SMALL SEPARATIONS IN PINCH-GRAPHIC MATROIDS

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ABSTRACT. Even-cycle matroids are elementary lifts of graphic matroids. Pinch-graphic matroids are even-cycle matroids that are also elementary projections of graphic matroids. In this paper we analyze the structure of 1-, 2-, and 3-separations in these matroids. As a corollary we obtain a polynomial-time algorithm that reduces the problem of recognizing pinch-graphic matroids to internally 4-connected matroids. Combining this with earlier results [5, 6] we obtain a polynomial-time algorithm for recognizing even-cycle matroids and we obtain a polynomial-time algorithm for recognizing even-cut matroids.

1. INTRODUCTION

By a cycle in a graph we mean a subset of the edges with the property that every vertex of the subgraph formed by these edges has even degree. An inclusion-wise minimal non-empty cycle in a graph is a *polygon*. A matroid M is graphic if its circuits are precisely the polygons of some graph G. Then we write $M = \mathrm{cycle}(G)$. A *signed-graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq EG$. A cycle C is *even* (resp. *odd*) if $|C \cap \Sigma|$ is even (resp. odd). Γ is a *signature* of (G, Σ) if (G, Σ) and (G, Γ) have the same set of even cycles or equivalently if $\Gamma = \Sigma \,\Delta\, \delta(U)$ for some cut $\delta(U) := \{uv \in EG : u \in U, v \notin U\}$ [8]. (Note, $A \,\Delta\, B = (A \cup B) - (A \cap B)$ where $A - B = \{a \in A : a \notin B\}$.) Replacing the signature in a signed graph by a new signature is called *resigning*. M is an *even-cycle matroid* if its circuits are precisely the inclusion-wise minimal non-empty even cycles of some signed-graph (G, Σ). Then we write $M = \mathrm{ecycle}(G, \Sigma)$ and we say that (G, Σ) is a *representation* of M. Even-cycle matroids are binary and are elementary lifts of graphic matroids [11] § 2.5. A pair of vertices a, b of a signed-graph (G, Σ) is a *blocking pair* if every odd polygon of (G, Σ) uses at least one of a or b. An even-cycle matroid M is *pinch-graphic* if it has a representation with a blocking pair. If $\Sigma = \emptyset$ then the circuits of $\mathrm{ecycle}(G, \Sigma)$ are the polygons of

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G, thus graphic matroids are pinch-graphic. Pinch-graphic matroids are also elementary projections of graphic matroids [11], Remark 2.9.

We denote by r_M the rank function of M and write $r(M)$ for the rank of M, i.e. $r(M) = r_M(EM)$. The *connectivity function* takes $X \subseteq EM$ as input and returns $\lambda_M(X) := r_M(X) + r_M(EM - X) - r(M)$. (For both r, λ we omit the index when unambiguous.) Consider $X \subseteq EM$ where $X \neq \emptyset$ and $X \neq EM$ and let k be a positive integer. Then X is k*-separating* when $\lambda(X) \leq k - 1$. It is *exactly* k-separating if equality holds. X is a k*-separation* if it is exactly k-separating and $|X|, |EM - X| \geq k$. A 3-separation X is proper if $|X|, |EM| \geq 4$. A matroid is 2*-connected* if it has no 1-separation, it is 3*-connected* if it 2-connected and has no 2-separation, it is *internally* 4*-connected* if it is 3-connected and there are no proper 3-separations. In this paper we analyze the structure of 1-separations, 2-separations, and proper 3-separations in pinch-graphic matroids.

1.1. **An application.** A consequence of our results will be the following algorithm,

(1) *We present a polynomial algorithm that checks if a matroid M is a pinch-graphic matroid or gives an internally* 4*-connected matroid N that is isomorphic to a minor of M such that M is pinch-graphic if and only if N is pinch-graphic.*

It is proved in [6] that this implies,

(2) *There is a polynomial algorithm that will check if a matroid M is a pinch-graphic matroid.*

It is proved in [5] that is in turn implies,

(3) *There is a polynomial algorithm that will check if a matroid M is an even-cycle matroid.*
(4) *There is a polynomial algorithm that will check if a matroid M is an even-cut matroid.*

For each of (1)-(4) we assume that M is a binary matroid given by its $0, 1$ matrix representation A and by polynomial we mean in the number of entries of A. A *graft* is a pair (G, T) where G is a graph and $T \subseteq VG$ where for every component C of G, $|VC \cap T|$ is even. A cut $\delta(U)$ is *even* if $|T \cap U|$ is even. M is an *even-cut matroid* if its circuits are precisely the inclusion-wise minimal non-empty even cuts of some graft (G, T).

1.2. **Reducible separations.** A matroid M has a 1-separation if and only if M can be expressed as a 1*-sum*, $M_1 \oplus_1 M_2$. A 2-connected matroid has a 2-separation if and only if M can be expressed as a 2*-sum*, $M_1 \oplus_2 M_2$ [1, 2, 13]. A 3-connected binary matroid has a proper 3-separation if and if it can be expressed as a 3*-sum*, $M_1 \oplus_3 M_2$ where $|EM_i - EM_{3-i}| \geq 4$ for $i \in [2]$ [13]. (Sums will be defined formally in Section 2.3.) Consider a binary matroid M where $M = M_1 \oplus_k M_2$ for

$k \in [3]$. If M_1 or M_2 is graphic we say that the k-separation $X = EM_1 - EM_2$ of M is *reducible*. The following result will justify the term,

Proposition 1. *Let $M = M_1 \oplus_k M_2$ for $k \in [3]$ where M_1 is graphic. If $k = 2$ assume that M is 2-connected and if $k = 3$ assume that M is 3-connected. Then M is pinch-graphic if and only if M_2 is pinch-graphic.*

In addition, we have the following useful property,

Proposition 2. *Every 1- and 2-separation of a pinch-graphic matroid is reducible.*

Now suppose that you wish to recognize if a binary matroid M is pinch-graphic. If M has a 1- or 2-separation X, then you may assume it is reducible otherwise by Proposition 2 you can deduce M is not pinch-graphic. Then for some $k \in [2]$ you can express M as $M_1 \oplus_k M_2$ where $X = EM_1 - EM_2$ and M_i is graphic for some $i \in [2]$. Finally, because of Proposition 1 it suffices to check if M_{3-i} is pinch-graphic. This allows you to reduce the recognition problem to 3-connected matroids.

1.3. **Non-reducible 3-separations.** If every proper 3-separation of a pinch-graphic matroid was reducible, Proposition 1 would yield the promised algorithm (1). Alas it is not true. Consider the signed-graphs illustrated in Figure 1 (i) and (ii). The shaded region corresponds to edges X. In black

(i) (ii)

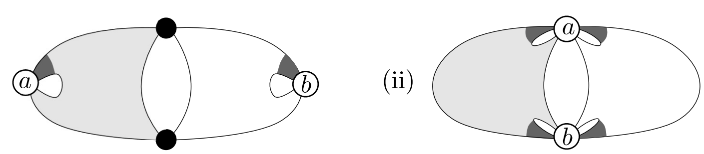


FIGURE 1. Examples of 3-separations that are not reducible.

we indicate a signature with all edges incident to the blocking pair a, b. Then X is a 3-separation of the corresponding pinch-graphic matroid and in general X is not reducible. (i) is an example of a *compliant* 3-separation X and (ii) is an example of a *recalcitrant* 3-separation X. Given a 3-separation X, we can get a new 3-separation Y by moving elements that are in the co-closure of either side of the separation. We say that X and Y are *homologous*. We will prove that in a 3-connected pinch-graphic matroid every proper 3-separation is homologous to a 3-separation that is reducible, compliant or recalcitrant. This will be the basis for algorithm (1).

An extended abstract of the content of this paper and [5, 6] appeared in [9].

1.4. **Organization of the paper.** In Section 2 we give a catalogue of 1-, 2- and 3-sums in pinch-graphic matroids in terms of the associated blocking pair representation. We use this to prove Proposition 1. In Section 3 we prove that 1- and 2-separations in pinch-graphic matroids are reducible, i.e. we show Proposition 2. We give a structure theorem for proper 3-separations in 3-connected pinch-matroids in Section 4. We further analyze the compliant and recalcitrant outcomes of the structure theorem in Section 5. Algorithm (1) is described in Section 6.

2. A CATALOGUE OF 1,2- AND 3-SUMS

2.1. **Minors.** Consider a signed-graph (G,Σ) and $I,J \subseteq EG$ where $I \cap J = \emptyset$. The minor $(G,\Sigma)/I \setminus J$ is the signed-graph defined as follows: If there exists an odd polygon of (G,Σ) contained in I then $(G,\Sigma)/I \setminus J = (G/I \setminus J, \emptyset)$, otherwise there exists a signature Γ where $\Gamma \cap I = \emptyset$ and $(G,\Sigma)/I \setminus J = (G/I \setminus J, \Gamma - J)$. Note, minors are only defined up to resigning. Consider an even-cycle matroid M with a representation (G,Σ). Then $(H,\Gamma) = (G,\Sigma)/I \setminus J$ is a representation of the minor $N = M/I \setminus J$ [11], page 21. In particular, the class of even-cycle matroids is minor-closed. Moreover, observe that having a blocking pair in a signed-graph is closed under minor. Hence, the class of pinch-graphic matroids is also minor-closed.

2.2. **Auxiliary graph.** Given a graph G and $X \subseteq EG$, we write $G|X$ for the subgraph of G with edges X and vertices that correspond to endpoints of edges of X. We denote $\partial(X)$ the set of vertices common to $G|X$ and $G|EG-X$. Consider a graph G and a set $X \subseteq EG$ where $X \neq \emptyset$ and $X \neq EG$. We define the *auxiliary graph H for the pair G and X* as follows: H is bipartite with bipartition U,W where vertices in U correspond to components of $G|X$ and vertices in W correspond to components in $G|EG-X$. For every $v \in \partial_G(X)$ we have an edge e_v of H with endpoints $u \in U$ and $w \in W$ where u corresponds to the unique component of $G|X$ containing v and w corresponds to the unique component of $G|EG-X$ containing v. We give an example in Figure 2. For each of (i), (ii), (iii) we have the auxiliary graph H on top and G where the shaded region correspond to edges in X on the bottom.

Given a signed graph (G,Σ) and $X \subseteq EG$ we denote by $(G,\Sigma)|X$ the signed-graph induced by the edges X, i.e. $(G,\Sigma)|X = (G|X, \Sigma \cap X)$. A signed-graph is *bipartite* if it has no odd cycle or equivalently if $\emptyset$ is a signature [8] see also [11], page 23. Given a signed-graph (G,Σ) we define,

$$p[(G,\Sigma)] := \begin{cases} 0 & \text{if } (G,\Sigma) \text{ is bipartite} \\ 1 & \text{otherwise.} \end{cases}$$

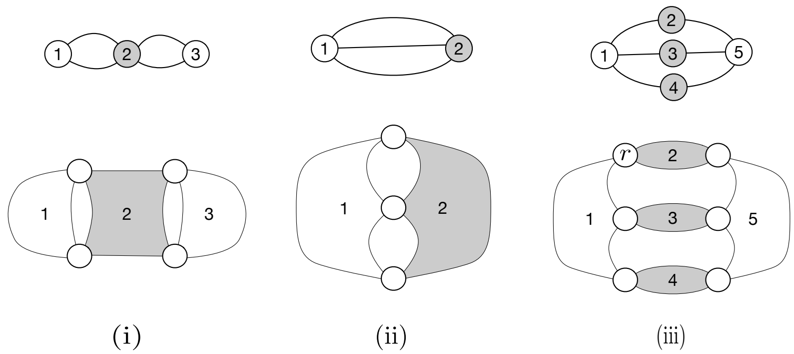


FIGURE 2. Auxiliary graph

For a graph H we denote the number of components of H by $\kappa(H)$. We are now ready to relate the connectivity function with the auxiliary graph,

Proposition 3 (Proposition 11, [5], see also [7])**.** *Consider an even-cycle matroid M with a non-bipartite representation (G,Σ) where G is connected. Let X, Y be a partition of EM where X, Y are non-empty. Then*

(a) $$\lambda_M(X) = |\partial_G(X)| - \kappa(G|X) - \kappa(G|Y) + p[(G,\Sigma)|X] + p[(G,\Sigma)|Y].$$

