Chapter 7

Product Measures

7.1 The Product Measure Theorem

Problem 7.1.1. Let \((X, A)\) and \((Y, B)\) be measurable spaces. Is there a natural way to define a measure on the space \(X \times Y\) which reflects the structure of the original measure space?

Definition 7.1.2. Let \((X, A)\) and \((Y, B)\) be measurable spaces. A measurable rectangle is a set of the form \(A \times B\), where \(A \in A\) and \(B \in B\). Let \(Z = X \times Y\) and

\[
Z_0 = \left\{ \prod_{i=1}^{n} A_i \times B_i \mid A_i \in A, B_i \in B \right\}
\]

Lemma 7.1.3. \(Z_0\) is an algebra in \(\mathcal{P}(Z)\).

Let \(Z\) be the \(\sigma\)-algebra generated by \(Z_0\). We write \(Z = \mathcal{A} \times \mathcal{B}\). Assume that \((X, A, \mu)\) and \((Y, B, \lambda)\) are measure spaces. A measure \(\pi\) on \((Z, Z)\) is called a product measure if \(\pi(A \times B) = \mu(A)\lambda(B)\).

Theorem 7.1.4 [Product Measure Theorem]. Let \((X, A, \mu)\) and \((Y, B, \lambda)\) be measure spaces. Then there exists a measure \(\pi\) on \((X \times Y, \mathcal{A} \times \mathcal{B})\) such that \(\pi(A \times B) = \mu(A)\lambda(B)\). Moreover, if \(\mu\) and \(\lambda\) are \(\sigma\)-finite, then \(\pi\) is unique and \(\sigma\)-finite.

In the case where \(\mu\) and \(\lambda\) are \(\sigma\)-finite, we denote the uniquely obtained measure by

\[
\pi = \mu \times \lambda
\]

and call the measure the product of \(\mu\) and \(\lambda\).

Proof. Suppose that \(A \times B\) can be written as \(\sum_{i=1}^{\infty} A_i \times B_i\), where each of the measurable rectangles \(A_i \times B_i\) are disjoint. Then

\[
\chi_A(x)\chi_B(y) = \chi_{A \times B}(x, y) = \sum_{i=1}^{\infty} \chi_{A_i}(x)\chi_{B_i}(y)
\]

for all \(x \in X\) and \(y \in Y\). Fix \(x\) and integrate with respect to \(\lambda\).
\[ \int_Y \chi_{A \times B} (x, y) \, d\lambda(y) = \int_Y \sum_{i=1}^{\infty} \chi_{A_i} (x) \chi_{B_i} (y) \, d\lambda(y) \]

\[ \int_Y \chi_A (x) \chi_B (y) \, d\lambda(y) = \sum_{i=1}^{\infty} \chi_{A_i} (x) \int_Y \chi_{B_i} (y) \, d\lambda(y) \]

Further integrating with respect to \( \mu \) yields (again by MCT)
\[ \mu(A) \lambda(B) = \sum_{i=1}^{\infty} \mu(A_i) \lambda(B_i) \quad (\ast). \]

Define \( \pi_0 \) on \( Z_0 \) by \( \pi_0(\cup_{i=1}^{n} A_i \times B_i) = \sum_{i=1}^{n} \mu(A_i) \lambda(B_i) \). Then \( \pi_0 \) is a measure on \( Z \) by (\ast) (the only nontrivial issue to check was countable additivity). Caratheodory’s Extension Theorem gives us a measure \( \pi \) defined on at least \( Z \) that extends \( \pi_0 \). If \( \mu \) and \( \lambda \) are \( \sigma \)-finite, then \( \pi_0 \) is \( \sigma \)-finite, so Hahn’s Extension Theorem tells us that \( \pi \) is unique.

**Definition 7.1.5.** Let \( E \subseteq Z = X \times Y \). An \( x \)-section of \( E \) is the set \( E_x = \{ y \in Y | (x, y) \in E \} \). A \( y \)-section is \( E^y = \{ x \in X | (x, y) \in E \} \). Let \( f : Z \rightarrow [-\infty, \infty] \) and \( x \in X \). The \( x \)-section of \( f \) is \( f_x(y) = f(x, y) \). For \( y \in Y \), the \( y \)-section of \( f \) is \( f^y(x) = f(x, y) \).

**Lemma 7.1.6.** If \( E \subseteq Z \) is measurable in the product measurable space \( (Z, \mathcal{Z} = A \times B) \).

