geq 0 \quad \forall e \in EG \\ & y_v \geq 0 \quad \forall v \in VG \\ & Y_{\bar{H}} \geq 0 \quad \forall \bar{H} \in \mathcal{H} \end{aligned}$$

has an integer optimal solution. To prove that $x(EH) \leq \lfloor\tfrac{1}{2}b^{(H,c)}(VH)\rfloor$ is not in the Schrijver system for $P(G,b,c)$ it suffices to show that for each integer vector w the linear program (5.11) has an integer optimal solution with $Y_H = 0$.

Let w be an integer vector and let (z,y,Y) be an integer optimal solution to (5.11). Suppose that $Y_H > 0$. The inequality (5.10) implies that

$$(5.12) \qquad b^{(H,c)}(S) \le Q_c^{(H,c)}(S,T)+|\mathcal{D}_1^{(H,c)}(S,T,c)|,$$

where $Q_c^{(H,c)}(S,T)$ and $\mathcal{D}_1^{(H,c)}(S,T,c)$ are $Q_c(S,T)$ and $\mathcal{D}_1(S,T,c)$ with respect to $b^{(H,c)}$ and H. Let $\mathcal{D}$ be the set of connected components of $H_c^T(VH\backslash S)$ that contain at least two nodes (so $\mathcal{D}_1^{(H,c)}(S,T,c) \subseteq \mathcal{D}$). Each connected component $\tilde{H}_i$ of $H_c^T(VH\backslash S)$ corresponds to a subgraph H_i of H. Note that if $\tilde{H}_i$ is in $\mathcal{D}$ then $b^{(H_i,c)}(VH_i) \le b^{(H,c)}(V\tilde{H}_i)$.

From the definition of $Q_c(S,T)$ and $\mathcal{D}$ the following equality holds:

$$(5.13) \qquad b^{(H,c)}(VH) = b^{(H,c)}(S)+Q_c^{(H,c)}(S,T)+b^{(H,c)}(\mathcal{D}),$$

where $b^{(H,c)}(\mathcal{D}) = \Sigma\{b^{(H,c)}(V\tilde{H}_i): \tilde{H}_i \in \mathcal{D}\}$. Now (5.12) implies that

$$(5.14) \qquad b^{(H,c)}(VH) \ge b^{(H,c)}(VH)+b^{(H,c)}(S)-Q_c^{(H,c)}(S,T)-|\mathcal{D}_1^{(H,c)}(S,T,c)|.$$

Combining (5.13) and (5.14) gives

$$(5.15) \qquad b^{(H,c)}(VH) \ge 2b^{(H,c)}(S)+b^{(H,c)}(\mathcal{D})-|\mathcal{D}_1^{(H,c)}(S,T,c)|.$$

It follows that

$$(5.16) \qquad \lfloor\tfrac{1}{2}b^{(H,c)}(VH)\rfloor \ge b^{(H,c)}(S)+\tfrac{1}{2}b^{(H,c)}(\mathcal{D})-\tfrac{1}{2}|\mathcal{D}_1^{(H,c)}(S,T,c)|$$

which is equivalent to

$$(5.17)\qquad \lfloor\tfrac{1}{2}b^{(H,c)}(VH)\rfloor \geq b^{(H,c)}(S)+\Sigma\{\lfloor\tfrac{1}{2}b^{(H,c)}(V\tilde{H}_i)\rfloor : \tilde{H}_i \in \mathcal{D}\}.$$

This inequality implies that

$$(5.18)\qquad \lfloor\tfrac{1}{2}b^{(H,c)}(VH)\rfloor \geq b^{(H,c)}(S)+\Sigma\{\lfloor\tfrac{1}{2}b^{(H_i,c)}(VH_i)\rfloor : \tilde{H}_i \in \mathcal{D}\}.$$

Using this it is easy to construct an integer optimal solution to (5.11) with $Y_H = 0$. Define $(\bar{z},\bar{y},\bar{Y})$ as follows

$\bar{Y}_H = 0$

$\bar{Y}_{H_i} = Y_{H_i}+Y_H$ for each $\tilde{H}_i \in \mathcal{D}$

$\bar{Y}_K = Y_K$ for each other $K \in \mathcal{H}$

$\bar{y}_v = y_v+Y_H$ for each $v \in S$ with $b_v^{(H,c)} = b_v$

$\bar{y}_v = y_v$ for each other $v \in VG$

$\bar{z}_e = z_e+Y_H$ for each $e \in EH$ which meets a node $u \in S$ with $b_u^{(H,c)} < b_u$.