Moreover, denote by H be the auxiliary graph for the pair G and X and assume M is connected. Then

(b) $$|EH| = |VH| + \lambda_M(X) - p[(G,\Sigma)|X] - p[(G,\Sigma)|Y] \geq |VH| - 1.$$

2.3. **Sums.** Let M_1, M_2 be matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 1$. Suppose that $E_1 \cap E_2 = \emptyset$. Then, we define the *1-sum* M of M_1, M_2, denoted by $M_1 \oplus_1 M_2$, as follows: the ground set of M is $E := E_1 \cup E_2$ and a subset C of E is a circuit of M if and only if C is either a circuit of M_1 or a circuit of M_2. Let M_1, M_2 be matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 3$. Suppose that $E_1 \cap E_2 = \{\Omega\}$ and that Ω is not a loop and not a coloop of M_i for $i \in [2]$. Then, we define the *2-sum* M of M_1, M_2, denoted by $M_1 \oplus_2 M_2$, as follows: the ground set of M is $E := E_1 \,\Delta\, E_2$ and a subset C of E is a circuit of M if and only if either C is a circuit of $M_1 \setminus \Omega$ or $M_2 \setminus \Omega$, or $C = C_1 \,\Delta\, C_2$ where for $i \in [2]$, C_i is a circuit of M_i containing Ω. Let M_1, M_2 be binary matroids on ground sets E_1, E_2, respectively where $|E_1|, |E_2| \geq 7$. Suppose that $E_1 \cap E_2 = D$ where $|D| = 3$ where for $i \in [2]$, D is a circuit of M_i and D contains no cocircuit of M_i. Then, we define the *3-sum* M of M_1, M_2, denoted by $M_1 \oplus_3 M_2$, as follows: the ground set of M is $E := E_1 \,\Delta\, E_2$ and a subset C of E is a circuit of M if and only if either C is a circuit of $M_1 \setminus D$ or $M_2 \setminus D$ or $C = C_1 \,\Delta\, C_2$ where for some $e \in D$ and for $i \in [2]$, C_i is a circuit of M_i with $C_i \cap D = \{e\}$. Note that 3-sums are defined for binary matroids only. If we have matroid $M = M_1 \oplus_1 M_2$ then M_1 and M_2 are restrictions of M and in particular are minors of M. Analogous results hold for 2- and 3-sums [13], namely

Proposition 4. *Suppose that $M = M_1 \oplus_2 M_2$ where M is 2-connected, or that $M = M_1 \oplus_3 M_2$ where M is 3-connected and binary. Then M_1 and M_2 are isomorphic to minors of M.*

2.4. **Completion.** We first require the following folklore observation,

Proposition 5. *Let M be a matroid with matrix representation A and let $X \subseteq EM$. We denote by $\langle X\rangle$ the vector space spanned by the columns of A indexed by X. Then*

$$\lambda_M(X) = \dim\left[\langle X\rangle \cap \langle EM - X\rangle\right].$$

Let M be a binary matroid with matrix representation A and let X be a 2-separation of M. Then $\lambda_M(X) = 1$. It follows from Proposition 5 that $\dim\left[\langle X\rangle \cap \langle EM - X\rangle\right] = 1$. Thus there exists a unique non-zero $0,1$ vector p for which $\langle p\rangle = \langle X\rangle \cap \langle EM - X\rangle$. Let A^+ be obtained from matrix A by adding column p and let N be the binary matroid represented by matrix A^+. Then N is the *completion of M with respect to the 2-separation X*. Let M be a binary matroid with matrix representation A and let X be a 3-separation of M. Then $\lambda_M(X) = 2$. By Proposition 5, $dim\left[\langle X\rangle \cap \langle EM - X\rangle\right] = 2$. Thus there exists non-zero $0,1$ vectors p, q for which $\langle \{p, q\}\rangle = \langle X\rangle \cap \langle EM - X\rangle$. Let A^+ be obtained from matrix A by adding columns p, q and $r = p + q$ (where the sum is taken over the two element field). Note that the set $\{p, q, r\}$ is uniquely determined by $\langle X\rangle \cap \langle EM - X\rangle$. Let N be the binary matroid represented by matrix A^+. Then N is the *completion of M with respect to the 3-separation X*.

Next we explain the relevance of the notion of completion,

Proposition 6 (Proposition 14 in [5])**.** *Let M be a binary matroid with a 2-separation X. Let N be the completion of M with respect to X. Then $M = (N \setminus X) \oplus_2 (N \setminus EM - X)$.*

Proposition 7. *Let M be a binary matroid with a proper 3-separation X. Let N be the completion of M with respect to X. Then $M = (N \setminus X) \oplus_3 (N \setminus EM - X)$.*

The proof of Proposition 7 is easy and similar to that of Proposition 6 so we shall omit it. The following straightforward observation will allow us to construct completions for 3-separations,

Remark 8. *Let M be a binary matroid with a proper 3-separation X. Let N be a binary matroid where $M = N \setminus \{\Omega_1, \Omega_2, \Omega_3\}$ and where $\{\Omega_1, \Omega_2, \Omega_3\}$ is a circuit of N. Suppose for $i \in [2]$ we have cycles C_i and D_i of N where $\Omega_i \in C_i \cap D_i$ and $C_i \subseteq X \cup \Omega_i$, $D_i \subseteq (EM - X) \cup \Omega_i$ then N is the completion of M with respect to X.*

2.5. **Examples of 2-sums.** We will require the following results,

Proposition 9 (Proposition 16 [5])**.** *Let $M = \text{ecycle}(G, \Sigma)$ with a 2-separation X and let $Y = EM - X$. Suppose that $p[(G, \Sigma)|X] = p[(G, \Sigma)|Y] = 1$. Let (G_1, Σ_1) be obtained from $(G, \Sigma)|X$ by adding an odd loop Ω and let (G_2, Σ_2) be obtained from $(G, \Sigma)|Y$ by adding an odd loop Ω. Then $M = \text{ecycle}(G_1, \Sigma_1) \oplus_2 \text{ecycle}(G_2, \Sigma_2)$.*

Proposition 10 (Proposition 17 [5])**.** *Let $M = \text{ecycle}(G, \Sigma)$ with a 2-separation X and let $Y = EM - X$. Suppose that $p[(G, \Sigma)|X] = 1$, $p[(G, \Sigma)|Y] = 0$, that $G|X$, $G|Y$ are connected and that $\partial_G(X) = \{a, b\}$ where a, b are distinct vertices. Then we may assume, after possibly resigning, that $\Sigma \subseteq X$. Let G_1 (resp. G_2) be obtained from $G|X$ (resp. $G|Y$) by adding edge $\Omega = (a, b)$. Then $M = \text{ecycle}(G_1, \Sigma) \oplus_2 \text{cycle}(G_2)$.*

2.6. **Examples of 3-sums.**

Proposition 11. *Let $M = \text{ecycle}(G, \Sigma)$ with a proper 3-separation X and let $Y = EM - X$. Suppose that $p[(G, \Sigma)|X] = p[(G, \Sigma)|Y] = 1$, that $G|X$, $G|Y$ are connected and that $\partial_G(X) = \{a, b\}$ where a, b are distinct vertices. Let (G_1, Σ_1) (resp. (G_2, Σ_2)) be obtained from $(G, \Sigma)|X$ (resp. $(G, \Sigma)|Y$) by adding an even edge $\Omega_1 = (a, b)$, an odd edge $\Omega_2 = (a, b)$ and an odd loop Ω_3. Then $M = \text{ecycle}(G_1, \Sigma_1) \oplus_3 \text{ecycle}(G_2, \Sigma_2)$.*

Proof. Let (H, Γ) be the signed-graph obtained from (G, Σ) by adding an even edge $\Omega_1 = (a, b)$, an odd edge $\Omega_2 = (a, b)$ and an odd loop Ω_3. Since $(G, \Sigma)|X$ is connected and non-bipartite, it has an even ab-join J_1 and an odd ab-join J_2 and since $(G, \Sigma)|Y$ is connected and non-bipartite, it has an even ab-join K_1 and an odd ab-join K_2. For $i \in [2]$, let $C_i = J_i \cup \Omega_i$ and let $D_i = K_i \cup \Omega_i$. Then for $i \in [2]$, C_i and D_i are cycles of $N := \text{ecycle}(H, \Gamma)$ where $\Omega_i \in C_i \cap D_i$ and $C_i \subseteq X \cup \Omega_i$, $D_i \subseteq Y \cup \Omega_i$. Moreover, $\{\Omega_1, \Omega_2, \Omega_3\}$ is a circuit of N. It follows from Remark 8 that N is the completion of M with respect to X. By Proposition 7, $M = (N \setminus Y) \oplus_3 (N \setminus X)$. Moreover, $N \setminus Y = \text{ecycle}(H, \Gamma) \setminus Y = \text{ecycle}(G_1, \Sigma_1)$ and $N \setminus X = \text{ecycle}(H, \Gamma) \setminus X = \text{ecycle}(G_2, \Sigma_2)$. □

Proposition 12. *Let* $M = \mathrm{ecycle}(G, \Sigma)$ *with a proper* 3*-separation* X *and let* $Y = EM - X$*. Suppose that* $p[(G, \Sigma)|X] = 1$*,* $p[(G, \Sigma)|Y] = 0$*, that* $G|X$*,* $G|Y$ *are connected and that* $\partial_G(X) = \{a, b, c\}$ *where* a, b, c *are distinct vertices. Then we may assume, after possibly resigning, that* $\Sigma \subseteq X$*. Let* G_1 *(resp.* G_2*) be obtained from* $G|X$ *(resp.* $G|Y$*) by adding edges* $\Omega_1 = (a, b)$*,* $\Omega_2 = (b, c)$ *and* $\Omega_3 = (a, c)$*. Then* $M = \mathrm{ecycle}(G_1, \Sigma) \oplus_3 \mathrm{cycle}(G_2)$*.*

Proof. Since $(G, \Sigma)|Y$ is bipartite, there exists a signature contained in X, hence we may assume $\Sigma \subseteq X$. Let H be the graph obtained from G by adding edges $\Omega_1 = (a, b)$, $\Omega_2 = (b, c)$ and $\Omega_3 = (a, c)$. Since $(G, \Sigma)|X$ is connected and non-bipartite, it has an even ab-join J_1 and an even ab-join J_2 and since $G|Y$ is connected, it has an ab-join K_1 and an ab-join K_2. For $i \in [2]$, let $C_i = J_i \cup \Omega_i$ and let $D_i = K_i \cup \Omega_i$. Then for $i \in [2]$, C_i and D_i are cycles of $N := \mathrm{ecycle}(H, \Sigma)$ where $\Omega_i \in C_i \cap D_i$ and $C_i \subseteq X \cup \Omega_i$, $D_i \subseteq Y \cup \Omega_i$. Moreover, $\{\Omega_1, \Omega_2, \Omega_3\}$ is a circuit of N. It follows from Remark 8 that N is the completion of M with respect to X. By Proposition 7, $M = (N \setminus Y) \oplus_3 (N \setminus X)$. Moreover, $N \setminus Y = \mathrm{ecycle}(H, \Sigma) \setminus Y = \mathrm{ecycle}(G_1, \Sigma)$ and $N \setminus X = \mathrm{ecycle}(H, \Sigma) \setminus X = \mathrm{cycle}(G_2)$. □

2.7. **Applications.** In this section we prove Proposition 1 by proving Propositions 13, 14, 15.