1. \( E_x, E^y \) are measurable in the factors for each \( x \in X \) and \( y \in Y \).
2. If \( f : Z \rightarrow \mathbb{R}^* \) is \( \mathcal{Z} \)-measurable, then \( f_x, f^y \) are measurable in each factor for every \( x \in X, y \in Y \).

**Proof.** i) First observe that it is a routine exercise to show that the set

\[ S = \{ E \in \mathcal{Z} | E_x \text{ is measurable} \} \]

is a \( \sigma \)-algebra. However if \( E = A \times B \) where \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), then

\[ E_x = \begin{cases} B & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A \end{cases} \]

so \( A \times B \in S \) and hence \( S = \mathcal{Z} \).

A similar argument works for the sections \( E^y \).

**Proof: ii) Let \( f : Z \rightarrow \mathbb{R}^* \) be measurable, \( x \in X \) and \( \alpha \in \mathbb{R} \). Then

\[ \{ y \in Y | f_x(y) > \alpha \} = \{ y \in Y | f(x, y) > \alpha \} = \{ (w, y) \in X \times Y | f(w, y) > \alpha \} \]

which is measurable by i).

A similar argument shows that \( f^y \) is measurable.

**7.2 The Fubini’s Theorem**

**Definition 7.2.1.** A monotone class is a non-empty collection \( M \subseteq \mathcal{P}(X) \) such that

1. If \( \{ E_n \}_{n=1}^{\infty} \subseteq M \) with \( E_n \subseteq E_{n+1}, \) then \( \bigcup_{n=1}^{\infty} E_n \in M. \)
2. If \( \{ E_n \}_{n=1}^{\infty} \subseteq M \) with \( E_n \supseteq E_{n+1} \), then \( \bigcap_{n=1}^{\infty} E_n \in M. \)
Every $\sigma$-algebra is a monotone class. If $A \subseteq \mathcal{P}(X)$ is any collection of subsets, then there is a smallest monotone class $M(A)$ that contains $A$. Simply take the intersection of all monotone classes that contain $A$. With this in mind, it is clear that $M(A) \subseteq \sigma(A)$, the smallest $\sigma$-algebra that contains $A$. In fact, the reverse inclusion also holds when $A$ is an algebra.

**Lemma 7.2.2 [Monotone Class Lemma].** If $A \subseteq \mathcal{P}(X)$ is an algebra, then $M(A) = \sigma(A)$.

**Proof.** We need only show that $M = M(A)$ is an algebra, since this combined with the fact that $M$ is closed under countable unions implies that $M$ is closed under countable unions. For $E \in M$ define $M(E) = \{ F \in M | E \setminus F, E \cup F, F \setminus E \in M \}$. Then $\emptyset \in M(E)$ and $E \in M(E)$. Further, if $F \in M(E)$ then $E \in M(F)$ by the symmetry of the definition. $M(E)$ is a monotone class since complementation, union, and intersection play nicely together.

Suppose that $E \in A$. Then since $A$ is an algebra, $A \subseteq M(E)$. But $M(E)$ is also a monotone class, so $M \subseteq M(E) \subseteq M$ and $M = M(E)$. It follows that $A \subseteq M(F)$ for every $F \in M$, and again we have $M = M(F)$. But $\emptyset, X \in A \subseteq M(E)$, so this implies that $M$ is closed under intersections and finite unions. 

**Lemma 7.2.3.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be $\sigma$-finite. If $E \in Z = A \times B$, then $f(x) = \lambda(E_x)$ and $g(y) = \mu(E^y)$ are measurable and

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\lambda$$

**Proof:** **Case 1** Assume that $\mu$ and $\lambda$ are finite.

Let $M$ denote the collection of all such $E$ for which the lemma holds. We claim that $M$ is a monotone class containing $Z_0$ and that as such $M = Z$.

Let $E = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then

$$f(x) = \chi_A(x)\lambda(B) \quad \text{and} \quad g(y) = \mu(A)\chi_B(y)$$

so

$$\int_X f \, d\mu = \mu(A)\lambda(B) = \int_Y g \, d\lambda.$$ 

Since $Z_0$ consists of disjoint unions of such sets, $Z_0 \subseteq M$.