$\bar{z}_e = z_e$ for each other $e \in EG$.

It is straightforward to check that $(\bar{z},\bar{y},\bar{Y})$ is a feasible solution to (5.11) (note that since $b^{(H,c)}(VH)$ is odd, H is not H_i for some $\tilde{H}_i \in \mathcal{D}$). The inequality (5.18) implies that it is also an optimal solution to (5.11). □

Lemma 5.12: Let H be a connected subgraph of G with $b^{(H,c)}(VH)$ even and $|VH| \geq 3$. The inequality $x(EH) \leq \lfloor\tfrac{1}{2}b^{(H,c)}(VH)\rfloor$ is in the Schrijver system for $P(G,b,c)$ only if either it is in (5.2)-(ii) or

H is a (b,c)-bicritical, (b,c)-nonseparable graph.

Proof: Suppose that $x(EH) \le \lfloor \frac{1}{2} b^{(H,c)}(VH) \rfloor$ is not in (5.2)-(ii) and that H is not a (b,c)-bicritical, (b,c)-nonseparable graph. Again, if H is (b,c)-separable then Lemma 2.7 implies that the inequality is not in the Schrijver system for $P(G,b,c)$. Suppose that H is not (b,c)-bicritical. Let $u \in VG$ be a node which which there does not exist a perfect (b',c)-matching of H where $b'_u = b_u^{(H,c)}-2$ and $b'_v = b_v^{(H,c)}$ for each $v \in VH\backslash\{u\}$ (it may be assumed that $b'_u \ge 0$). Let $S \subseteq VH$ and $T \subseteq VH\backslash S$ be sets such that

(5.19) $$b'(S) < Q'_c(S,T)+|\mathcal{D}'_1(S,T,c)| .$$

This inequality implies that

(5.20) $$b^{(H,c)}(S) \le Q_c^{(H,c)}(S,T)+|\mathcal{D}_1^{(H,c)}(S,T,c)|+1.$$

Now since $b^{(H,c)}(VH)$ is even, (5.13) implies that $b^{(H,c)}(S)+Q_c^{(H,c)}(S,T)+|\mathcal{D}_1^{(H,c)}(S,T,c)|$ is even. So (5.20) implies (5.12). So the lemma can be proven in the same manner as Lemma 5.11. □

The above two lemmas together imply Lemma 4.9, which completes the proof of Theorem 4.10.

The following result characterizes the Schrijver system for $P(G,b,c)$.

Theorem 5.13: The unique minimal totally dual integral system with integer left hand sides which defines the convex hull of the c-capacitated b-matchings of G is (5.8) together with

(5.21) $x(EH) \le \frac{1}{2} b^{(H,c)}(VH)$ for all (b,c)-bicritical,

(b,c)-nonseparable subgraphs H of G such that there does not exist an edge $(u,v) \in EG \setminus EH$ with $u,v \in VH$ and with $b_u^{(H,c)} = b_u$ and $b_v^{(H,c)} = b_v$ and such that there does not exist an edge $(u,v) \in EG$ with $u \in VH$, $v \notin VH$ and either $c_{(u,v)} = 1$ or $b_v = 1$.

Proof: If an inequality is not in (5.2), then it is not in the Schrijver system for $P(G,b,c)$. It is easy to check that an inequality in (5.2)-(i) or (5.2)-(ii) is in the Schrijver system for $P(G,b,c)$ only if it is in (5.8)-(i), (5.8)-(ii), or (5.8)-(iii). Conversely, by Lemma 5.5, Lemma 5.6 and Lemma 5.7, each inequality in (5.8)-(i), (5.8)-(ii), and (5.8)-(iii) is essential for $P(G,b,c)$ and hence is in the Schrijver system for $P(G,b,c)$.