Proposition 13. *Let* $M = M_1 \oplus_1 M_2$ *where* M_1 *is graphic. Then* M *is pinch-graphic if and only if* M_2 *is pinch-graphic.*

Proof. If M is pinch-graphic so is $M_2 = M \setminus EM_1$ as pinch-graphic matroids form a minor closed class. Suppose that M_2 is pinch-graphic, i.e. $M_2 = \mathrm{ecycle}(G_2, \Sigma)$ for some signed-graph (G_2, Σ) with a blocking pair, say v, w. Since M_1 is graphic, $M_1 = \mathrm{cycle}(G_1)$ for some graph G_1. Then $M = \mathrm{ecycle}(G, \Sigma)$ where G is the union of G_1 and G_2. As v, w is a blocking pair of (G, Σ), M is pinch-graphic. □

Proposition 14. *Let* $M = M_1 \oplus_2 M_2$ *where* M_1 *is graphic and* M *is* 2*-connected. Then* M *is pinch-graphic if and only if* M_2 *is pinch-graphic.*

Proof. If M is pinch-graphic then so is M_2 since M_2 is isomorphic to a minor of M (Proposition 4) and pinch-graphic matroids form a minor closed class. Suppose that M_2 is pinch-graphic, i.e. $M_2 = \mathrm{ecycle}(G_2, \Sigma_2)$ for some signed-graph (G_2, Σ_2) with a blocking pair, say v, w. Since M_1 is graphic, $M_1 = \mathrm{cycle}(G_1)$ for some graph G_1. Denote by e the unique element in $EM_1 \cap EM_2$. By definition of 2-sums, e is not a loop or a co-loop of M_1 or M_2. In particular, e is not a loop of G_1 and not a bridge of G_1 or G_2.

Case 1. e is not a loop of G_2.

After possibly resigning (G_2, Σ_2) we may assume $e \notin \Sigma_2$. Let G be obtained from G_1 and G_2 by identifying edge e and then deleting e. Let $M' = \text{ecycle}(G, \Sigma_2)$. Proposition 3 (a) and the fact that e is not a bridge of G_1, G_2 implies that $\lambda_{M'}(X) = 1$, thus X is a 2-separation of M'. By Proposition 10 $M' = \text{cycle}(G_1) \oplus_2 \text{ecycle}(G_2, \Sigma_2) = M_1 \oplus_2 M_2$. Thus $M = M'$ and in particular (G, Σ_2) is a representation of M. Finally, observe that v, w is a blocking pair of (G, Σ_2), hence, M is pinch-graphic.

Case 2. e is a loop of G_2.

Since e is not a loop of M_2, $e \in \Sigma_2$, thus e is incident to v or w in G_2. Suppose r, s denote the ends of e in G_1. Then let $\Sigma_1 = \delta_{G_1}(r)$ and let G_1' be the graph obtained from G_1 by identifying r and s. Note, e is an odd loop of (G_1', Σ_1) with ends $r = s$. Let G be obtained from G_1 and G_2 by identifying the vertex incident to e and then deleting e. Let $M' = \text{ecycle}(G, \Sigma_1 \cup \Sigma_2 - e)$. Proposition 3 (a) and the fact that e is not a bridge of G_1, implies that $\lambda_{M'}(X) = 1$, thus X is a 2-separation of M'. By Proposition 9 $M' = \text{ecycle}(G_1', \Sigma_1) \oplus_2 \text{ecycle}(G_2, \Sigma_2) = M_1 \oplus_2 M_2$. Thus $M = M'$ and in particular (G, Σ_2) is a representation of M. Finally, observe that v, w is a blocking pair of (G, Σ_2), hence, M is pinch-graphic. □

Proposition 15. *Let $M = M_1 \oplus_3 M_2$ where M_1 is graphic and M is 3-connected. Then M is pinch-graphic if and only if M_2 is pinch-graphic.*

Proof. Sufficiency follows as in Proposition 14. Suppose that M_2 is pinch-graphic, i.e. $M_2 = \text{ecycle}(G_2, \Sigma_2)$ for some signed-graph (G_2, Σ_2) with a blocking pair, say v, w. Since M_1 is graphic, $M_1 = \text{cycle}(G_1)$ for some graph G_1. Denote by $D = \{e, f, g\} = EM_1 \cap EM_2$. By definition of 3-sum, D is a circuit of M_1 of M_2. In particular, D is a polygon of G_1. Also by definition of 3-sum D does not contain a cocircuit of M_1 or M_2. Hence, D does not contain a cut of G_1 or G_2. After possibly interchanging the roles of e, f, g we may assume that one of Case 1 or Case 2 occurs.

Case 1. D is a polygon of G_2.

After possibly resigning (G_2, Σ_2) we may assume that $\Sigma_2 \cap D = \emptyset$. Let G be obtained from G_1, G_2 by identifying e identifying f and identifying g and deleting e, f, g. Let $M' = \text{ecycle}(G, \Sigma_2)$. Proposition 3 (a) and the fact that D does not contain a cut of G_1, G_2, implies that $\lambda_{M'}(X) = 2$, thus X is a 3-separation of M'. By Proposition 12 $M' = \text{cycle}(G_1) \oplus_3 \text{ecycle}(G_2, \Sigma_2) = M_1 \oplus_3 M_2$.

Thus $M = M'$ and in particular (G, Σ_2) is a representation of M. Finally, observe that v, w is a blocking pair of (G, Σ_2), hence, M is pinch-graphic.

Case 2. e, f are parallel and g is a loop in G_2. Moreover, $\{e, f, g\} \cap \Sigma_2 = \{f, g\}$.

Let r be the vertex of G_1 incident to g, f and let s be the vertex of G_1 incident to g, e. Then let $\Sigma_1 = \delta_{G_1}(r)$ and let G_1' be the graph obtained from G_1 by identifying r and s. Note that g is an odd loop of (G_1', Σ_1) with ends $r = s$. Since $\{e, f\}$ is an odd polygon of (G_2, Σ_2) we may assume that one of the end of e, f is vertex v of the blocking pair v, w and that g is incident to v in G_2. Let G be obtained from G_1 and G_2 by identifying vertex $r = s$ of G_1 with vertex v of G_2, by identifying the other end of e, f, and then deleting e, f, g. Let $\Gamma = (\Sigma_1 \cup \Sigma_2) - \{e, f, g\}$. Let $M' = \mathrm{ecycle}(G, \Gamma)$. Proposition 3 (a) and the fact that D does not contain a cut of G_1, G_2 implies that $\lambda_{M'}(X) = 2$, thus X is a 3-separation of M'. By Proposition 11 $M' = \mathrm{ecycle}(G_1', \Sigma_1) \oplus_3 \mathrm{ecycle}(G_2, \Sigma_2) = M_1 \oplus_3 M_2$. Thus $M = M'$ and in particular (G, Γ) is a representation of M. Note that v, w is a blocking pair of (G, Γ), hence, M is pinch-graphic. $\square$

3. 1- AND 2-SEPARATIONS ARE REDUCIBLE

The goal of this section is to prove Proposition 2. First we require some definitions.

3.1. **Flips.** Consider a graph G with a partition X, Y of its edge set where $G|X$ and $G|Y$ are connected and where $\partial(X)$ consists of two vertices v_1 and v_2. Let G' be obtained from G by identifying, for $i \in [2]$, vertex v_i of $G|X$ with with vertex v_{3-i} of $G|Y$. We say that G' is obtained from G by a 2-*flip* on the set X (resp. Y). We call a 1-*flip* the identification of two vertices in distinct components or splitting two blocks into different components. Two graphs are *equivalent* if they are related by a sequence of 1-flips and 2-flips.

Theorem 16 (Whitney [17])**.** *A pair of graphs have the same set of polygons (viewed as edge sets) if and only if they are equivalent.*

Two signed graphs are *equivalent* if they are related by a sequence of 1-flips, 2-flips and resignings. In particular, if (G, Σ) is a representation of an even-cycle matroids then so is every equivalent signed-graph (G', Σ'). Note, however that 1- and 2-flips do not preserve blocking pairs, i.e. the set of blocking pair representations is not closed under equivalence.

3.2. **Warm-up.** Recall that if a bipartite signed-graph is a representation of an even-cycle matroid then that matroid is graphic.

Proposition 17. *If X is a 1-separation of an even-cycle matroid M, then X is reducible.*

Proof. Then $M = M_1 \oplus_1 M_2$ for some M_1, M_2 where $X = EM_1$ and let $Y = EM_2$. We may assume that M is not graphic for otherwise so is M_1 and X is reducible. Thus M has a non-bipartite representation (G, Σ). After possible 1-flips we may assume G is connected. Let H be the auxiliary graph for X and (G, Σ). Since $\lambda_M(X) = 0$, Proposition 3 (b) implies, $|VH| + 0 - p[(G, \Sigma)|X] - p[(G, \Sigma)|Y] \geq |VH| - 1$ or equivalently, $p[(G, \Sigma)|X] + p[(G, \Sigma)|Y] \leq 1$. Thus $(G, \Sigma)|X$ or $(G, \Sigma)|Y$ is bipartite. As $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are representations of M_1 and M_2 respectively, at least one of M_1, M_2 is graphic, i.e. X is reducible. □

It remains to prove the analogous result for 2-separations, namely,

Proposition 18. *If X is a 2-separation of a 2-connected pinch-graphic matroid M, then X is reducible.*

Before we can proceed with the proof we will require some preliminaries.

3.3. **Preliminaries.** Consider a graph H with two distinct vertices r, s that consists of three internally disjoint rs-paths P_1, P_2, P_3 (all vertices of H except r, s have degree two). Then we say that H is a *theta*. If $|P_1| = |P_2| = |P_3| = k$ for some integer k (where P_i are viewed as subsets of edges) then the theta graph is *k-uniform*. Consider now a graph H that is obtained from two disjoint polygons C_1, C_2 by identifying a vertex of C_1 with a vertex of C_2. Then we say that H is a *double ear*. If $|C_1| = |C_2| = k$ for some integer k (where C_i are viewed as subsets of edges) then the double ear is *k-uniform*. In Figure 2(i) - top graph, we have a 2-uniform double ear, in (ii) top graph, we have a 1-uniform theta graph, and in (iii) top graph, we have a 2-uniform theta graph.

Remark 19. *A connected bridgeless graph H where $|EH| = |VH| + 1$ is a double ear or a theta.*

Consider a signed-graph (G, Σ) and vertices $v_1, v_2 \in V(G)$ where every edge of Σ is incident to either v_1 or v_2. So v_1, v_2 is a blocking pair of (G, Σ). Consider first the case where there is no odd edge between v_1 and v_2 and there is no odd loop. Let H be obtained from G by, for $i \in [2]$, splitting v_i into v_i', v_i'' such that $\delta_H(v_i') = \delta_G(v_i) \cap \Sigma$. Then let G' be obtained from H by identifying v_1' and v_2'' to a new vertex w_1 and by identifying v_2' and v_1'' to a new vertex w_2. If (G, Σ) has an odd loop f, then f will have ends w_1, w_2 in G' and if (G, Σ) has an odd edge g with ends v_1, v_2 then g will be an odd loop of (G', Σ). We then say that (G', Σ) is obtained from (G, Σ) by a *Lovász-flip* on v_1, v_2 and that w_1, w_2 is the *resulting blocking pair*. Informally, G' is obtained from G by exchanging the odd edges incident to v_1 with the odd edges incident to v_2 where odd loops and odd edges between v_1 and v_2 behave like odd walks of length two. It is not difficult to see that Lovász-flips preserve even-cycles [11, 3]. In Figure 3 we illustrate a pair of signed-graphs related by a Lovász-flip. Vertices v_1 and v_2 are indicated in white. Odd edges correspond to dashed lines. Even edges are unchanged.

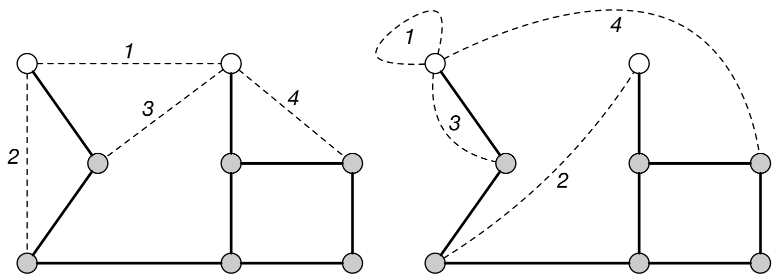


FIGURE 3. Lovász-flip

A vertex a of a signed-graph (G, Σ) is a *blocking-vertex* if every odd polygon of (G, Σ) uses a.