Let $\{E_n\} \subseteq M$ with $E_n \subseteq E_{n+1}$ and let

$$E = \bigcup_{n=1}^{\infty} E_n.$$ 

Then

$$f_n(x) = \lambda((E_n)_x) \quad \text{and} \quad g_n(y) = \mu((E_n)^y)$$

are measurable with

$$\int_X f_n \, d\mu = \pi(E_n) = \int_Y g_n \, d\lambda.$$ 

If

$$f(x) = \lambda(E_x) \quad \text{and} \quad g(y) = \mu(E^y),$$

then $f_n \nearrow f$ and $g_n \nearrow g$ so the Monotone Convergence Theorem and continuity from below for $\pi$ shows that

$$\int_X f \, d\mu = \pi(E) = \int_Y g \, d\lambda.$$ 

Given that $\pi$ is finite, we can argue in much the same way using the continuity from above for $\pi$ and the Lebesgue Dominated Convergence Theorem that if $\{E_n\} \subseteq M$ with $E_{n+1} \subseteq E_n$ and

$$E = \bigcap_{n=1}^{\infty} E_n,$$

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Case 2) If the measures are $\sigma$-finite, we let

$$Z = \bigcup_{n=1}^{\infty} Z_n$$

where $Z_n \subseteq Z_{n+1}$ and $\pi(Z_n) < \infty$. We then apply Case 1) to $E \cap Z_n$ and derive the final result from the MCT.

**Theorem 7.2.4 [Tonelli’s Theorem].** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be $\sigma$-finite. Let $F : Z = X \times Y \to [0, \infty]$ be measurable. Then the functions defined by $f(x) = \int_Y F(x, y) \, d\lambda$ and $g(y) = \int_X F(x, y) \, d\mu$ are measurable and

$$\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\lambda,$$

where $\pi = \mu \times \lambda$. This is to say that

$$\int_X \left( \int_Y F(x, y) \, d\lambda(y) \right) \, d\mu(x) = \int_Z F \, d\pi = \int_Y \left( \int_X F(x, y) \, d\mu(x) \right) \, d\lambda(y).$$

**Proof.** If $F = \chi_E$ for some $E \in Z = \mathcal{A} \times \mathcal{B}$, then the theorem is exactly the previous lemma. It follows immediately that the theorem holds for all non-negative measurable simple functions. If $F$ is arbitrary, we can find a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of non-negative simple functions such that $\Phi_n \uparrow F$. Let $\varphi_n(x) = \int_Y (\Phi_n)_x \, d\lambda$ and $\psi_n(y) = \int_X (\Phi_n)_y \, d\mu$. Then $\varphi_n$ and $\psi_n$ are measurable and monotonic in $n$. By the Monotone Convergence Theorem,

$$\lim_{n \to \infty} \varphi_n(x) = f(x) \quad \text{and} \quad \lim_{n \to \infty} \psi_n(y) = g(y).$$

Again, by the Monotone Convergence Theorem,

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X \varphi_n \, d\mu = \lim_{n \to \infty} \int_Z \Phi_n \, d\pi = \int_Z F \, d\pi$$

and similarly, $\int_Y g \, d\lambda = \int_Z F \, d\pi$.

**Theorem 7.2.5 [Fubini’s Theorem].** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be $\sigma$-finite and let $\pi = \mu \times \lambda$. If $F$ is integrable with respect to $\pi$ on $Z = X \times Y$, then the extended real valued functions defined almost everywhere by $f(x) = \int_Y F(x, y) \, d\lambda$ and $g(y) = \int_X F(y, x) \, d\mu$ have finite integrals and

$$\int_X f \, d\mu = \int_Z F \, d\pi = \int_Y g \, d\lambda.$$

That is to say,

$$\int_X \left( \int_Y F(x, y) \, d\lambda(y) \right) \, d\mu(x) = \int_Z F \, d\pi = \int_Y \left( \int_X F(x, y) \, d\mu(x) \right) \, d\lambda(y).$$

**Proof.** Since $F$ is $\pi$-integrable, so are $F^+$ and $F^-$. Apply Tonelli’s Theorem to establish that $f^+$ and $f^-$ have finite integrals and hence are finite almost everywhere. Therefore $f = f^+ - f^-$ is defined almost everywhere and $\int_X f \, d\mu = \int_Z F \, d\pi$. Similarly, we can show that $g = g^+ - g^-$ is defined almost everywhere and $\int_Y g \, d\lambda = \int_Z F \, d\pi$. 

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