Let H be a connected subgraph of G with $|VH| \ge 3$. Suppose that $b^{(H,c)}(VH)$ is odd. Using Lemma 5.11, it is easy to check that

(5.22) $x(EH) \le \lfloor \frac{1}{2} b^{(H,c)}(VH) \rfloor$

is in the Schrijver system for $P(G,b,c)$ only if it is in (5.8)-(iv). Conversely, Lemma 2.7 implies that each inequality in (5.8)-(iv) is in the Schrijver system for $P(G,b,c)$. Suppose that $b^{(H,c)}(VH)$ is even. Lemma 5.12 implies that (5.22) is in the Schrijver system for $P(G,b,c)$ only if H is (b,c)-bicritical, (b,c)-nonseparable. Suppose that H is (b,c)-bicritical, (b,c)-nonseparable. Lemma 2.7 implies that (5.22) is in the Schrijver system for $P(G,b,c)$ only if it is in (5.21). Conversely, Lemma 2.7, together with Lemma 5.10, implies that each inequality in (5.21) is in the Schrijver system for $P(G,b,c)$. □

Theorem 4.10 may be obtained from the above result by letting $c_e = 1$ for each $e \in EG$. To obtain the characterization of the Schrijver system for the convex hull of the b-matchings of G, given in Theorem 3.3, from the above result let $c_e = \alpha$ for each e EG where $\alpha = 2 \cdot \max\{b_v : v \in VG\}$. Now a vector x is a b-matching of G if and only if x is a (b,c)-matching of G. So (5.8) together with (5.21) is the Schrijver system for the convex hull of the b-matchings of G. It is straightforward to check that this linear system is identical to (3.3) together with (3.4).

It is possible to test in polynomial time whether or not a given subgraph of G is (b,c)-critical or (b,c)-bicritical. Indeed, using Edmonds' blossom algorithm (Edmonds [65,65a]) and a "scaling" argument similar to the one used by Edmonds and Karp [72] to solve min cost flows, Cunningham and Marsh (see Marsh [79]) found a polynomial-time algorithm for the b-matching problem. Using the construction of Tutte, this algorithm can be used to solve the (b,c)-matching problem in polynomial time, which implies that it is possible to test in polynomial time whether or not a graph is (b,c)-critical or (b,c)-bicritical. The polynomial-time algorithm of Cunningham and Marsh, together with Theorem 3.3, also gives a polynomial-time algorithm to test whether or not a graph is b-separable and, hence, whether or not an inequality is in the Schrijver system for the convex hull of the b-matchings of G. A characterization of (b,c)-nonseparability which yields a polynomial-time test for (b,c)-nonseparability is given in Cook and Pulleyblank [83].

<u>Remark 5.14</u>: An alternative polynomial-time algorithm for the b-matching problem has been found by Anstee[83]. Anstee's algorithm uses a polynomial-time min cost flow algorithm as a subroutine and thus avoids a separate "scaling" argument. □

6. Triangle-free 2-matchings

Let G be a graph. A 2-matching of G is a b-matching with $b_v = 2$ for each $v \in VG$. Motivated by the fact that the 2-matching problem is a relaxation of the travelling salesman problem, Cornuéjols and Pulleyblank [80] considered a constrained variation of 2-matchings called triangle-free 2-matchings. A 2-matching, x, is a triangle-free 2-matching if $x_{e_1}+x_{e_2}+x_{e_3} \leq 2$ for each triple of edges $\{e_1,e_2,e_3\} \subseteq EG$ which form the edges of a triangle of G (a triangle is a circuit of length 3). Cornuéjols and Pulleyblank found a polynomial-time algorithm for solving the triangle-free 2-matching problem and characterized the unique minimal defining system for T(G), the convex hull of the triangle-free 2-matchings of G. The Schrijver system for T(G) will be characterized in this section.

Let $\mathcal{T}$ be the set of all triangles of G. The following result is a consequence of the Cornuéjols-Pulleyblank algorithm.