Remark 20 (Lemma 6, [7]). *If an even-cycle matroid has a representation with a blocking vertex then it is graphic.*

3.4. **Proof of Proposition 18.** Since X is a 2-separation of M, $M = M_1 \oplus_2 M_2$ for some matroids M_1, M_2 where $X = EM_1 - EM_2$ and let $Y = EM_2 - EM_1$. We may assume that M_1, M_2 are not graphic for otherwise X is reducible and we are done. Let e denote the unique element in $EM_1 \cap EM_2$. Since M is an even-cycle matroid it has a representation (G, Σ). Note that (G, Σ) is not bipartite for otherwise M is graphic and then by Proposition 4 so is M_1, a contradiction. Among all possible connected representation (G, Σ) of M pick one according to the following priorities,

(m1) (G, Σ) has a blocking pair and $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are both non-bipartite,

(m2) $(G, \Sigma)|X$ is bipartite and $(G, \Sigma)|Y$ is non-bipartite and we minimize $\kappa(G|X) + \kappa(G|Y)$.

(m3) $(G, \Sigma)|X$ and $(G, \Sigma)|Y$ are both non-bipartite and we minimize $\kappa(G|X) + \kappa(G|Y)$.

Note for (m2) and (m3) we do not require that (G, Σ) have a blocking pair. Let H denote the auxiliary graph for X and (G, Σ). Since G is connected, so is H.

Claim 1. (G, Σ) *is not picked according to (m1).*

Subproof. Suppose otherwise. Proposition 3 (b) implies, $|EH| = |VH| + 1 - 1 - 1 = |VH| - 1$. Since H is connected H is a tree. Denote by $X_1, \ldots, X_p$ the connected components of $G|X$ and by $Y_1, \dots, Y_q$ the connected components of $G|Y$. For all $j \in [p]$, $(G,\Sigma)|X_j$ is non-bipartite for otherwise X_j is a 1-separation of M by Proposition 3 (a), a contradiction. Similarly, $(G,\Sigma)|Y_j$ is non-bipartite for all $j \in [q]$. If for all $j \in [p]$, $(G,\Sigma)|X_j$ has a blocking vertex and for all $j \in [q]$, $(G,\Sigma)|Y_j$ has a blocking vertex then after a sequence of 1-flips there exists a blocking vertex of (G,Σ), a contradiction as Remark 20 then implies that M is graphic. Thus we may assume that $(G,\Sigma)|X_1$ has no blocking vertex and that $(G,\Sigma)|X_1$ has a blocking pair v, w of (G,Σ). Note that $p = 1$ since otherwise $(G,\Sigma)|X_2$ contains an odd polygon that avoids v, w, a contradiction. After possible 1-flips we may assume that v is vertex of $G|Y_j$ for all $j \in [q]$, in particular, $q = 1$. Thus $G|X$ and $G|Y$ are both connected and $v \in \partial_G(X)$. Let G_1 be obtained from $G|X$ by adding a loop e_1 incident to vertex v and let G_2 be obtained from $G|Y$ by adding a loop e_2 incident to vertex v. Let $\Sigma_1 = (\Sigma \cap X) \cup e_1$ and $\Sigma_2 = (\Sigma \cap Y) \cup e_2$. It follows by Proposition 9 that $M = \mathrm{ecycle}(G,\Sigma) = \mathrm{ecycle}(G_1,\Sigma_1) \oplus_2 \mathrm{ecycle}(G_2,\Sigma_2)$. Thus $M_2 = \mathrm{ecycle}(G_2,\Sigma_2)$. Finally, note that (G_2,Σ_2) has a blocking vertex v. It follows from Remark 20 that M_2 is graphic, a contradiction. $\Diamond$

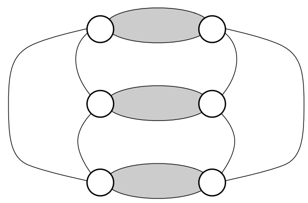
Claim 2. (G,Σ) *is not picked according to (m2).*

Subproof. Suppose otherwise. Since $(G,\Sigma)|X$ is bipartite, we may assume after possibly resigning, that $\Sigma \subseteq Y$. Proposition 3 (b) implies, $|EH| = |VH| + 1 - 0 - 1 = |VH|$. We picked (G,Σ) that minimizes $\kappa(G|X) + \kappa(G|Y)$. Since we can rearrange G by 1-flips, H is bridgeless, hence H is a polygon. Since we can rearrange G by 2-flips, $|VH| = 2$, i.e. $G|X$ and $G|Y$ are connected and $\partial_G(X) = \{a, b\}$ for some distinct vertices a, b. Let G_1 be obtained from $G|X$ by adding an edge e_1 between a, b and let G_2 be obtained from $G|Y$ by adding an edge e_2 between a, b. It follows by Proposition 10 that $M = \mathrm{ecycle}(G,\Sigma) = \mathrm{cycle}(G_1) \oplus_2 \mathrm{ecycle}(G_2,\Sigma)$. Hence, $M_1 = \mathrm{cycle}(G_1)$ is graphic, a contradiction. $\Diamond$

It follows from Claim 1 and Claim 2 that $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are both bipartite. Proposition 3 (b) implies that $|EH| = |VH| + 1 - 0 - 0 = |VH| + 1$. Hence, by Remark 19 H is either a theta or a double ear. By the choice (m3) and since we can rearrange G by 1-flips, H is bridgeless. (Note, that here we are free to perform 1-flip and 2-flips as we are not trying to preserve blocking pairs).

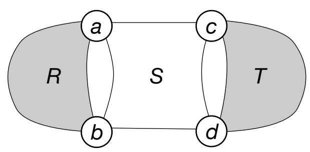
Case 1. H is a theta.

H consists of three internally disjoint path P_1, P_2, P_3. By (m3) and since we can rearrange G by 2-flips, for $j \in [3]$, $|P_j| \in [2]$. As H is bipartite, $|P_1|, |P_2|, |P_3|$ have the same parity. Hence, the theta H is 1 or 2-uniform. Consider first the case where H is 1-uniform. Then $G|X$ and $G|Y$ are connected and $|\partial_G(X)| = 3$. Since $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are bipartite some vertex of $\partial_G(X)$ is a blocking vertex. This implies by Remark 20 that M is graphic, a contradiction. Consider now the case where H is 2-uniform. After possibly interchanging the role of X and Y, (G,Σ) is of the form given in the next figure where X corresponds to the shaded region. Since $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are bipartite some vertex of $\partial_G(X)$ is a blocking vertex. Again, this implies by Remark 20 that M is graphic, a contradiction.



Case 2. H is a double ear.

H consists of two polygons C_1, C_2 sharing a single vertex. Since H is bipartite $|C_1|, |C_2| \geq 2$. By (m3) and since we can rearrange G by 2-flips, $|C_1| = |C_2| = 2$, i.e. the double ear H is 2-uniform. Thus $VH = \{r,s,t\}$, $EH = \{a = rs, b = rs, c = st, d = st\}$. After possibly interchanging the role of X and Y, we may assume $X = R \cup T$ where R and T are the components of $G|X$ corresponding to $r,t \in VH$, and that $Y = S$ where S is the component of $G|Y$ corresponding to $s \in VH$. Then $a,b,c,d \in EH$ correspond to vertices in $\partial_G(X)$ where, $\partial_G(R) = \{a,b\}$ and $\partial_G(T) = \{c,d\}$. We illustrate (G,Σ) in the next figure where X correspond to the shaded region,



Since $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are bipartite after possibly resigning we have, $\Sigma = [\delta_G(a) \cap R] \cup [\delta_G(c) \cap S]$. Let (G',Σ) be obtained from a Lovász-flip on the blocking pair a,c. Then observe that $\partial_{G'}(X) = \{b,d\}$, that $(G',\Sigma)|X$ is bipartite and that $(G',\Sigma)|Y$ is non-bipartite. But then (G',Σ) is a representation as in (m2) contradicting our choice of representation.

3.5. **Representation of 3-connected even-cycle matroid.** We will use the following multiple times,

Proposition 21. *Let M be a 3-connected even cycle matroid with representation (G,Σ). Then (G,Σ) has at most one loop, which is odd. Moreover, if we let G' be obtained from G by deleting that loop (if it exists) then G' is 2-connected and every cut of G contains at least three edges.*

Proof. Since M is 3-connected, M has no loops and parallel elements. Suppose e is a loop of G. Then $e \in \Sigma$ for otherwise, e is a loop of M. Moreover, if G has distinct loops e, f then $\{e, f\}$ is a circuit of M, i.e. e, f are parallel, a contradiction. We may assume that G' is connected by identifying components as we will show that G' is in fact 2-connected. Suppose that G' has a cut vertex v. Then there exists $X \subseteq EG'$ such that $\partial_{G'}(X) = \{v\}$ and $G'|X$ and $G'|EG' - X$ are both connected. By moving the loop e of G (if it exists) we may assume that $G|X$ and $G|EG - X$ are also connected. Then by Proposition 3 (a) $\lambda_M(X) = |\partial_G(X)| - \kappa(G|X) - \kappa(G|EM - X) + p[(G, \Sigma)|X] + p[(G, \Sigma)|EG - X] \leq 1 - 1 - 1 + 1 + 1 = 1$, thus X is 2-separating. If $|X| = 1$, then the unique element in X is a bridge of G, i.e. a coloop of M, a contradiction. Thus $|X| \geq 2$ and similarly, $|EG - X| \geq 2$ and it follows that X is a 2-separation, a contradiction. Finally, if D is a cut of G then D is cocycle of M [11], Remark 2.1. It follows that $|D| \geq 3$. □

4. Structure theorem for 3-separations

In this section we characterize the structure of 3-separations in pinch-graphic matroids.

4.1. **The statement.** We first require a few definitions. Given a matroid M and $X \subseteq EM$ denote by $\mathrm{cl}_M(X)$ the closure of X for matroid M. Recall, that M^* denotes the dual of M. Let M be a matroid and let $X \subseteq EM$ be a proper 3-separation. Suppose $|X| \geq 5$ and suppose that there exists $e \in X$ with $e \in \mathrm{cl}_{M^*}(EM - X)$ and $e \in \mathrm{cl}_{M^*}(X - e)$. Then observe that $X - e$ is also a proper 3-separation. We say that $X - e$ is *homologous* to X and so is any set that is obtained by repeat application of the aforementioned procedure.

Given a graph G and $X \subseteq EG$ we write $\mathcal{I}_G(X)$ for the vertices of $G|X$ that are not in $\partial_G(X)$. Let (G, Σ) be a connected signed-graph and consider $X \subseteq EG$. The triple (G, Σ, X) is a *Type I* or *Type II configuration* if $|X|, |EG - X| \geq 4$, $G|X, G|EG - X$ are both connected, and $|\partial_G(X)| = 2$. In addition for Type I, there exists a blocking pair u, v where $u \in \mathcal{I}_G(X)$ and $v \in \mathcal{I}_G(EG - X)$ and for Type II, $\partial_G(X)$ is a blocking pair. (See Figure 1 (i) for a representation of a Type I configuration and (ii) for a representation of a Type II configuration.) Consider a pinch-graphic matroid M with a proper 3-separation X. We say that X is *compliant* if there exists a representation (G, Σ) for which (G, Σ, X) is a Type I configuration. We say that X is *recalcitrant* if there exists a representation (G, Σ) for which (G, Σ, X) is a Type II configuration.

Here is the promised characterization,

Proposition 22. *Let M be a 3-connected pinch-graphic matroid and let X' be a proper 3-separation. Then there exists a homologous proper 3-separation X that is reducible, compliant, or recalcitrant.*

The proof of this result will share some commonality with that of Proposition 18, namely we will analyze the auxiliary graph H for a representation (G,Σ) and separation X homologous to X'. However, when $(G,\Sigma)|X$ and $(G,\Sigma)|EM-X$ are both bipartite, then $|EH| = |VH| + 3$ and a simple-minded analysis of all possible graphs H becomes very complicated. Instead we will prove a key property in Section 4.5 that will bypass most of the case analysis. The proof of Proposition 22 will be organized as follows: Section 4.2 presents basic results about closure, Section 4.3 indicates how to pick a suitable representation (G,Σ) and a separation X of M. A key property of the auxiliary graph is derived in Sections 4.4 and 4.5, finally in Section 4.6 we analyze the auxiliary graph, completing the proof.

4.2. **Closure and small-separations.** For 3-connected matroids we have the following characterization of homologous separations.

Proposition 23 (Lemma 3.1 [10])**.** *Let M be a 3-connected matroid where $|EM| \geq 9$ and let $X \subseteq EM$ be a proper 3-separation. Suppose $|X| \geq 5$ and that there exists $e \in X$ with $e \in \mathrm{cl}_{M^*}(EM-X)$. Then $X-e$ is homologous to X.*

Observe that we dropped one of the condition from the definition of homologous separations.

Next we describe what it means for an element to be in the co-closure of a set, for the case of even-cycle matroids.

Remark 24. *Let $M = \mathrm{ecycle}(G,\Sigma)$ and let $X \subseteq EM$, $e \in EM-X$. Then the following are equivalent,*

(a) $e \in \mathrm{cl}_{M^*}(X)$,
(b) there exists a signature D of (G,Σ) or a cut D of G where $D-X=\{e\}$.

Proof. Clearly, $e \in \mathrm{cl}_{M^*}(X)$ if and only if there exists a cocircuit D of M with $D-X=\{e\}$. Moreover, cocircuits of M are signatures of (G,Σ) and cuts of G [11] Remark 2.1. ☐

In the proof of Proposition 22 we will consider a proper 3-separations X. We will want to check if $X-e$ is an homologous proper 3-separation for some $e \notin X$. A trivial reason this is not the case is if $|X|=4$. Let us study such 3-separations,

Proposition 25. *Let M be a 3-connected binary matroid with a 3-separation X where $|X|=4$. Then*

(a) $r_M(X) = r_{M^*}(X) = 3$, and

(b) X contain both a circuit and a cocircuit where each have at least three elements.

Proof. Recall, that $r_{M^*}(X) = |X| - \big[r(M) - r_M(EM - X)\big]$.

$$r_M(X) + r_{M^*}(X) = r_M(X) + |X| - r(M) + r_M(EM - X) = |X| + \lambda_M(X) = 4 + 2 = 6.$$

Observe that $r_M(X) \geq 3$, for otherwise as $|X| = 4$, M would have a loop or a pair of parallel elements contradicting the fact that M is 3-connected. Similarly $r_{M^*}(X) \geq 3$ and thus (a) holds. Finally (a) and $|X| > 3$ implies (b). □

4.3. **Choice of representation and separation.** Throughout the proof of Proposition 22, M denotes a 3-connected pinch-graphic matroid and X' is a proper 3-separation. Since M is pinch-graphic there exists a signed-graph (G, Σ) such that

i. (G, Σ) is a representation of M,
ii. (G, Σ) has a blocking pair, and
iii. G is connected.

We make the following *minimality assumption*. Among all choices of (G, Σ) that satisfy (i)-(iii) and among all choices of homologous separations X of X' with $Y = EG - X$ we pick one that minimizes,

$$\kappa(G|X) + \kappa(G|Y). \qquad (\dagger)$$

Throughout the remainder of the proof of Proposition 22, (G, Σ) will denote the representation of M and X, Y the partition selected according to (†). Next we give some easy properties,

Proposition 26. *We may assume that (G, Σ) is non-bipartite and has no blocking vertex.*

Proof. Otherwise by Remark 20, M is graphic. Express M as a 3-sum, i.e. $M = M_1 \oplus_3 M_2$ for some matroids M_1, M_2 where $X = EM_1 - EM_2$. Proposition 4 implies that M_1 is isomorphic to a minor of M. Since M is graphic then so is M_1. Thus X is reducible and Proposition 22 holds. □

4.4. **Edge condition.** The following describes necessary conditions for which a signed-graph stops having a blocking pair after a 2-flip.

Proposition 27. *Let (H, Γ) be a signed-graph with a blocking pair v, w where H is 2-connected. Let Z be a 2-separation of H and let $\partial_H(Z) = \{u_1, u_2\}$. Let H' be obtained from H by a 2-flip on Z and assume that (H', Γ) does not have a blocking pair. Then, exactly one of u_1, u_2 is in $\{v, w\}$*

(say $u_1 = v$), and there exist odd circuits C_1, C_2, C_3 of (H, Γ) where $C_1 \subseteq Z$ contains u_1, and avoids w, u_2; $C_2 \subseteq EH - Z$ contains u_1, and avoids w, u_2; and C_3 contains w, and avoids u_1, u_2.

Proof. Label the vertices of H' so that vertices distinct from $\partial_{H'}(Z)$ have the same label as H and vertices in $H|Z$ and $H'|Z$ have the same label. Then $\{v, w\} \cap \partial_H(Z) \neq \emptyset$ for otherwise v, w would be a blocking pair of H' and $\{v, w\} \neq \{u_1, u_2\}$ for otherwise u_1, u_2 would be a blocking pair of H'. We may assume, $u_1 = v$. Then C_1, C_2, C_3 exist because u_1, w is a blocking pair of (H, Γ) and for C_1 we have that u_2, w is not a blocking pair of (H', Γ), for C_2 we have that u_1, w is not a blocking pair of (H', Γ), and for C_3 we have that u_1, u_2 is not a blocking pair of (H', Γ). □

We shall also require the following observation about Lovász-flip,

Remark 28. *Let (H, Γ) be a signed-graph with a blocking pair v_1, v_2 and $\Gamma \subseteq \delta_H(v_1) \cup \delta_H(v_2)$. Consider $Z \subseteq EH$ such that $H|Z$ is connected. Suppose for $i = 1, 2$,*

$$\delta_H(v_i) \cap Z \in \{\emptyset, \delta_H(v_i) \cap \Sigma, \delta_H(v_i) - \Sigma\}.$$

Let (H', Γ) be obtained from (H, Γ) by a Lovász-flip on v_1, v_2. Then $H'|Z$ is also connected.

Note, it suffices to observe that if two edges of Z are incident in H then they will be incident in H'.

The next result will be key,

Proposition 29. *No component of $G|X$ or $G|Y$ consists of a single edge.*

Proof. Suppose otherwise, i.e. there exists a component of $G|X$ or $G|Y$ that consists of a single edge e. Without loss of generality, we may assume that $e \in X$. Denote by u, v a blocking pair of (G, Σ).

Claim 1. $e \in \text{cl}^*(Y)$.

Subproof. First consider the case where e not a loop. Since G is connected there exists an end, say x, of e that is a vertex of $G|Y$. Then $D = \delta_G(x)$ is a cut of G where $D - Y = \{e\}$. Hence, by Remark 24, $e \in \text{cl}^*(Y)$ as required. Now consider the case where e is a loop. By Proposition 21, $e \in \Sigma$. Suppose that $(G, \Sigma)|X - e$ is non-bipartite. Then $G|X - e$ contains one of the two vertices of the blocking pair, say u. Let G' be obtained from G by moving e to u. Then (G', Σ) and X contradict the minimality assumption (†). Thus $(G, \Sigma)|X - e$ is bipartite and it follows that there exists a signature Γ of (G, Σ) where $\Gamma - Y = \{e\}$. Thus by Remark 24, $e \in \text{cl}^*(Y)$ as required. ◊

Claim 2. $|X| = 4$

Subproof. Suppose for a contradiction that $|X| \geq 5$. Then by Claim 1 and Proposition 23, $X - e$ is homologous to X. But then (G, Σ) and $X - e$ violate the minimality assumption (†). $\Diamond$

Claim 3. *Circuits of M contained in X avoid e.*

Subproof. Since M is binary, circuits and cocircuits have an even number of common elements. By Claim 1 there is a cocircuit D of M with $D - Y = \{e\}$. Let C be a circuit of M where $C \subseteq X$. Then, $C \cap D \subseteq \{e\}$. It follows that $C \cap D = \emptyset$, i.e. $e \notin C$. $\Diamond$

Denote the elements of X by e, f, g, h. By Proposition 25 there exists a circuit $C \subseteq X$ of M with $|C| \geq 3$. By Claim 3, $e \notin C$, thus $C = \{f, g, h\}$. By Proposition 25 there exists a cocircuit $D \subseteq X$ of M with $|D| \geq 3$. Since $|C \cap D|$ is even we may assume, $D = \{e, f, g\}$.

Claim 4. *e is not a loop of G.*

Subproof. Suppose for a contradiction e is a loop. Then D is not a cut of G. It follows that $D = \{e, f, g\}$ is a signature of (G, Σ). By Proposition 21 e is the only loop of G. Thus $C = \{f, g, h\}$ is a polygon of G. Let w denote the common end of f and g. Let G' be obtained from G by moving e to w. Then (G', Σ) has a blocking vertex, contradicting Proposition 26. $\Diamond$

There are two cases for D namely, D is a cut of G or a signature of (G, Σ). There are two cases for C namely, C is a polygon of G or C consists of two parallel edges (exactly one of which is odd) and an odd loop. We will consider all four possible combinations. Let x, y denote the ends of edge e.

Case 1. D is a cut and C is a polygon.

Let p, q, r denote the vertices of the polygon C in G where p is incident to f, g. As C is a cycle of M, C is even in (G, Σ). By exchanging the roles of x, y if needed, there exists a non-trivial partition Y_1, Y_2 of Y such that $\partial(Y_1) = \{x, p\}$ and $\partial(Y_2) = \{y, q, r\}$. Let $Z = Y_1 \cup e$, then $\partial(Z) = \{p, y\}$. Let G' be obtained from G by a 2-flip on Z. Suppose for a contradiction that (G', Σ) has no blocking pair. Recall, that u, v denotes the blocking pair of (G, Σ). Then by Proposition 27 we may assume (i) $u = p$ or (ii) $u = y$. Moreover, if (i) occurs then we have an odd circuit $C_2 \subseteq EG - Z$ that uses p and avoids v. It follows that C_2 uses edges f, g. But then $C_2 - \{f, g\} \cup h$ is an odd circuit of (G, Σ) that avoids both u, v, a contradiction. If (ii) occurs we must have a circuit $C_1 \subseteq Z$ that uses y, a contradiction as e is the only edge of Z incident to y. Hence, (G', Σ) has a blocking pair and together with X it contradicts the minimality assumption (†).

Case 2. D is a cut and C is not a polygon.

Then as $D = \{e, f, g\}$, C consists of parallel edges f, g and loop h. Denote by p and q the ends of f, g. By exchanging the roles of x, y if needed, there exists a non-trivial partition Y_1, Y_2 of Y such that $\partial(Y_1) = \{x, p\}$ and $\partial(Y_2) = \{y, q\}$. For $i = 1, 2$ let $Z_i = Y_i \cup e$ and let G^i be obtained from G by a 2-flip on Z_i. Suppose for a contradiction that neither (G^1, Σ) nor (G^2, Σ) have a blocking pair. Since (G^1, Σ) does not have a blocking pair it follows from Proposition 27 that (i) $u = p$ or (ii) $u = y$. However, (ii) does not occur for otherwise we must have an odd circuit $C_1 \subseteq Z_1$ that uses y, a contradiction as e is the only edge of Z_1 incident to y. Hence, (i) holds, i.e. $u = p$. Applying the same argument to (G^2, Σ) we deduce that $v = q$, i.e. p, q is a blocking pair of (G, Σ). Since (G^1, Σ) has no blocking pair, Proposition 27 implies that there exists an odd circuit of (G, Σ) contained in $EG - Z$ that uses p but avoids q, a contradiction as no such circuit exists. Hence, at least one of (G^1, Σ), (G^2, Σ) has a blocking pair and with X contradicts the minimality assumption (†).

Case 3. D is a signature and C is a polygon.

Let p be the vertex common to f, g in G. Then x and p is a blocking pair. Let $\Gamma = D \,\Delta\, \delta(x)$. Let (G', Γ) be obtained from (G, Γ) by a Lovász-flip on x and p. Clearly (G', Γ) has a blocking pair. Moreover, $\kappa(G'|X) = 1$ and by applying Remark 28 to each component of $G|Y$ we deduce $\kappa(G'|Y) \leq \kappa(G|Y)$. Hence, (G', Σ) together with X, contradicts the minimality assumption (†).

Case 4. D is a signature and C is not a polygon.

Since $C = \{f, g, h\}$ is a circuit of M but not a polygon, it is the union of two odd polygons of (G, Σ). As $D = \{e, f, g\}$, one of f or g is a loop. Without loss of generality we may assume f is a loop and thus h, g are parallel. Note that f is not a component of $G|X$ for otherwise moving it to an end of g, h preserves blocking pairs and contradicts the minimality assumption (†). Thus, f, g, h are incident to a common vertex, say p. Then end x of e and p is a blocking pair. Let $\Gamma = D \,\Delta\, \delta(x)$. Let (G', Γ) be obtained from (G, Γ) by a Lovász-flip on x and p. Clearly (G', Γ) has a blocking pair. Moreover, $\kappa(G'|X) = 1$ and by Remark 28 $\kappa(G'|Y) \leq \kappa(G|Y)$. Hence, (G', Σ) together with X, contradicts the minimality assumption (†). □

4.5. **Degree condition.** Throughout the remainder of the proof of Proposition 22 we let H denote the auxiliary graph for G and X which are selected according the minimality assumption (†). Next are the key properties of the auxiliary graph H.