Theorem 6.1: A totally dual integral defining system for T(G) is

$$(6.1)\quad \begin{array}{ll} x_e \geq 0 & \forall e \in EG \\ x(\delta(v)) \leq 2 & \forall v \in VG \\ x(ET) \leq 2 & \forall T \in \mathcal{T} \\ x(\gamma(S)) \leq |S| & \forall S \subseteq VG. \end{array}$$

□

The following result of Cornuéjols and Pulleyblank [80] follows very easily from the above theorem.

Theorem 6.2: The unique minimal defining system (unique up to positive scalar multiples of the inequalities) for the convex hull of the triangle free 2-matchings of G is

(6.2)
$x_e \geq 0 \quad \forall e \in EG$

$x(\delta(v)) \leq 2$ for all $v \in VG$ such that either $d_G(v) \geq 3$ <u>or</u> $d_G(v)=2$ and v is not the node of a triangle <u>or</u> $d_G(v)=1$ and v is in a 2-node connected component of G.

$x(ET) \leq 2 \quad \forall T \in \mathcal{T}$. □

If G is a circuit of length 5, then (6.2) is not totally dual integral. This is an example of a triangle-free bicritical graph. If x is a triangle-free 2-matching such that $x(\delta(v)) = 2$ for each $v \in VG$, then x is a perfect triangle-free 2-matching. If G is connected and $|VG| \geq 4$, then G is <u>triangle-free-bicritical</u> if for each $v \in VG$ the graph obtained by deleting v from G has a perfect triangle-free 2-matching. A triangle T of a connected graph G is a <u>pendent triangle</u> of G if T contains a cutnode of G and T contains two nodes v_1, v_2 with $d_G(v_1) = d_G(v_2) = 2$.

<u>Theorem 6.3:</u> The unique minimal totally dual integral system with integer left hand sides that defines the convex hull of the triangle-free 2-matchings of G is (6.2) together with

(6.3) $x(\gamma(S)) \leq |S|$ for all $S \subseteq VG$ such that $G[S]$ is triangle-free-bicritical and contains no triangle T which is a pendent triangle of $G[S]$.

<u>Proof:</u> Let $S \subseteq VG$. Suppose that $G[S]$ is not triangle-free-bicritical. Let $v \in S$ be a node such that $G[S\backslash\{v\}]$ does not have a perfect triangle-free 2-matching. The inequality $x(\gamma(S)) \leq |S|$ can be obtained by summing the valid inequalities $x(\delta_{G[S]}(v)) \leq 2$ and $x(\gamma(S\backslash\{v\})) \leq |S|-2$.

So $x(\gamma(S)) \leq |S|$ is not in the Schrijver system for $T(G)$. Using this, it follows from Theorem 6.1 that (6.2) together with (6.3) is a totally dual integral defining system for $T(G)$. So an inequality is in the Schrijver system for $T(G)$ only if it is in (6.2) or (6.3).

Each inequality in (6.2) is essential for $T(G)$ and hence is in the Schrijver system for $T(G)$. Let $S \subseteq VG$ and suppose that $G[S]$ is a triangle-free-bicritical graph which contains no triangle T which is a pendent triangle of $G[S]$. Let $V' \subseteq VG$, $\mathcal{T}' \subseteq \mathcal{T}$, and $\mathcal{S}' \subseteq \{R: R \subseteq VG\}$. Call $(V',\mathcal{T}',\mathcal{S}')$ a cover of $G[S]$ if for each edge $e \in \gamma(S)$ either $e \in \delta(v)$ for some $v \in V'$ or $e \in ET$ for some $T \in \mathcal{T}'$ or $e \in \gamma(S')$ for some $S' \in \mathcal{S}'$. If $(V',\mathcal{T}',\mathcal{S}')$ is a cover of $G[S]$ then the weight of $(V',\mathcal{T}',\mathcal{S}')$ is $w(V',\mathcal{T}',\mathcal{S}') = 2|V'| + 2|\mathcal{T}'| + \Sigma\{|S'|: S' \in \mathcal{S}'\}$. To show that $x(\delta(S)) \leqslant |S|$ is in the Schrijver system for $T(G)$ it suffices to show that the weight of any cover of $G[S]$ such that $S \notin \mathcal{S}'$ is greater than $|S|$. Let $(V',\mathcal{T}',\mathcal{S}')$ be a cover of $G[S]$ of minimum weight (since $(\emptyset,\emptyset,\{S\})$ is a cover of $G[S]$, $w(V',\mathcal{T}',\mathcal{S}') \leq |S|$). Suppose that $S \notin \mathcal{S}'$ and assume that of all minimum weight coverings of $G[S]$ with $S \notin \mathcal{S}'$, $|\mathcal{T}'|$ is as small as possible.