Proposition 30. *H is bridgeless, in particular every vertex has degree at least two. Moreover, if a vertex has degree exactly two then the corresponding component of $(G,\Sigma)|X$ or $(G,\Sigma)|Y$ is non-bipartite.*

Proof. Note that H is connected since G is connected. Suppose for a contradiction that H has a bridge e. Since H is connected $H \setminus e$ has two components H_1 and H_2. Let Z be the set of edges of G in components of $G|X$ and $G|Y$ corresponding to the vertices of H_1. Then $\partial_G(Z) = \{v\}$ for some $v \in VG$. By Proposition 3 (a), $\lambda_M(Z) = |\partial_G(Z)| - \kappa(G|Z) - \kappa(G|EM - Z) + p[(G,\Sigma)|Z] + p[(G,\Sigma)|EM - Z] \leq 1 - 1 - 1 + 1 + 1 = 1$, thus Z is 2-separating. Moreover, by Proposition 29, $|Z|, |EM - Z| \geq 2$, a contradiction as M is 3-connected. Let p be a degree two vertex of H and let Z denote the edges in the component of $G|X$ or $G|Y$ corresponding to p. Then $|\partial_G(Z)| = 2$. Proposition 29 implies that $|Z|, |EM - Z| \geq 2$. Since M is 3-connected, Z is not 2-separating, i.e. $2 \leq \lambda_M(Z) \leq 2 - 1 - 1 + p[(G,\Sigma)|Z] + p[(G,\Sigma)|EM - Z]$. Thus $p[(G,\Sigma)|Z] = 1$, i.e. $(G,\Sigma)|Z$ is non-bipartite. □

4.6. **Proof of Proposition 22.** There are three cases to consider depending on whether each of $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are bipartite (as we can interchange the role of X and Y). We consider each of these in Propositions 31, 32 and 34.

Proposition 31. *If $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are non-bipartite then X is reducible, compliant, or recalcitrant.*

Proof. Proposition 3 (b) implies that $|EH| = |VH| + \lambda_M(X) - p[(G,\Sigma)|X] - p[(G,\Sigma)|Y] = |VH| + 2 - 1 - 1 = |VH|$. Moreover, H is connected and by Proposition 30 it is bridgeless. It follows that H is a polygon. Since every vertex of H has degree two, Proposition 30 implies that, each component of $G|X$ and $G|Y$ is non-bipartite. It follows that each component of $G|X$ and $G|Y$ contains one of u, v where u, v is the blocking pair of (G,Σ). This implies that $|VH| \in \{2, 4\}$. Consider first the case where $|VH| = 4$ then we have proper partitions X_1, X_2 of X and Y_1, Y_2 or Y where $u \in \partial(X_1) \cap \partial(Y_1)$ and $v \in \partial(X_2) \cap \partial(Y_2)$. Let G' be obtained from G by a 2-flip of X_1, Y_2. Then (G',Σ) has a blocking pair and (G',Σ) and X contradict our minimality assumption (†). Hence $|VH| = 2$, i.e. $G|X$ and $G|Y$ are connected and $\partial(X) = \{a, b\}$ for some vertices a, b of G.

Let us now analyze the possible location of the blocking pair u, v. If $u \in \mathcal{I}(X)$ and $v \in \mathcal{I}(Y)$ or vice-versa then (G,Σ,X) is a Type I configuration and X is compliant. If $\{u, v\} = \{a, b\}$ then (G,Σ,X) is a Type II configuration and X is recalcitrant. Since $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are non-bipartite, we may thus assume, after possibly interchanging the role of X, Y and u, v and a, b that

$u = a$ and $v \in \mathcal{I}(Y)$. After possibly resigning, edges of Σ are incident to u or v. Let G_1 (resp. G_2) be obtained from $G|X$ (resp. $G|Y$) by adding parallel edges f, g between a and b and adding a loop h incident to a. Let $\Sigma_1 = \Sigma \cap X \cup \{f, h\}$ and let $\Sigma_2 = \Sigma \cap Y \cup \{f, h\}$. Then a is a blocking vertex of (G_1, Σ_1) which implies by Remark 20 that $\mathrm{ecycle}(G_1, \Sigma_1)$ is graphic. Finally, Proposition 12 implies that $M = \mathrm{ecycle}(G_1, \Sigma_1) \oplus_3 \mathrm{ecycle}(G_2, \Sigma_2)$. Hence, X is reducible. □

Proposition 32. *If $(G, \Sigma)|X$ is bipartite and $(G, \Sigma)|Y$ is non-bipartite then X is reducible.*

Proof. Proposition 3(b) implies that $|EH| = |VH| + \lambda_M(X) - p[(G, \Sigma)|X] - p[(G, \Sigma)|Y] = |VH| + 2 - 0 - 1 = |VH| + 1$. Recall that H is connected and Proposition 30 implies that H is bridgeless. It follows from Remark 19 that H is a theta or a double ear.

Case 1. H is a theta.

The theta graph H consists of three internally disjoint paths P_1, P_2, P_3. Consider first the case where H is 1-uniform, i.e. $|P_1| = |P_2| = |P_3| = 1$. Then $G|X$ and $G|Y$ are connected. Proposition 12 implies that $M = \mathrm{cycle}(G') \oplus_3 M_2$ for some graph G' and for some matroid M_2 where $EG' = X$. But then X is reducible. Thus we may assume H is not 1-uniform, i.e. $|P_i| \geq 2$ for some $i \in [3]$. Since $(G, \Sigma)|X$ is bipartite, Proposition 30 implies that every internal vertex of P_i corresponds to a component of $G|Y$. Thus $|P_i| = 2$ and as H is bipartite and is not 1-uniform, $|P_i| = 2$ for all $i \in [3]$. By Proposition 30 it follows that the degree two vertex v_i of P_i correspond to a component $G|Y_i$ of $G|Y$ where $(G, \Sigma)|Y_i$ is non-bipartite. Hence, (G, Σ) has three pairwise vertex disjoint odd circuits, a contradiction as there exists a blocking pair.

Case 2. H is a double ear.

The double ear graph H consists of two polygons C_1 and C_2 that share exactly one vertex. Proposition 30 implies that H is 2-uniform and that for $i \in [2]$ the degree two vertex v_i of C_i corresponds to a component $G|Y_i$ of $G|Y$ where $(G, \Sigma)|Y_i$ is non-bipartite. Thus we may assume for the blocking pair u, v of (G, Σ) that u and v are vertices of $G|Y_1$ and $G|Y_2$ respectively. Since $(G, \Sigma)|X$ is bipartite, we may assume $\Sigma \subseteq Y$ and that all edges of Σ are incident to u or v. Let (G', Σ) be obtained from (G, Σ) by a Lovász-flip on u, v. Then $G'|X$ and $G'|Y$ are connected as $G|Y_i$ contains an odd polygon for each $i \in [2]$. Thus (G', Σ) and X contradict our minimality assumption (†). □

Consider a graph F. A *ear* of F is a walk P where the two endpoints of P may coincide, but every other vertex of P has degree two. An *ear decomposition* of F is a partition of its edges into a sequence of ears, such that the one or two endpoints of each ear belong to earlier ears in the sequence and such that the internal vertices of each ear do not belong to any earlier ear. Additionally, the first ear in the sequence must be a polygon.

Theorem 33 ([12])**.** *A graph is connected and bridgeless if and only if it has an ear decomposition.*

Proposition 34. *If $(G,\Sigma)|X$ and $(G,\Sigma)|Y$ are bipartite then X is reducible, compliant, or recalcitrant.*

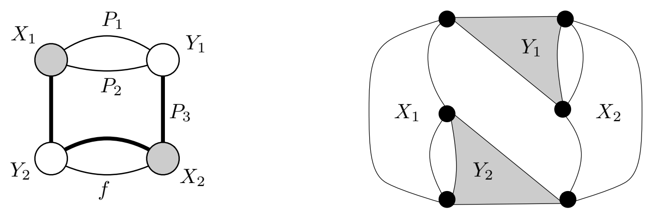
Proof. Proposition 3 (b) implies that $|EH| = |VH| + \lambda_M(X) - p[(G,\Sigma)|X] - p[(G,\Sigma)|Y] = |VH| + 2 - 0 - 0 = |VH| + 2$. Proposition 30 implies that the minimum degree $\delta(H)$ of H is at least 3. $\delta(H) \geq 3$ and $|EH| = |VH| + 2$ implies by Theorem 33 that H is obtained from a polygon C by adding a sequence of two ears say Q_1, Q_2. Let H' be the graph obtained from C by adding ear Q_1. Then H' is either a double ear or a theta (Remark 19). Moreover, $\delta(H) \geq 3$ implies that Q_2 consists of a single edge, say f. Consider first the case where H' is a double ear that consists of polygons C and C' joined at a vertex w. Then f has ends in C and C' (distinct from w). Since $\delta(H) \geq 3$, C and C' have each exactly one vertex distinct from w that is incident to f. But then H has a triangle, a contradiction as H is bipartite. Consider now the case where F is a theta that is formed by internally disjoint paths P_1, P_2, P_3. We may assume f is not incident to an internal vertex of P_1. As $\delta(H) \geq 3$, P_1 consist of a single edge. It follows that P_2 and P_3 each have an odd number of edges. As $\delta(H) \geq 3$ we can assume that P_2 has a single edge and that $|P_3| \in \{1, 3\}$.

Case 1. $|P_3| = 1$.

Then the ends of f correspond to the degree 3 vertices of H, i.e. H consists of four parallel edges. Then $G|X$ and $G|Y$ are connected and denote by a, b, c, d the vertices of $\partial_G(X)$. By Remark 20 there is no blocking vertex of (G,Σ). Thus we may assume, after possibly resigning and exchanging the roles of a,b,c,d if needed, that $\Sigma = (\delta_G(a) \cap X) \cup (\delta_G(b) \cap Y)$. Let (G',Σ') be obtained from (G,Σ) by a Lovász-flip on a and b. Observe that $G'|X$ and $G'|Y$ remain connected, but $(G',\Sigma)|X$ and $(G',\Sigma)|Y$ are non-bipartite. Thus (G',Σ) and X satisfy the minimality assumption and thus by Proposition 31, X is reducible, compliant, or recalcitrant.

Case 2. $|P_3| = 3$.

Since $\delta(H) \geq 3$ the ends of f must correspond to internal vertices of P_3. It follows that H is the graph obtained from a polygon with four edges by replacing edges in a matching by two parallel edges. We illustrate the auxiliary graph H in the next figure (left) with the corresponding graph G (right). Let G' be obtained from G by a 2-flip on $X_1 \cup Y_2$. Then, by Proposition 27, (G', Σ) has a



blocking pair. Then (G', Σ) and X contradict our minimality assumption (†). □

5. Properties of compliant and recalcitrant separations

5.1. **Compliant separations.** Let us motivate the term "compliant". Given a 3-connected binary matroid M described by its $0,1$ matrix representation and given $X \subseteq EM$ we will show that we can check in polytime if X is compliant. The key to that result is the next proposition.

Proposition 35. *Let $M = M_1 \oplus_3 M_2$ be a 3-connected binary matroid. Let $X = EM_1 - EM_2$ and assume X is not reducible. Then the following are equivalent,*

(a) X is compliant.
(b) There exists $e \in EM_1 \cap EM_2$ for which both $M_1 \setminus e$ and $M_2 \setminus e$ are graphic.

The algorithmic details on how to use this result are in Section 6.