Since $G[S]$ is triangle-free-bicritical, V' must be the empty set. Suppose that $\mathcal{S}'$ is also the empty set. Since $|\mathcal{T}'|$ is as small as possible, $|VT_1 \cap VT_2| \leq 1$ for all $T_1,T_2 \in \mathcal{T}'$. Now since there is no triangle T in $G[S]$ which is a pendent triangle of $G[S]$ and since $|S| \geqq 4$, there exists a circuit C in $G[S]$ which is not a triangle in $\mathcal{T}'$. Let $\{v_0,\ldots,v_{k-1}\}$ be the node set of C, where v_i is adjacent to v_{i+1}, for $i=0,\ldots,k-1$ (subscripts should be taken modulo k). There exist distinct triangles $T_0,\ldots,T_{k-1}$ in $\mathcal{T}'$ such that $VT_i = \{v_i,v_{i+1},u_i\}$, where $u_i \notin VC$ and $u_i \neq u_j$ for $i \neq j$. Let G' be the graph with node set $\cup\{VT_i: i=0,\ldots,k-1\}$ and edge set

$\cup\{ET_i: i=0,\dots,k-1\}$. The graph obtained by deleting v_0 from G' does not have a perfect triangle-free 2-matching. So $G[S]$ is not identical to G'. Thus $S' = \cup\{VT_i: i = 0,\dots,k-1\}$ is not S. Letting $\mathcal{S}'' = \{S'\}$ and $\mathcal{T}'' = \mathcal{T}'\backslash\{T_0,\dots,T_{k-1}\}$, a covering of $G[S]$ is obtained with $S \notin \mathcal{S}''$, $w(V',\mathcal{T}'',\mathcal{S}'') = w(V',\mathcal{T}',\mathcal{S}')$, and $|\mathcal{T}''| < |\mathcal{T}'|$, contrary to assumption. So $\mathcal{S}'$ is nonempty. If S_i and S_j are distinct elements in $\mathcal{S}'$, then $S_i \cap S_j = \emptyset$, since $(V',\mathcal{T}',\mathcal{S}')$ is a minimum weight cover. Let $S_1 \in \mathcal{S}'$. By the assumption on the cardinality of $\mathcal{T}'$, there does not exist any triangle $T \in \mathcal{T}'$ with $VT \cap S_1 \neq \emptyset$. Since $G[S]$ is connected, there exists an edge $e = (u,v) \in \gamma(S)$ with $u \in S_1$ and $v \notin S_1$. So for all $S' \in \mathcal{S}'$, $e \notin \gamma(S')$ and for all $T \in \mathcal{T}'$, $e \notin ET$. So $(V',\mathcal{T}',\mathcal{S}')$ is not a cover of $G[S]$, a contradiction. So there does not exist a covering of $G[S]$ of weight less than or equal to $|S|$ with $S \in \mathcal{S}'$. □

<u>Corollary 6.4:</u> If G is triangle-free-bicritical with no pendent triangle, then G has a perfect triangle-free 2-matching.

<u>Proof:</u> Suppose that G is triangle-free-bicritical with no pendent triangle. Theorem 6.3 implies that $x(EG) \leq |VG|$ is in the Schrijver system for $T(G)$. Hence, G must have a triangle-free 2-matching of size $|VG|$. □

<u>Corollary 6.5:</u> A graph G is triangle-free nonseparable if and only if either G is isomorphic to $K_{1,n}$ for some n <u>or</u> G is triangle -

free-bicritical with no pendent triangle.

Proof: This follows from Theorem 6.3 and Lemma 2.7. □

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