Consider a binary matroid M with an element e and a cycle C. Observe that every cycle of M is a cycle avoiding e or the symmetric difference of C and a cycle avoiding e. It thus follows that,

Remark 36. *Let M, N be binary matroids with the same ground set containing an element e. Then $M = N$ if and only if all the cycles avoiding e are the same in M and N and at least one cycle using e is the same in M and N.*

Proof of Proposition 35. Suppose (a) holds. Then (G, Σ, X) is a Type I configuration for some representation (G, Σ) with blocking pair u, v where $u \in \mathcal{I}_G(X)$ and v not a vertex $G|X$. Let G_1 be obtained from $G|X$ by adding a pair of parallel edges e, f between $a, b \in \partial_G(X)$ and by adding a loop g to be incident to u. Let $\Sigma_1 = \Sigma \cap X \cup \{e, g\}$. By Proposition 12, $M_1 = \text{ecycle}(G_1, \Sigma_1)$. Moreover, $M_1 \setminus e = \text{ecycle}(G_1 \setminus e, \Sigma_1 - e)$. Then $(G_1 \setminus e, \Sigma_1 - e)$ has blocking vertex u. Thus, Remark 20 implies that $M_1 \setminus e$ is graphic. By interchanging the role of X and $EM - X$ we similarly prove that $M_2 \setminus e$ is graphic.

Suppose (b) holds. Then $M_1 \setminus e = \text{cycle}(H)$ for some graph H. Since M has no loop neither does $M_1 \setminus e$. Hence, H has no loop. Let r, s denote the ends of g in H. Since f, g are not parallel, we may assume that r is not incident to f. Let G_1 be obtained from H by identifying r and s into t_1 and adding an edge e parallel to f. Let $\Sigma_1 = \delta_H(r) \cup \{e\}$. Observe that $\{e, f, g\}$ is an even cycle of (G_1, Σ_1) and a cycle of M_1. It follows from Remark 36 that $M_1 = \text{ecycle}(G_1, \Sigma_1)$. Similarly, construct (G_2, Σ_2) by identifying r and s into t_2 where $M_2 = \text{ecycle}(G_2, \Sigma_2)$ where e, f are parallel edges of G_2 and g is a loop G_2. Let G be obtained from G_1 and G_2 by identifying e, f and deleting e, f, g. Let $\Sigma = [\Sigma_1 \cup \Sigma_2] - \{e, g\}$. Then Proposition 11 implies that $M = \text{ecycle}(G, \Sigma)$. Since X is not reducible, $t_1 \in \mathcal{I}_G(X)$ and $t_2 \in \mathcal{I}_G(EG - X)$. Finally, observe that (G, Σ, X) is a Type I configuration, i,.e. (a) holds. □

5.2. **Recalcitrant separations.** Consider a 3-connected binary matroid M described by its $0, 1$ matrix representation. A natural approach for algorithm (1) in Section 1 is to design a subroutine to check if a proper 3-separation is recalcitrant in polynomial time. However, this seems to be harder than checking if X is compliant, so instead we will either: establish that X is recalcitrant, or find another proper 3-separation that is reducible. We develop the necessary tools in this section, the algorithmic details will appear in Section 6. Throughout this section, M denotes a 3-connected matroid with a proper 3-separation X and $Y = EM - X$,

5.2.1. *Working with the completion.* Next we relate representations of M and its completion N with respect to a recalcitrant separation.

Remark 37. *Suppose that (G, Σ, X) is a Type II configuration and that (G, Σ) is a representation of M. Let a, b denote the vertices of $\partial_G(X)$. Let (H, Γ) be the signed graph obtained from (G, Σ) by adding an odd loop e_1, an even edge e_2 with ends a, b and an odd edge e_3 with ends a, b. Then*

(a) $N = \text{ecycle}(H, \Gamma)$ is the completion of M with respect to X,

(b) if $N = \text{ecycle}(H', \Gamma')$ and $(H', \Gamma')|EM = (G, \Sigma)$ then (H, Γ) and (H', Γ') are isomorphic up to moving e_1.

Proof. (a) is shown in the proof of Proposition 11. (b) Let $e \in EN - EM$. Then by definition of completion there exists cycles C and D of N where $C \subseteq X \cup e$, $D \subseteq (EM - X) \cup e$ and $e \in C \cap D$. Thus $C \,\Delta\, D = (C - e) \cup (D - e)$ is a cycle of M and in particular an even-cycle of (G, Σ). It follows that $C - e$ and $D - e$ are either both odd cycles of (G, Σ) or both ab-joins of G. Hence, e is either an odd loop of (H', Γ') or has ends a, b in H'. As elements in $EN - EM$ form a circuit of N they form an even cycle of (H', Γ') and the result follows. □

Throughout this section N shall denote the completion of M with respect to X. Moreover, e_1, e_2, e_3 denote the elements of $EN - EM$.

The next result shows that it suffices to check X is recalcitrant for N.

Remark 38. *X is a recalcitrant separation of M if and only if X is a recalcitrant separation of N.*

Proof. Suppose that X is a recalcitrant separation of M. Then there exists a representation (G, Σ) of M for which (G, Σ, X) is a Type II configuration with $\{a, b\} = \partial_G(X)$. Then the signed graph (H, Γ) obtained from (G, Σ) in Remark 37 is a representation of N. Since a, b is a blocking pair of (G, Σ) it is a blocking pair of (H, Γ). Thus, (H, Γ, X) is a Type II configuration and X is a recalcitrant separation of N. Suppose that X is a recalcitrant separation of N. Then there exists a representation (H, Γ) of N for which (H, Γ, X) is a Type II configuration with $\{a, b\} = \partial_H(X)$. Let $(G, \Sigma) = (H, \Gamma) \setminus EN - EM$. Then (G, Σ) is a representation of M. As X is a proper 3-separation, $|X|, |EG - X| \geq 4$, moreover, as M is 3-connected, $G|X$, $G|EG - X$ are both connected and $\partial_G(X) = \{a, b\}$. Finally, as a, b is a blocking pair of (H, Γ) it is also a blocking pair of (G, Σ). Thus (G, Σ, X) is a Type II configuration and X is a recalcitrant separation of M. □

5.2.2. *Bilateral representations.* A representation (H, Γ) of N is *bilateral* if for some $\{i, j, k\} = [3]$,

i. $H|X$ and $H|Y$ are both connected,
ii. $\partial_H(X) = \{a, b\}$,
iii. e_i is a loop and the ends of e_j, e_k are a, b,
iv. $e_i, e_j \in \Gamma$, $e_k \notin \Gamma$.

Remark 39. *If (H, Γ, X) is a Type II configuration where $N = \text{ecycle}(H, \Gamma)$ then (H, Γ) is a bilateral representation of N. Moreover, if (H, Γ) is a bilateral representation of N with blocking pair a, b then (H, Γ, X) is a Type II configuration.*

Next we show that if X is recalcitrant, then we can construct a bilateral representation. Moreover, it suffices to find an equivalent representation for which $\partial(X)$ is a blocking pair to certify that the representation is recalcitrant.

Proposition 40. *Suppose X is a recalcitrant separation of N and pick $\{i, j, k\} = [3]$. Then*

(a) $N/e_i = \text{cycle}(G)$ for some graph G.

Pick an arbitrary graph G' equivalent to G and let H be obtained from G' by adding loop e_i. Let Γ be a cocircuit of N using e_i. Then the following also hold, *(b)* (H, Γ) *is a representation of* N.

(c) There exists (H', Γ') *equivalent to* (H, Γ) *for which* (H', Γ', X) *is a Type II configuration.*

(d) (H, Γ) *is a bilateral representation of* N.

Before we can proceed with the proof we require a number of preliminaries.

Proposition 41. *If* X *is a recalcitrant separation of* N *then for every* $i \in [3]$ *there exists a Type II configuration* (H, Γ, X) *where* (H, Γ) *is a representation of* N *and where* e_i *is an odd loop.*

Proof. Since X is a recalcitrant separation of N there exists a Type II configuration (H', Γ, X) where (H', Γ) is a representation of N. Denote by a, b the vertices in $\partial_H(X)$. By Remark 37, we have that e_1, e_2, e_3 are are either loops or have ends a, b. We may assume that e_i has ends a, b. After possibly resigning, we have Γ with $e_i \in \Gamma$ and $\Gamma \subseteq \delta_{H'}(a) \cup \delta_{H'}(b)$. Let (H, Γ) be obtained from (H', Γ) by a Lovász-flip on the blocking pair a, b. Observe that e_i is an odd loop of H. Then (H, Γ, X) is the required Type II configuration. □

We will also require the following immediate consequence of Theorem 16 (see [7], Lemma 17),

Remark 42. *Two signed-graphs with the same even cycles and a common odd cycle are equivalent.*

We are now ready for the main proof of this section.

Proof of Proposition 40. (a) Since X is a recalcitrant separation of N it follows by Proposition 41 that there exists a Type II configuration (H', Γ', X) where $N = \mathrm{ecycle}(H', \Gamma')$ and where e_i is an odd loop of H'. Since e_i is an odd loop, $N/e_i = \mathrm{ecycle}(H', \Gamma')/e_i = \mathrm{cycle}(H'/e_i)$. Then let $G = H'/e_i = H' \setminus e_i$. (b) Let $N' = \mathrm{ecycle}(H, \Gamma)$. Recall that by Theorem 16, $\mathrm{cycle}(G') = \mathrm{cycle}(G)$. Then

$$N'^* \setminus e_i = (N'/e_i)^* = (\mathrm{ecycle}(H, \Gamma)/e_i)^* = \mathrm{cycle}(G')^* = (N/e_i)^* = N^* \setminus e_i.$$

By hypothesis Γ is a cocircuit of N. Moreover, Γ is a cocircuit of N' as Γ is a signature of (H, Γ) and signatures correspond to cocircuits. It follows from Remark 36 that $N'^* = N^*$ i.e. that $N = N' = \mathrm{ecycle}(H, \Gamma)$. (c) By (b), (H, Γ) and (H', Γ') have the same set of odd cycles. Moreover, e_i is an odd loop of both signed graphs. It follows from Remark 42 that (H, Γ) and (H', Γ') are equivalent, proving (c). (d) By Remark 39, (H', Γ') is a bilateral representation of N. Remark 37 implies that e_j, e_k are joining the ends of $\delta_{H'}(X)$. Moreover, H is 2-connected by Proposition 21. We proved in (c) that H and H' are equivalent. Hence, H is obtained from H' by a sequence of 2-flips on sets $U \subseteq X$ or $U \cap X = \emptyset$. Thus (H, Γ) is also a bilateral representation of N. □

5.2.3. *Solid separations.* Proposition 40 suggests the following procedure for recognizing if X is a recalcitrant separation of N, pick $i \in [3]$ and check if N/e_i is graphic. If it is not, then X is not recalcitrant. Otherwise $N/e_i = \text{cycle}(G')$ for some graph G', and construct a representation (H, Γ) of N as described in Proposition 40. If H is 3-connected, then $H = H'$ and it follows from Remark 39 that X is recalcitrant if and only if the ends of e_j (resp. e_k) form a blocking pair of (H, Γ). Alas H need not be 3-connected. Moving forward, our strategy will be to analyze the 2-separations of H. In this section we analyze one such type of separations.

Let (H, Γ) be a bilateral representation of N. Then $Z \subseteq EH$ is a *solid separation* of H if the following hold,

i. $Z \cap \{e_1, e_2, e_3\} = \emptyset$,
ii. $H|Z$ and $H|EH - Z$ are connected,
iii. $|\partial_H(Z)| = 2$ and $\partial_H(X) \neq \partial_H(Z)$,
iv. there exists internally disjoint path P_1, P_2 in $H|Z$ with ends $\partial_H(Z)$ and $|P_1|, |P_2| \geq 2$.

Next we present the key result of this section.

Proposition 43. *Suppose X is a recalcitrant separation of N and (H, Γ) is a bilateral representation of N. If Z is a solid separation then Z is a reducible separation of M.*

First we require the following observation used in [3].

Remark 44. *Let H be a graph that contains a theta subgraph consisting of internally disjoint path P_1, P_2, P_3. Let H' be a graph equivalent to H. Then P_1, P_2, P_3 are paths of H' that form a theta subgraph. Note, the order of the edges in P_i need not be the same in H and H'.*

Proof of Proposition 43. Let u, v denote the vertices in $\partial_H(Z)$. Because edge e_j (resp. e_k) has ends u, v, $Z \subset X$ or $Z \subset Y$ and we may assume the former. Since Z is solid there exists internally disjoint uv-paths P_1, P_2 in $H|Z$. Since $H|EH - Z$ is connected there exists a uv-path P_3 in $H|EH - Z$. Observe that P_1, P_2, P_3 form a theta graph of H. By Proposition 40 there exists H' equivalent to H such that (H', Γ, X) is a Type II configuration. By Remark 44 P_1, P_2, P_3 form a theta graph of H'. Denote by u', v' the ends of P_1, P_2, P_3 in H'. By Proposition 21 H, H' are 2-connected (up to a single loop). It follows that $\partial_{H'}(Z) = \{u', v'\}$ and that $H'|Z$ connected.

Claim 1. *If u' or v' is a blocking vertex of $(H', \Gamma)|Z$ then Z is reducible for M.*

Subproof. Suppose for a contradiction that u' is blocking vertex of (H', Γ'). Let (G, Σ) be obtained from $(H', \Gamma)|Z$ by adding an odd loop f_1, an even edge f_2 with ends $\partial_{H'}(Z)$ and an odd edge f_3

with ends $\partial_{H'}(Z)$. It follows from Proposition 3 (a) that Z is 3-separating in M. As $|P_1|, |P_2| \geq 2$, $|Z| \geq 4$. Note, $Z \subseteq X$ hence $|EM - Z| \geq 4$. Since M is 3-connected, Z must be exactly 3-separating and Z is a proper 3-separation of M. It follows that $M = M_1 \oplus_3 M_2$ for some matroids M_1, M_2 where $EM_1 - EM_2 = Z$. By Proposition 11, M_1 is isomorphic to $\mathrm{ecycle}(G, \Sigma)$. Note, u' is a blocking vertex of (G, Σ). It follows from Remark 20 that M_1 is graphic, hence by definition Z is reducible. $\diamondsuit$

Because (H', Γ, X) is a Type II configuration, the ends of e_j (resp. e_k) is a blocking pair of (H', Γ). Suppose for a contradiction, Z is not reducible for M. Then by Claim 1 neither u' nor v' is a blocking vertex of $(H', \Gamma)|Z$. It follows that u', v' must be the ends of e_j. Observe that P_1, P_2, e_2 is theta graph. Hence, by Remark 44, P_1, P_2, e_j is a theta graph of H. It follows that u, v are the ends of e_j in H. But the ends of e_j are $\partial_H(X)$, hence $\partial_H(X) = \partial_H(Z)$, contradicting the definition of solid separation. $\square$

5.2.4. *Kernels.* We still need to consider bilateral representations (H, Γ) of N that are not 3-connected but do not have a solid separation. Consider a signed-graph (H, Γ) and let $Z = \{f_1, f_2, f_3\} \subseteq EH$ where $H|Z$ is connected and $\partial_H(Z) = \{u, v\}$ for some $u, v \in VH$. Suppose that f_1, f_2 form a u,v-path, that f_3 is parallel to f_2, and that $\{f_2, f_3\}$ is an odd polygon. Then Z is a *degenerate separation* of H. We say that $(H', \Gamma)/f_1$ is obtained from (H, Γ) by *reducing* the degenerate separation Z. The graph obtained from (H, Γ) by reducing all degenerate separations is the *kernel* of (H, Γ). We leave the following as an easy exercise,

Remark 45. *Consider a signed-graph (H, Γ) with kernel (H', Γ') and $g \in EG \cap EG'$. Then the ends of g form a blocking pair of (H, Γ) if and only if the ends of g form a blocking pair of (H', Γ').*

In a signed-graph a *double path* is obtained from an internally disjoint path by replacing every edge by an odd polygon of size two. The next proposition will be the key for recognizing recalcitrant separations,

Proposition 46. *Let (H, Γ) be a bilateral representation of N with loop e_i. Suppose that H has no solid separation. Then X is a recalcitrant separation of N if and only the ends of e_j (resp. e_k) form a blocking pair of the kernel of (H, Γ).*

Proof. Suppose first that the ends of e_2 form a blocking pair of the kernel of (H, Γ). It then follows that the ends of e_2 also form a blocking pair of (H, Γ) by Remark 45. Hence, (H, Γ, X) is a Type II configuration and by definition X is a recalcitrant separation of N. Suppose now that X is a recalcitrant separation of N. By Proposition 40 there exists a Type II configuration (H', Γ, X) where H and H' are equivalent. After possibly resigning we have kernel $(\hat{H}, \Gamma)$ of (H, Γ) and kernel $(\hat{H}', \Gamma)$ of (H', Γ) where $\hat{H}$ and $\hat{H}'$ are equivalent. Suppose that Z is a (not necessarily proper) 2-separation of $\hat{H}$ with $\{u, v\} = \partial_{\hat{H}}(Z)$.

Claim 1. *Then Z consists of a single edge or a double path with ends u, v.*

Subproof. Let us proceed by induction on $|Z|$. Clearly, the result holds if $\mathcal{I}_{\hat{H}}(Z) = \emptyset$. Since there is no solid separation we may assume that there exists $w \in \mathcal{I}_{\hat{H}}(Z)$ and a partition Z_1, Z_2 of Z such that Z_1, Z_2 are 2-separations with $\partial_{\hat{H}}(Z_1) = \{u, w\}$ and $\partial_{\hat{H}}(Z_2) = \{v, w\}$. But then apply induction on Z_1, Z_2. For $i \in [2]$, Z_i is either a single edge or a double path. If Z_1 and Z_2 are single edges, then we have a degree two vertex contradicting Proposition 21. If Z_i is an edge and Z_{3-i} is a double path for some $i \in [2]$ then we have a trivial separation, a contradiction. If Z_1 and Z_2 are both double paths, then so is Z as required. $\diamondsuit$

Claim 2. *Z is a double path of length two.*

Subproof. Otherwise one of the odd polygon will not be incident to the ends of e_2 in $\hat{H}'$ contradicting the fact that the ends of e_2 form a blocking pair of $(\hat{H}', \Gamma)$. $\diamondsuit$

The ends of e_2 form a blocking pair of (H', Γ). It follows by Remark 45 that the ends of e_2 form a blocking pair of $(\hat{H}', \Gamma)$. Since $\hat{H}$ and $\hat{H}'$ are equivalent and because of Claim 1, the ends of e_2 form a blocking pair of $(\hat{H}, \Gamma)$ as required. $\square$

6. THE ALGORITHM

In this section we will show that when trying to recognize if a matroid is pinch-graphic we can restrict ourselves to internally 4-connected matroids. We will describe a number of procedures that take a (binary) matroid M as input. In each case the matroid will be described by its $m \times n$, $0, 1$ matrix representation A. A procedure runs in polynomial time if its running time is bounded by a polynomial in m and n. We will rely on the following two results,

Proposition 47 ([15], [14])**.** *Given a matroid M described by an $m \times n$ $0, 1$ matrix A. Then in time polynomial in m and n we can either establish that M is not graphic, or construct a graph G such that $M = \text{cycle}(G)$.*

Proposition 48 ([2]). *Given a matroid M described by an $m \times n$ $0,1$ matrix A and let k, ℓ be fixed integers. Then in time polynomial in m and n we can either find a k-separating set X where $|X|, |EM - X| \geq \ell$ or establish that none exists.*

Next we describe algorithms that will analyze reducible, compliant, and recalcitrant separations.

6.1. **Algorithm A: reducible separations.** The procedure takes as input a pair M and $X \subseteq EM$ where either: (1) X is a 1-separation of M, or (2) X is a 2-separation of M and M is 2-connected, or (3) X is a proper 3-separation of M and M is 3-connected. In polynomial time the procedure will either indicate that: (a) X is not reducible, or (b) return a matroid N where $|EN| < |EM|$ and where N is pinch-graphic if and only if M is pinch-graphic. We proceed as follows: X is a k-separation for some $k \in [3]$. If $k = 1$ then for $i \in [2]$ we let $M_i = M \setminus EM_{3-i}$. If $k \in \{2, 3\}$ then we construct the completion N of M with respect to X and for $i \in [2]$ we let $M_i = N \setminus EM_{3-i}$. Then $M = M_1 \oplus_k M_k$ (see Propositions 6 and 7). Then we check if there exists $i \in [2]$ such M_i is graphic (see Proposition 47). If not we stop and indicate that X is not reducible. Otherwise we stop and return $N := M_{3-i}$ that is pinch-graphic if and only if M is pinch-graphic (see Propositions 13, 14, 15).

6.2. **Algorithm B: compliant separations.** The procedure takes as input a pair M and $X \subseteq EM$ where X is a proper 3-separation of M and M is 3-connected. In polynomial time the procedure will indicate whether X is compliant. We proceed as follows: first we construct the completion N of M with respect to X and for $i \in [2]$ we let $M_i = N \setminus EM_{3-i}$. Let $\{e_1, e_2, e_3\} = EM_1 \cap EM_2$. If for some $i \in [3]$, $M_1 \setminus e$ and $M_1 \setminus e$ are both graphic then we stop and indicate that X is compliant. Otherwise we stop and indicate that X is not compliant. Correctness follows from Proposition 35.

6.3. **Algorithm C: recalcitrant separations.** The procedure takes as input a pair M and $X \subseteq EM$ where X is a proper 3-separation of M and M is 3-connected. In polynomial time the procedure will do one of the following: (a) establish that X is recalcitrant, (b) establish that X is not recalcitrant, or (c) return a matroid N where $|EN| < |EM|$ and where N is pinch-graphic if and only if M is pinch-graphic. We proceed as follows: first we construct the completion N of M with respect to X and for $i \in [2]$ we let $M_i = N \setminus EM_{3-i}$. Let $\{e_1, e_2, e_3\} = EM_1 \cap EM_2$. If M/e_1 is not graphic then we stop and indicate that X is not recalcitrant. Correctness follows from Proposition 40 (a). Otherwise we find a graph G' for which $M/e = \text{cycle}(G')$. Construct H by adding loop e_1 and let Γ be any cocircuit of M using e_1. If (H, Γ) is not bilateral then we stop X and indicate that X is not recalcitrant. Correctness follows from Proposition 40 (c). We then check if H has a solid separation

Z. If it does we run procedure A with input M and Z. If procedure A says that Z is not reducible, we stop and indicate that X is not recalcitrant. Correctness follows from Proposition 43. Otherwise we stop and return the matroid N given by procedure A. Next we construct the kernel of (H, Γ). If the ends of e_2 (resp. e_3) of the kernel form a blocking pair, then we indicate that X is recalcitrant, otherwise we indicate that X not recalcitrant. Correctness follows from Proposition 46.

6.4. **Putting it together.** The procedure takes as input a matroid M. In polynomial time it will either (a) establish that M is pinch-graphic, (b) establish that M is not pinch-graphic, or (c) construct a matroid N that is internally 4-connected where N is isomorphic to a proper minor of M and where N is pinch-graphic if and only M is pinch-graphic. Note we can check if M has a 1-, 2-, or proper 3-separation in polynomial time (see Proposition 48). Also observe that if we establish that M has a compliant or a recalcitrant separation then by definition M is pinch-graphic. Finally note that a proper 3-separation of a 3-connected binary matroid has at most 8 homologous 3-separations.

We repeat the following steps until we stop,

(1) Try to find a 1-separation, or a 2-separation. If there exists such a k-separation pick one minimizing k. Use Algorithm A to check if such a separation X is reducible. If X is not reducible then stop, M is not pinch-graphic (see Propositions 14, 15). If X is reducible, then set $M := N$ where N is the matroid returned by Algorithm A and start again at the beginning of (1).

At this stage of the algorithm the matroid M is 3-connected.

(2) Try to find a proper 3-separation Y. If no such separation exists we stop and return M.
(3) For each separation X that is homologous to Y do the following
    (3.1) Use Algorithm A to check if X is reducible. If it is then set $M := N$ where N is the matroid returned by Algorithm A and start again at the beginning of (1).
    (3.2) Use Algorithm B to check if X is compliant. If it is then stop, M is pinch-graphic.
    (3.3) Use Algorithm C. If the algorithm indicates that X is recalcitrant then stop, M is pinch-graphic. If the algorithm returns N then set $M := N$ and start again at the beginning of (1).

At this stage none of the separations homologous to Y is either reducible, compliant or recalcitrant. Hence, by Proposition 22 M is not pinch-graphic. Hence,

(4) Stop, M is not pinch-graphic.

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